Tunable IIR Digital Filters

- We have described earlier two 1st-order and two 2nd-order IIR digital transfer functions with tunable frequency response characteristics
- We shall show now that these transfer functions can be realized easily using allpass structures providing independent tuning of the filter parameters

Tunable Lowpass and Highpass Digital Filters

- We have shown earlier that the 1st-order lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

and the 1st-order highpass transfer function

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

are doubly-complementary pair

Tunable Lowpass and Highpass Digital Filters

- Moreover, they can be expressed as

$$H_{LP}(z) = \frac{1}{2}[1 + A_1(z)]$$

$$H_{HP}(z) = \frac{1}{2}[1 - A_1(z)]$$

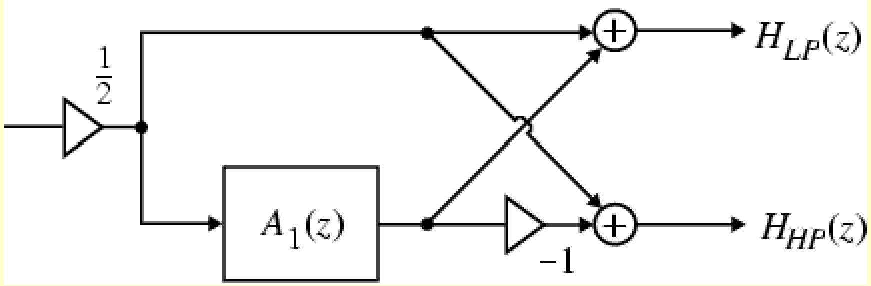
where

$$A_1(z) = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$$

is a 1st-order allpass transfer function

Tunable Lowpass and Highpass Digital Filters

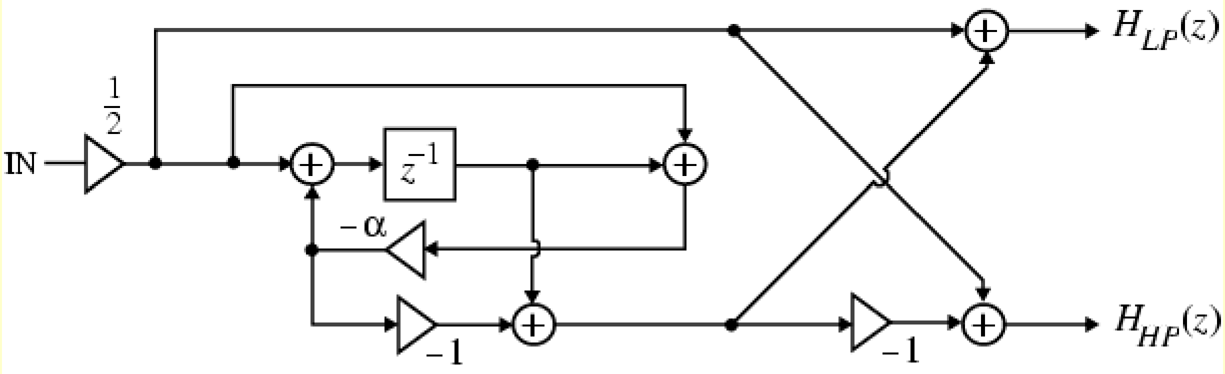
- A realization of $H_{LP}(z)$ and $H_{HP}(z)$ based on the allpass-based decomposition is shown below



- The 1st-order allpass filter can be realized using any one of the 4 single-multiplier allpass structures described earlier

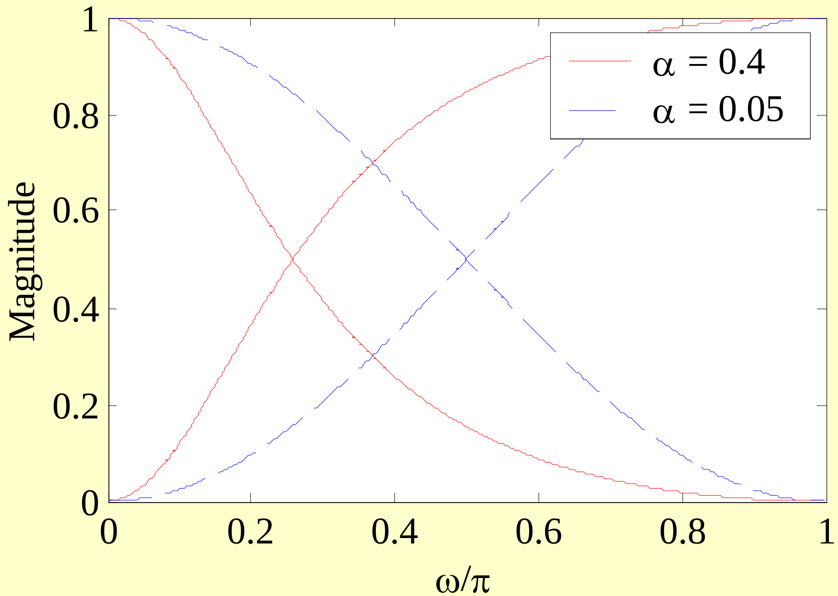
Tunable Lowpass and Highpass Digital Filters

- One such realization is shown below in which the 3-dB cutoff frequency of both lowpass and highpass filters can be varied simultaneously by changing the multiplier coefficient α



