Module 8 : Numerical Relaying I : Fundamentals

Lecture 28 : Sampling Theorem

Objectives

In this lecture, you will review the following concepts from signal processing:

- Role of DSP in relaying.
- Sampling theorem.
- Role of anti-iliasing filter.

28.1 Why Digital Signal Processing?

Digital relaying involves digital processing of one or more analog signals. It involves following three steps:

1. Conversion of analog signal to digital form.
2. Processing of digital form.
3. Boolean decision to trip or not to trip.

Usually in DSP, after processing information in discrete domain, it has to be converted back to analog domain. However, for us the step- 3 does not involve conversion of processed signals back to analog form.

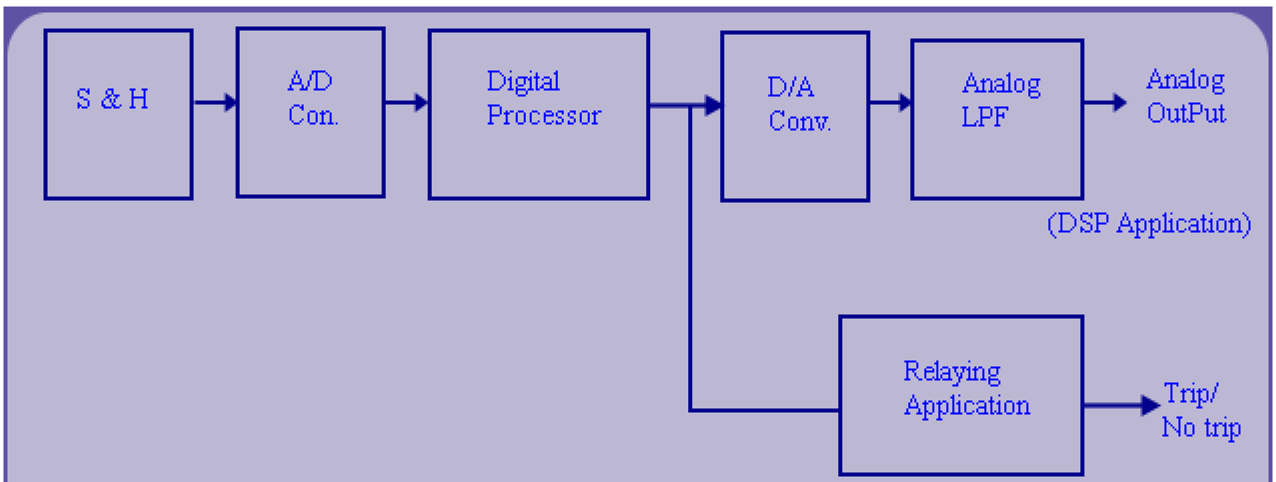

Fig 28.1 Block Diagram of DSP

In the previous lecture, we have discussed the step - 1 in detail. In this lecture we discuss the next step. At this point, a worthwhile observation is that direct analog signal processing is conceptually much simpler. However, advantages of digital processing far outweigh analog processing. Some of the advantages of digital processing are as follows:

- Operation of digital circuits do not depend on precise values of digital signals. As a result, a digital circuit is less
- sensitive to tolerances of component values.

- A digital circuit has little sensitivity to temperature, aging and other external parameters.
- In terms of economics of volume, a digital circuit can be reproduced easily in volume quantities. With VLSI circuits, it is

  possible to integrate highly sophisticated and complex digital signal processing systems on a single chip.
- In DSP, accuracy of computation can be increased by increasing word length. With the availability of floating point
- arithmetic in digital signal processors, dynamic ranges of signal and coefficients can be increased.
- A signal processor can process many signals, reducing processing cost per signal.
- Digital implementation allows the realization of certain characteristics not possible with analog implementation; such
- as polygon in R-X plane for distance relaying.
- Digital signals can be stored indefinitely without loss of accuracy.

There are also some disadvantages with DSP. One of them is that DSP contains active devices. Active devices are less reliable than passive components. Passive components consume less power than active devices. However, advantages of digital relays (i.e. relaying using digital signal processing) are far more significant than the disadvantages. In what follows, we discuss digital signal processing for relaying.