Tunable Lowpass and Highpass Digital Filters

- Figure below shows the composite magnitude responses of the two filters for two different values of α



Tunable Bandpass and Bandstop Digital Filters

- The 2nd-order bandpass transfer function

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

and the 2nd-order bandstop transfer function

$$H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

also form a doubly-complementary pair

Tunable Bandpass and Bandstop Digital Filters

- Thus, they can be expressed in the form

$$H_{BP}(z) = \frac{1}{2}[1 - A_2(z)]$$

$$H_{BS}(z) = \frac{1}{2}[1 + A_2(z)]$$

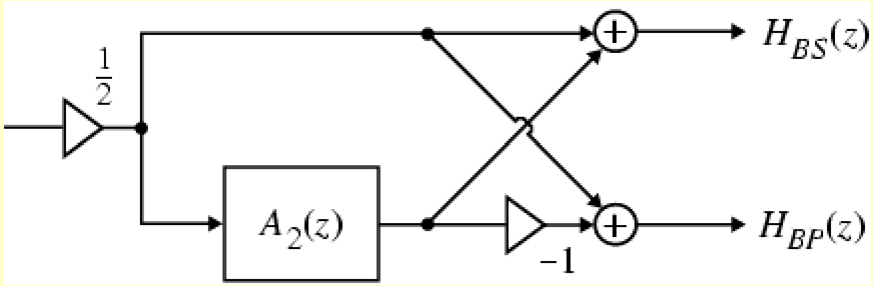
where

$$A_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

is a 2nd-order allpass transfer function

Tunable Bandpass and Bandstop Digital Filters

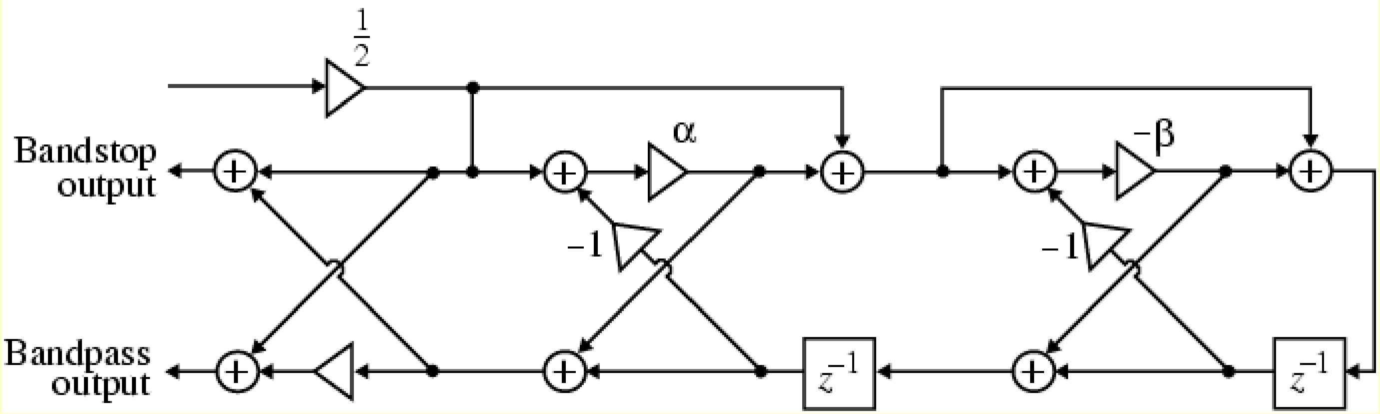
- A realization of $H_{BP}(z)$ and $H_{BS}(z)$ based on the allpass-based decomposition is shown below



- The 2nd-order allpass filter is realized using a cascaded single-multiplier lattice structure

Tunable Bandpass and Bandstop Digital Filters

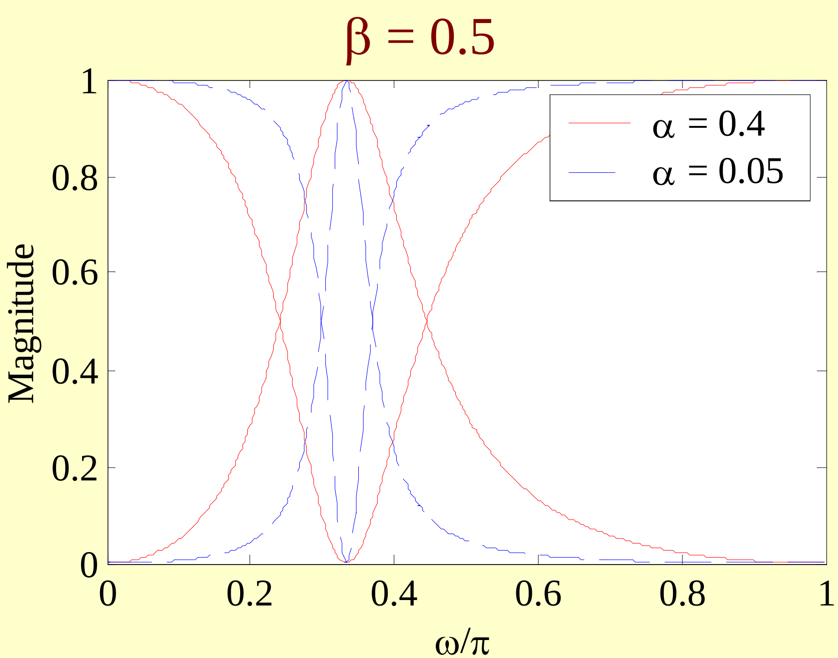
- The final structure is as shown below

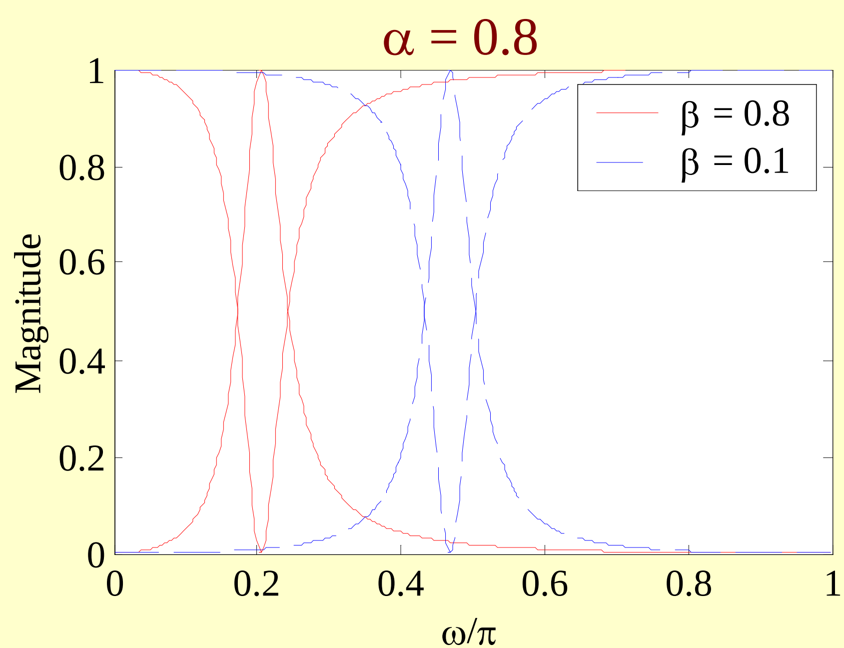


- In the above structure, the multiplier β controls the center frequency and the multiplier α controls the 3-dB bandwidth

Tunable Bandpass and Bandstop Digital Filters

- Figure below illustrates the parametric tuning property of the overall structure

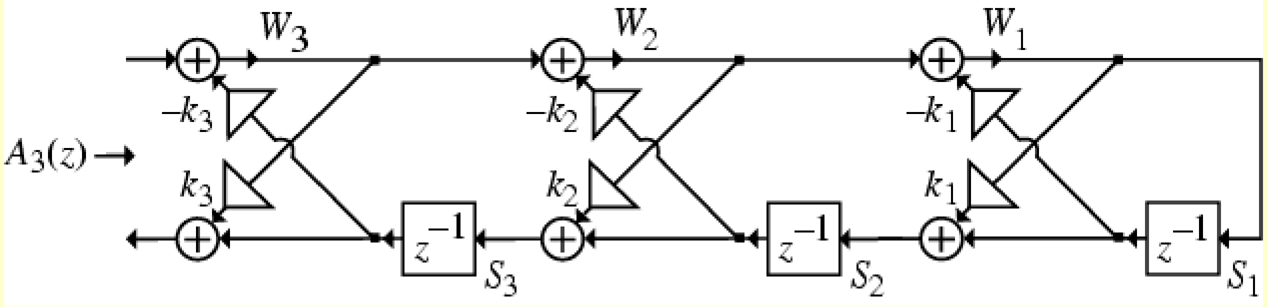




IIR Tapped Cascaded Lattice Structures

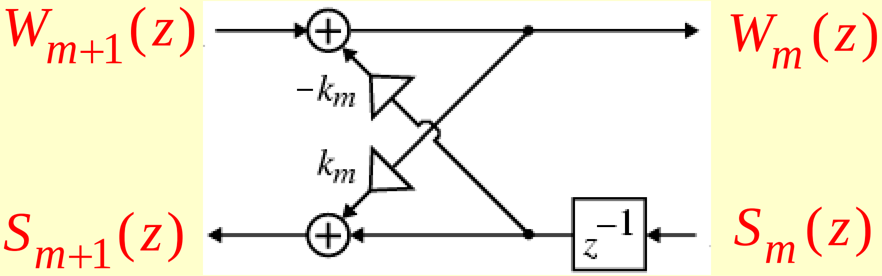
Realization of an All-pole IIR Transfer Function

- Consider the cascaded lattice structure derived earlier for the realization of an allpass transfer function



IIR Tapped Cascaded Lattice Structures

- A typical lattice two-pair here is as shown below



- Its input-output relations are given by

$$W_m(z) = W_{m+1}(z) - k_m z^{-1} S_m(z)$$

$$S_{m+1}(z) = k_m W_m(z) + z^{-1} S_m(z)$$

IIR Tapped Cascaded Lattice Structures

- From the input-output relations we derive the chain matrix description of the two-pair:

$$\begin{bmatrix} W_{i+1}(z) \\ S_{i+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i z^{-1} \\ k_i & z^{-1} \end{bmatrix} \begin{bmatrix} W_i(z) \\ S_i(z) \end{bmatrix}$$

- The chain matrix description of the cascaded lattice structure is therefore

$$\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \begin{bmatrix} 1 & k_3 z^{-1} \\ k_3 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} W_1(z) \\ S_1(z) \end{bmatrix}$$