28.2 Sampling

Consider a continuous time domain sinusoid signal as, $x(t) = \sin(2\pi f_0 t)$ .The sine wave has frequency f_0 e.g. 50 Hz. Let the waveform x(t) be sampled at a rate of f_s samples/sec, i.e. with time period $t_s = 1/f_s$ sec. Let the sampling process start at time $t = 0$. The samples of this sequence are given by

$$x(0) = \sin(2\pi f_0(0t_s)) \quad (1)$$

$$x(1) = \sin(2\pi f_0(1t_s)) \quad (2)$$

$$x(2) = \sin(2\pi f_0(2t_s)) \quad (3)$$

$$\cdots \qquad \cdots \quad (4)$$

$$x(n) = \sin(2\pi f_0(nt_s)) \quad (5)$$

Because of periodicity of sine wave, it is not possible to distinguish two samples with a phase difference equal to 2πm , where m is an integer. Therefore,

$$x(n) = \sin(2\pi f_0 nt_s) = sin(2\pi f_0 nt_s + 2\pi m) \quad (6)$$

$$= \sin\left(2\pi\left(f_0 + \frac{m}{nt_s}\right)nt_s\right) \quad (7)$$

If we choose m to be an integer multiple of n i.e. $m = kn$; k an integer and substitute ***$t_s = 1/f_s$***, the above equation transforms into the following:

$$x(n) = sin(2\pi(f_0 + kf_s)nt_s) \quad (8)$$

Note that in equation(7) m has to be varied from sample to sample, so as to maintain fixed value of k.

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s
values;

$$x(0) = \sin(2\pi f_0(0t_s)) \quad (1)$$

$$x(1) = \sin(2\pi f_0(1t_s)) \quad (2)$$

$$x(2) = \sin(2\pi f_0(2t_s)) \quad (3)$$

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Because of periodicity of sine wave, it is not possible to distinguish two samples with a phase difference equal to $2\pi m$, where m is an integer. Therefore,

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28.2 Sampling (contd..)

The equation (8), leads to a very interesting result viz.

When sampling at a rate of f_s samples/sec. if k is any positive or negative integer, we cannot distinguish between sampled values of sine wave of ***f_0 Hz*** and a sine wave of ***(f_0+kf_s) Hz.***

Fig 28.2 shows a ***7kHz*** signal being sampled at ***6khz*** with $f_0 = 7$, $f_s = 6$ and $k = -1$, we reach the conclusion that we cannot distinguish between signal of ***7kHz*** and ***1kHz*** with sampling frequency of ***6kHz***. This effect of ***7kHz*** signal taking an alias of ***1kHz*** signal is called aliasing. In this case, a high frequency signal has taken an alias of a low frequency signal.

28.2 Sampling (contd..)

An another example, consider a ***50Hz*** signal sampled at ***50Hz*** (see in fig 28.3). It can be seen that signal is aliased to dc signal.

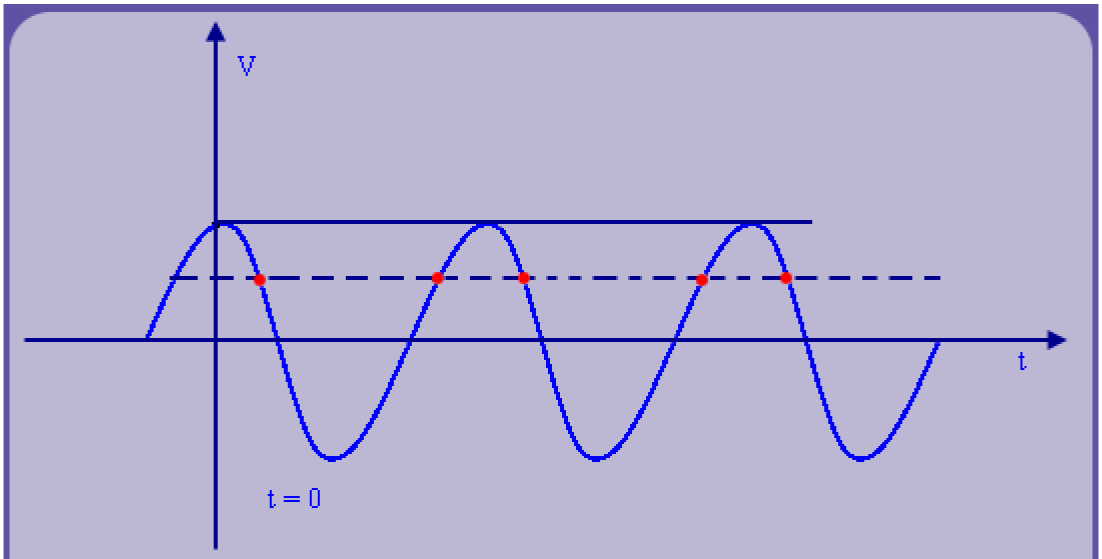

Fig 28.3 50Hz Signal Sampled at 50Hz

Similarly, sampling a 50 Hz signal at 51 Hz will alias it to 1 Hz.