IIR Tapped Cascaded Lattice Structures

- From the above equation we arrive at

$$X_1(z) = \{1 + [k_1(1+k_2) + k_2 k_3]z^{-1} + [k_2 + k_1 k_2(1+k_2)]z^{-2} + k_3 z^{-2}\}W_1(z)$$
$$= (1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3})W_1(z)$$

using the relation $S_1(z) = W_1(z)$ and the relations

$$k_1 = d_1'', \quad k_2 = d_2', \quad k_3 = d_3$$

IIR Tapped Cascaded Lattice Structures

- The transfer function $W_1(z)/X_1(z)$ is thus an all-pole function with the same denominator as that of the 3rd-order allpass function $A_3(z)$:

$$\frac{W_1(z)}{X_1(z)} = \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

IIR Tapped Cascaded Lattice Structures

Gray-Markel Method

- A two-step method to realize an Mth-order arbitrary IIR transfer function

$$H(z) = P_M(z) / D_M(z)$$

- Step 1: An intermediate allpass transfer function $A_M(z) = z^{-M} D_M(z^{-1}) / D_M(z)$ is realized in the form of a cascaded lattice structure

IIR Tapped Cascaded Lattice Structures

- Step 2: A set of independent variables are summed with appropriate weights to yield the desired numerator $P_M(z)$
- To illustrate the method, consider the realization of a 3rd-order transfer function

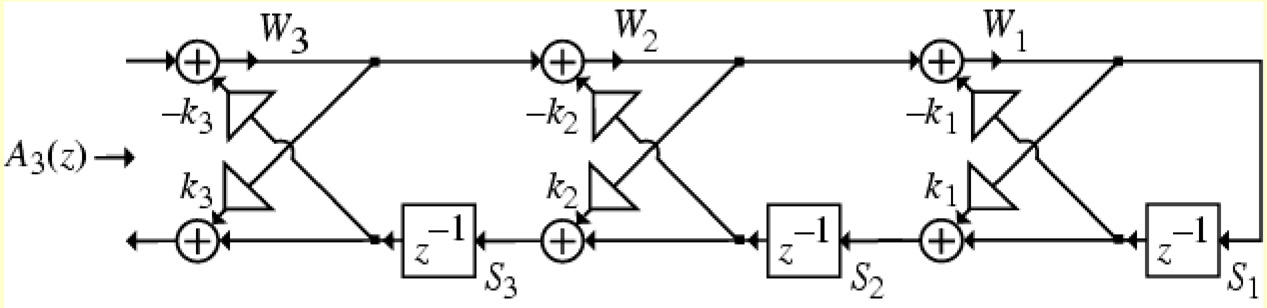
$$H(z) = \frac{P_3(z)}{D_3(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

IIR Tapped Cascaded Lattice Structures

- In the first step, we form a 3rd-order allpass transfer function

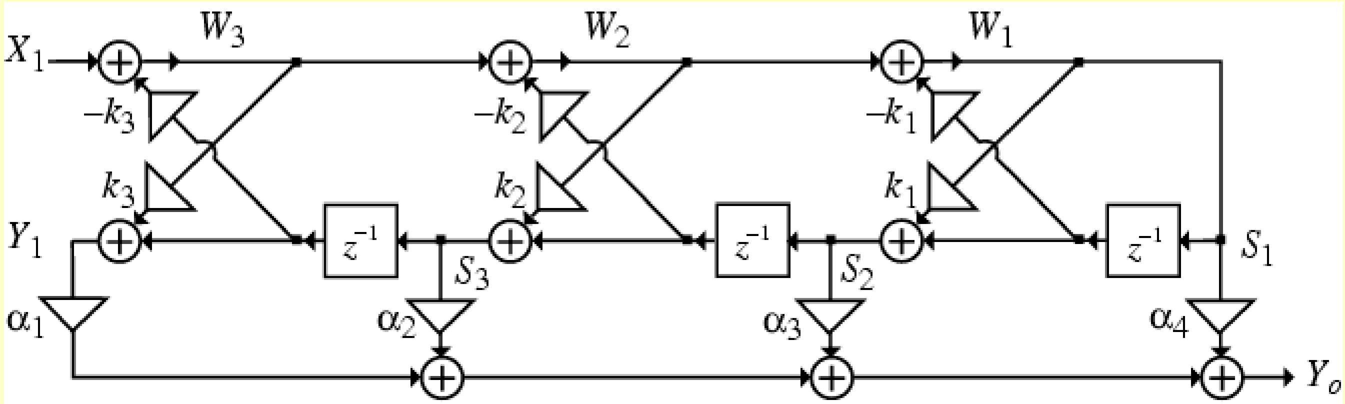
$$A_3(z) = Y_1(z) / X_1(z) = z^{-3} D_3(z^{-1}) / D_3(z)$$

- Realization of $A_3(z)$ has been illustrated earlier resulting in the structure shown below



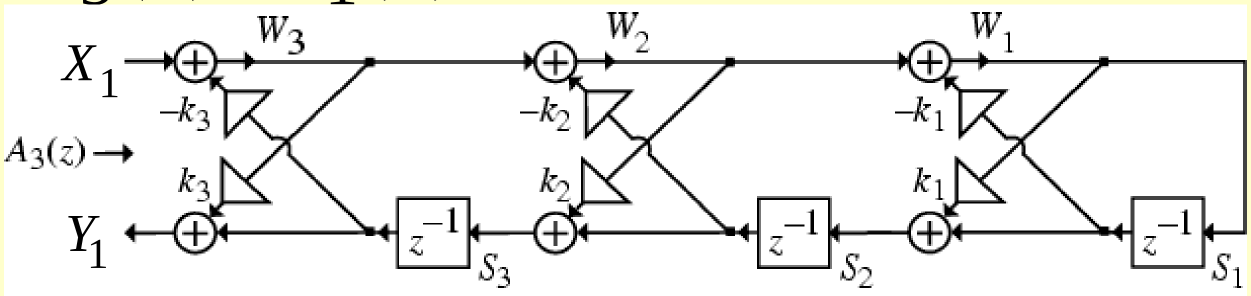
IIR Tapped Cascaded Lattice Structures

- Objective: Sum the independent signal variables Y_1, S_1, S_2, and S_3 with weights $\{\alpha_i\}$ as shown below to realize the desired numerator $P_3(z)$



IIR Tapped Cascaded Lattice Structures

- To this end, we first analyze the cascaded lattice structure realizing and determine the transfer functions $S_1(z)/X_1(z)$, $S_2(z)/X_1(z)$, and $S_3(z)/X_1(z)$



- We have already shown

$$\frac{S_1(z)}{X_1(z)} = \frac{1}{D_3(z)}$$

IIR Tapped Cascaded Lattice Structures

- From the figure it follows that

$$S_2(z) = (k_1 + z^{-1})S_1(z) = (d_1'' + z^{-1})S_1(z)$$

and hence

$$\frac{S_2(z)}{X_1(z)} = \frac{d_1'' + z^{-1}}{D_3(z)}$$

IIR Tapped Cascaded Lattice Structures

- In a similar manner it can be shown that

$$S_3(z) = (d'_2 + d'_1 z^{-1} + z^{-2}) S_1(z)$$

- Thus,

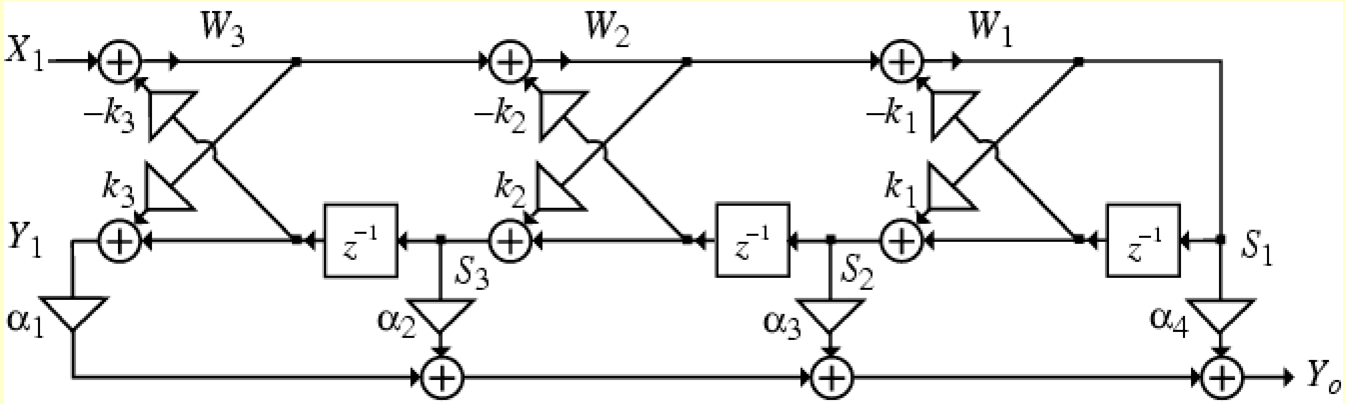
$$\frac{S_3(z)}{X_1(z)} = \frac{d'_2 + d'_1 z^{-1} + z^{-2}}{D_3(z)}$$

- Note: The numerator of $S_i(z)/X_1(z)$ is precisely the numerator of the allpass transfer function $A_i(z) = S_i(z)/W_i(z)$

IIR Tapped Cascaded Lattice Structures

- We now form

$$\frac{Y_o(z)}{X_1(z)} = \alpha_1 \frac{Y_1(z)}{X_1(z)} + \alpha_2 \frac{S_3(z)}{X_1(z)} + \alpha_3 \frac{S_2(z)}{X_1(z)} + \alpha_4 \frac{S_1(z)}{X_1(z)}$$



IIR Tapped Cascaded Lattice Structures

- Substituting the expressions for the various transfer functions in the above equation we arrive at

$$\frac{Y_o(z)}{X_1(z)} = \frac{\alpha_1(d_3 + d_2 z^{-1} + d_1 z^{-2} + z^{-3}) + \alpha_2(d_2' + d_1' z^{-1} + z^{-2}) + \alpha_3(d_1'' + z^{-1}) + \alpha_4}{D_3(z)}$$

IIR Tapped Cascaded Lattice Structures

- Comparing the numerator of $Y_o(z)/X_1(z)$ with the desired numerator $P_3(z)$ and equating like powers of z^{-1} we obtain

$$\alpha_1 d_3 + \alpha_2 d_2' + \alpha_3 d_1'' + \alpha_4 = p_0$$
$$\alpha_1 d_2 + \alpha_2 d_1' + \alpha_3 = p_1$$
$$\alpha_1 d_1 + \alpha_2 = p_2$$
$$\alpha_1 = p_3$$