28.2 Sampling (contd..)

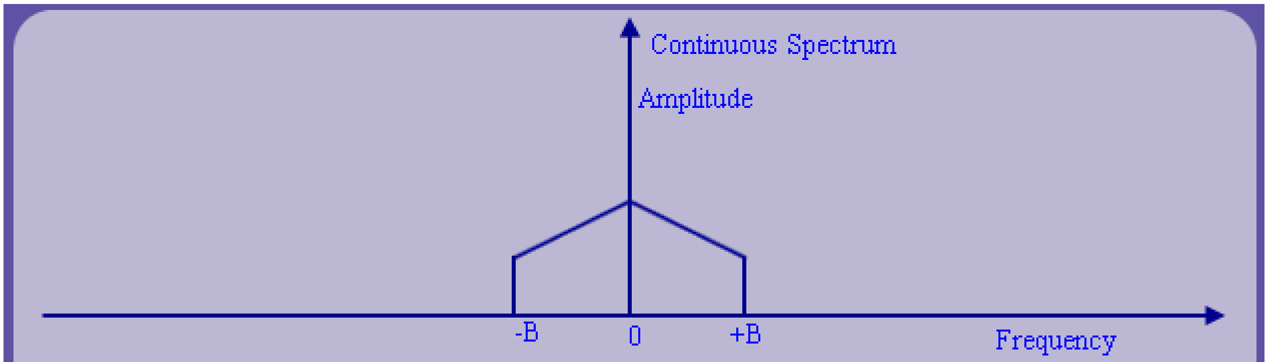

Fig 28.4 Original Continuous Signal Spectrum

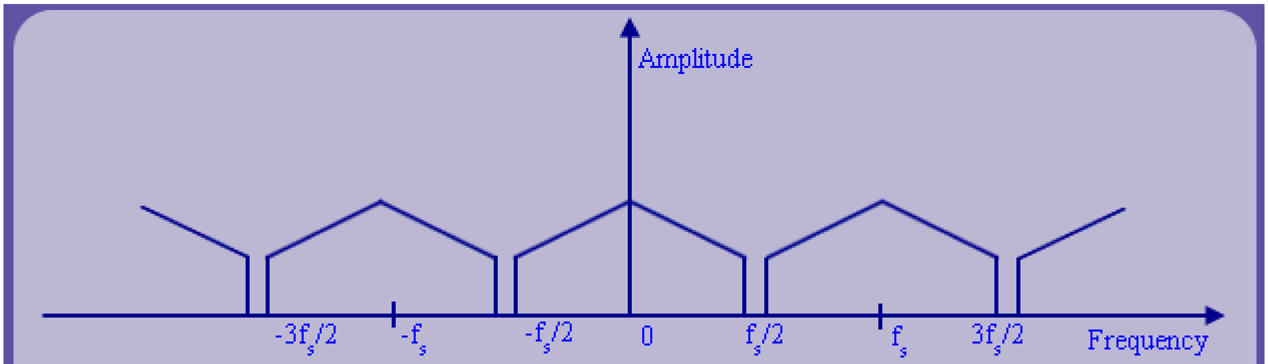

Fig 28.5 Spectral Representation for Signal Sampled at $f_s/2>B$

Fig 28.4 shows a signal with frequency content between ±B Hz . Such signals are said to be band limited signals. Note that because $\cos \omega t = (e^{+j\omega t} + e^{-j\omega t})/2$ and $\sin \omega t = (e^{+j\omega t} - e^{-j\omega t})/2j$ magnitude component of a real life signals have typically an even symmetry around dc signal. By the observation made in the previous slide that a signal of f_0 Hz can be aliased to ($f_0 \pm kf_s$) Hz { k = ±1, ± 2,--- } , it follows that post sampling in frequency domain, we will see repeating lobes (replicas) of original signal, each lobe being displaced by f_s Hz. In other words, after sampling we cannot distinguish the signal lobe from other replicated lobes.

An interesting analog can be drawn by considering a room having many mirrors each reflecting image from one to another. It is seen that if a person is standing in such a room, another observer cannot distinguish him from his image. The difficulty can be resolved if the observer has an idea of location or coordinates of the real person. In the same manner, we can identify the original lobe from replicated lobes if we have an idea of the frequency content of original signal. In fig 28.5, notice that lobes are distinctly separated because f_s > 2B Hz . On the other hand, if f_s = 2B Hz , then as seen in fig 28.6, lobes will just touch each other. If however, f_s < 2B Hz, then lobes will overlap (fig 28.7) and this will lead to distortion of replicated frequency spectrum. Thus, it is necessary that f_s the sampling frequency should atleast equal to 2B Hz.

28.2 Sampling (contd..)

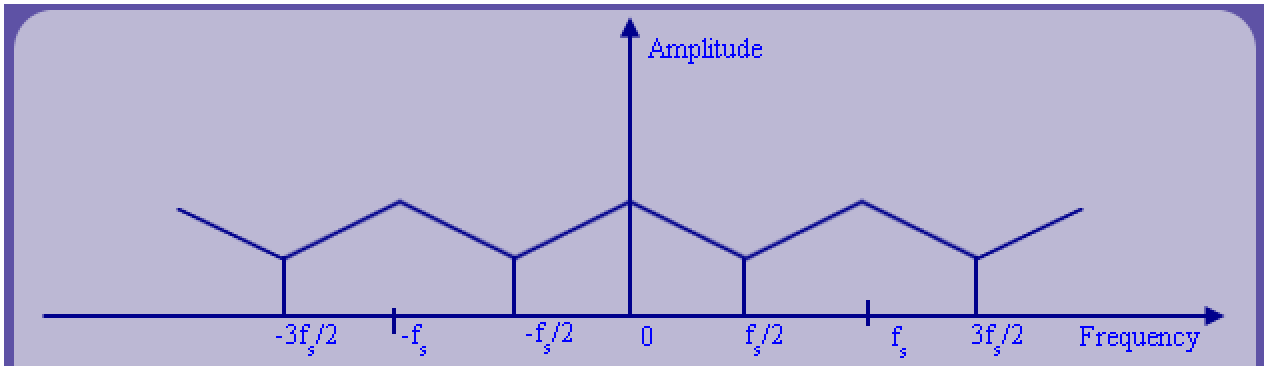

Fig 28.6 Spectral Representation for Signal Sampled at $f_s/2 = B$

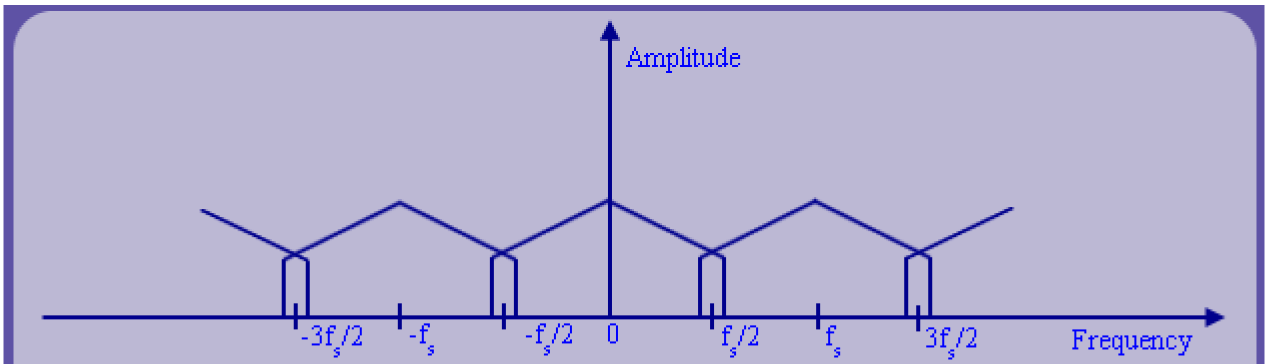

Fig 28.7 Spectral Representation for Signal Sampled at $f_s/2 < B$

Thus qualitatively, we can classify sampling frequency into three categories.

1. Sampling at a rate $f_s > 2B$
2. Sampling at a rate $f_s = 2B$
3. Sampling at a rate $f_s < 2B$

28.2 Sampling (contd..)

When sampling frequency, $f_s < 2B$, then there is an overlapping effect around frequency $f_s/2$, known as the **folding frequency**. As a consequence of superposition, the frequency domain information is distorted. Thus, we should choose $f_s > 2B$. This important result is a part of sampling theorem stated below in two equivalent ways.