IIR Tapped Cascaded Lattice Structures

- Solving the above equations we arrive at

$$\alpha_1 = p_3$$

$$\alpha_2 = p_2 - \alpha_1 d_1$$

$$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1^{'}$$

$$\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2^{'} - \alpha_3 d_1^{''}$$

IIR Tapped Cascaded Lattice Structures

- Example - Consider

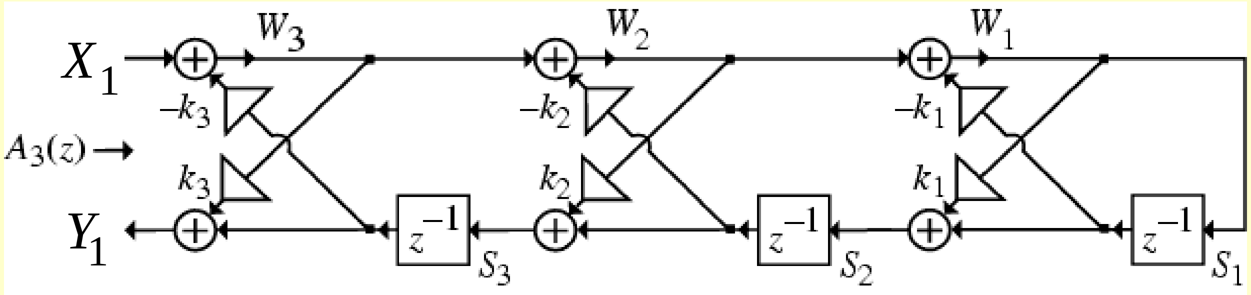
$$H(z) = \frac{P_3(z)}{D_3(z)} = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

- The corresponding intermediate allpass transfer function is given by

$$A_3(z) = \frac{z^{-3}D_3(z^{-1})}{D_3(z)} = \frac{-0.2 + 0.18z^{-1} + 0.0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

IIR Tapped Cascaded Lattice Structures

- The allpass transfer function $A_3(z)$ was realized earlier in the cascaded lattice form as shown below



- In the figure,

$$k_3 = d_3 = -0.2, \quad k_2 = d_2' = 0.2708333$$

$$k_1 = d_1'' = 0.3573771$$

IIR Tapped Cascaded Lattice Structures

- Other pertinent coefficients are:

$$d_1 = 0.4,\ d_2 = 0.18,\ d_3 = -0.2,\ d_1^{'} = 0.4541667$$

$$p_0 = 0,\ p_1 = 0.44,\ p_2 = 0.36,\ p_3 = 0.02,$$

- Substituting these coefficients in

$$\alpha_1 = p_3$$

$$\alpha_2 = p_2 - \alpha_1 d_1$$

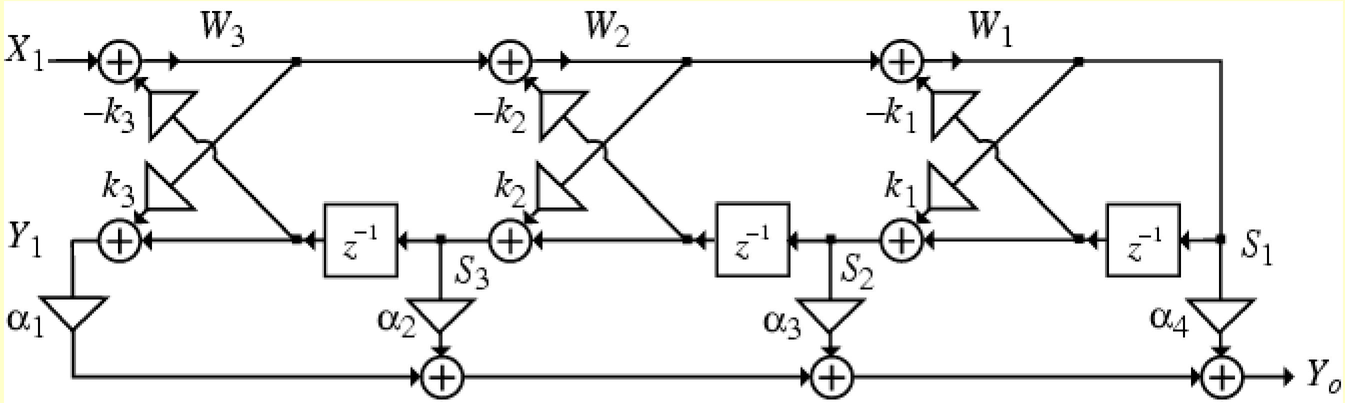
$$\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1^{'}$$

$$\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2^{'} - \alpha_3 d_1^{''}$$

IIR Tapped Cascaded Lattice Structures

$$\alpha_1 = 0.02, \;\; \alpha_2 = 0.352$$
$$\alpha_3 = 0.2765333, \;\; \alpha_4 = -0.19016$$

- The final realization is as shown below



$$k_1 = 0.3573771, \;\; k_2 = 0.2708333, \;\; k_3 = -0.2$$

Tapped Cascaded Lattice Realization Using MATLAB

- Both the pole-zero and the all-pole IIR cascaded lattice structures can be developed from their prescribed transfer functions using the M-file `tf2latc`
- To this end, Program 6_4 can be employed

Tapped Cascaded Lattice Realization Using MATLAB

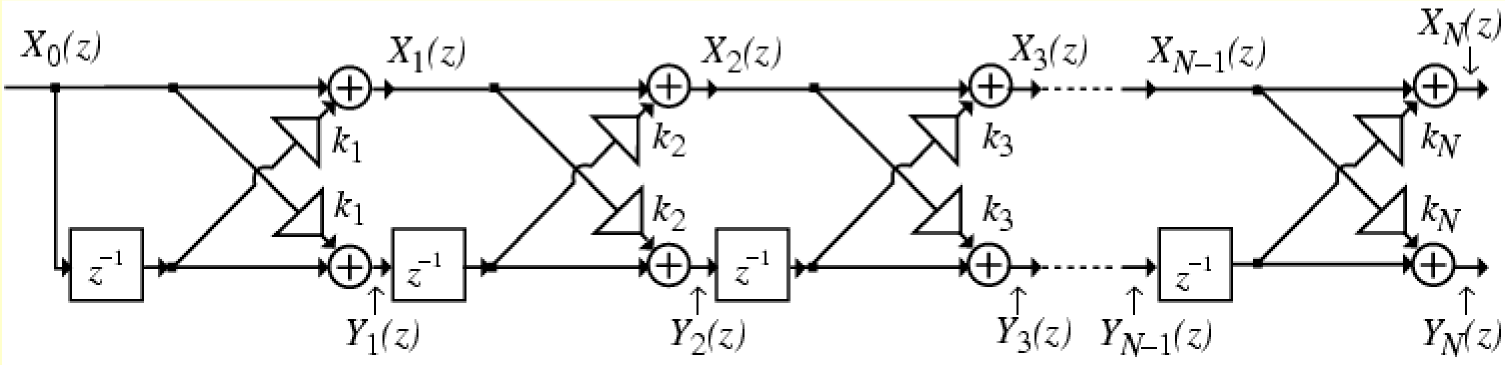
- The M-file `latc2tf` implements the reverse process and can be used to verify the structure developed using `tf2latc`
- To this end, Program 6_5 can be employed

FIR Cascaded Lattice Structures

- An arbitrary Nth-order FIR transfer function of the form

$$H_N(z) = 1 + \sum_{n=1}^{N} p_n z^{-n}$$

can be realized as a cascaded lattice structure as shown below



FIR Cascaded Lattice Structures

- From figure, it follows that

$$X_m(z) = X_{m-1}(z) + k_m z^{-1} Y_{m-1}(z)$$

$$Y_m(z) = k_m X_{m-1}(z) + z^{-1} Y_{m-1}(z)$$

- In matrix form the above equations can be written as

$$\begin{bmatrix} X_m(z) \\ Y_m(z) \end{bmatrix} = \begin{bmatrix} 1 & k_m z^{-1} \\ k_m & z^{-1} \end{bmatrix} \begin{bmatrix} X_{m-1}(z) \\ Y_{m-1}(z) \end{bmatrix}$$

where $m = 1, 2, \ldots, N$

FIR Cascaded Lattice Structures

- Denote

$$H_m(z) = \frac{X_m(z)}{X_0(z)}, \qquad G_m(z) = \frac{Y_m(z)}{X_0(z)}$$

- Then it follows from the input-output relations of the *m*-th two-pair that

$$H_m(z) = H_{m-1}(z) + k_m z^{-1} G_{m-1}(z)$$
$$G_m(z) = k_m H_{m-1}(z) + z^{-1} G_{m-1}(z)$$

FIR Cascaded Lattice Structures

- From the previous equation we observe

$$H_1(z) = 1 + k_1 z^{-1}, \quad G_1(z) = k_1 + z^{-1}$$

where we have used the facts

$$H_0(z) = X_0(z)/X_0(z) = 1$$

$$G_0(z) = Y_0(z)/X_0(z) = X_0(z)/X_0(z) = 1$$

- It follows from the above that

$$G_1(z) = z^{-1}(z\,k_1 + 1) = z^{-1}H_1(z^{-1})$$

- $\Rightarrow$ $G_1(z)$ is the mirror-image of $H_1(z)$

FIR Cascaded Lattice Structures

- From the input-output relations of the m-th two-pair we obtain for $m = 2$:

$$H_2(z) = H_1(z) + k_2 z^{-1} G_1(z)$$
$$G_2(z) = k_2 H_1(z) + z^{-1} G_1(z)$$

- Since $H_1(z)$ and $G_1(z)$ are 1st-order polynomials, it follows from the above that $H_2(z)$ and $G_2(z)$ are 2nd-order polynomials

FIR Cascaded Lattice Structures

- Substituting $G_1(z) = z^{-1}H_1(z^{-1})$ in the two previous equations we get

$$H_2(z) = H_1(z) + k_2 z^{-2} H_1(z^{-1})$$

$$G_2(z) = k_2 H_1(z) + z^{-2} H_1(z^{-1})$$

- Now we can write

$$G_2(z) = k_2 H_1(z) + z^{-2} H_1(z^{-1})$$

$$= z^{-2}[k_2 z^2 H_1(z) + H_1(z^{-1})] = z^{-2} H_2(z^{-1})$$

- $\Rightarrow$ $G_2(z)$ is the mirror-image of $H_2(z)$

FIR Cascaded Lattice Structures

- In the general case, from the input-output relations of the *m*-th two-pair we obtain

$$H_m(z) = H_{m-1}(z) + k_m z^{-1} G_{m-1}(z)$$
$$G_m(z) = k_m H_{m-1}(z) + z^{-1} G_{m-1}(z)$$

- It can be easily shown by induction that

$$G_m(z) = z^{-m} H_m(z^{-1}), \quad m = 1, 2, \ldots, N-1, N$$

- $\Rightarrow$ $G_m(z)$ is the mirror-image of $H_m(z)$

FIR Cascaded Lattice Structures

- To develop the synthesis algorithm, we express $H_{m-1}(z)$ and $G_{m-1}(z)$ in terms of $H_m(z)$ and $G_m(z)$ for $m = N, N-1, \ldots, 2, 1$ arriving at

$$H_{N-1}(z) = \frac{1}{(1-k_N^2)}\{H_N(z) - k_N G_N(z)\}$$

$$G_{N-1}(z) = \frac{1}{(1-k_N^2)z^{-1}}\{-k_N H_N(z) + G_N(z)\}$$

FIR Cascaded Lattice Structures

- Substituting the expressions for

$$H_N(z) = 1 + \sum_{n=1}^{N} p_n z^{-n}$$

and

$$G_N(z) = z^{-N} H_N(z^{-1}) = \sum_{n=0}^{N-1} p_n z^{-n} + z^{-N}$$

in the first equation we get

$$H_{N-1}(z) = \frac{1}{1-k_N^2}\{(1-k_N p_N) + \sum_{n=1}^{N-1}(p_n - k_n p_{N-n})z^{-n}$$

$$+ (p_N - k_N)z^{-N}\}$$

FIR Cascaded Lattice Structures

- If we choose $k_N = p_N$, then $H_{N-1}(z)$ reduces to an FIR transfer function of order $N-1$ and can be written in the form
$$H_{N-1}(z) = 1 + \sum_{n=1}^{N-1} p'_n z^{-n}$$
where $p'_n = \frac{p_n - k_N p_{N-n}}{1-k_N^2}, 1 \le n \le N-1$
- Continuing the above recursion algorithm, all multiplier coefficients of the cascaded lattice structure can be computed

FIR Cascaded Lattice Structures

- Example - Consider

$$H_4(z) = 1 + 1.2z^{-1} + 1.12z^{-2} + 0.12z^{-3} - 0.08z^{-4}$$

- From the above, we observe $k_4 = p_4 = -0.08$
- Using

$$p'_n = \frac{p_n - k_4 p_{4-n}}{1 - k_4^2},\ 1 \le n \le 3$$

we determine the coefficients of $H_3(z)$:

$$p'_3 = 0.2173913,\quad p'_2 = 1.2173913$$

$$p'_1 = 1.2173913$$

FIR Cascaded Lattice Structures

- As a result,

$$H_3(z) = 1 + 1.2173913\,z^{-1} + 1.2173913\,z^{-2} + 0.2173913\,z^{-3}$$

- Thus, $k_3 = p'_3 = 0.2173913$
- Using

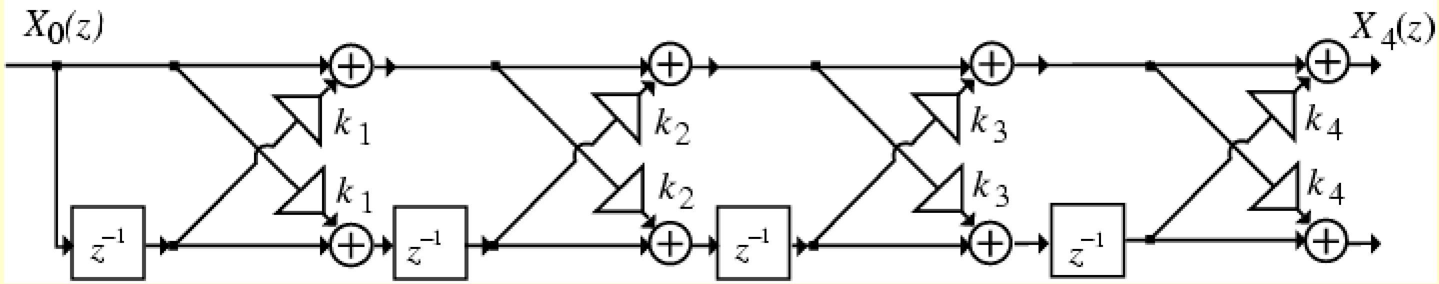
$$p''_n = \frac{p'_n - k_3 p'_{2-n}}{1 - k_3^2},\ 1 \le n \le 2$$

we determine the coefficients of $H_2(z)$:

$$p''_2 = 1.0, \qquad p''_1 = 1.0$$

FIR Cascaded Lattice Structures

- As a result, $H_2(z) = 1 + z^{-1} + z^{-2}$
- From the above, we get $k_2 = p''_2 = 1$
- The final recursion yields the last multiplier coefficient $k_1 = p''_1/(1 + k_2) = 0.5$
- The complete realization is shown below



$$k_1 = 0.5,\ k_2 = 1,\ k_3 = 0.2173913,\ k_4 = -0.08$$

FIR Cascaded Lattice Realization Using MATLAB

- The M-file `tf2latc` can be used to compute the multiplier coefficients of the FIR cascaded lattice structure
- To this end Program 6_6 can be employed
- The multiplier coefficients can also be determined using the M-file `poly2rc`