1. A band limited signal of finite energy, which has no frequency component higher than B Hz, is completely described

   by specifying the values of the signal at instants of time separated by $\frac{1}{2B}$ seconds.

2. A band limited signal of finite energy which has no frequency component higher than B Hz may be completely

   recovered from the knowledge of its samples taken at a rate of $2B$ per sec.

The sampling rate of 2B samples per sec is known as **Nyquist rate**.

In practice, even a band limited signal will contain noise. Noise reflects as high frequency component in the overall spectrum, (fig 28.8). Thus, even if we sample the signal at a rate say $2.1B$ Hz, we cannot reconstruct the correct frequency domain information. Noise is aliased to lower frequency. It distorts the frequency domain information by superposing an alias of noise on the original signal. To avoid this, in practice it is necessary to pass the continuous signal first through an analog low pass filter. Such a filter is known as anti-aliasing filter. Fig 28.9 illustrates this concept.

28.2 Sampling (contd..)

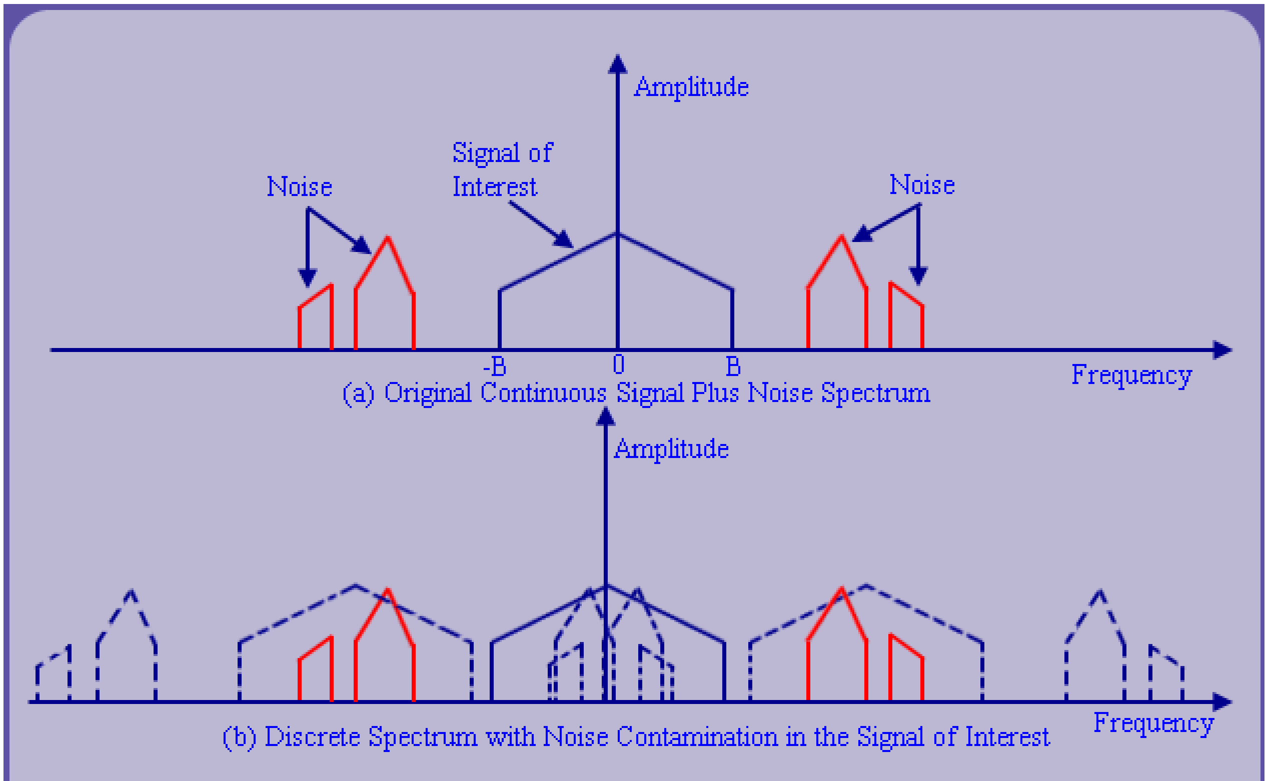

Fig 28.8 Spectral Representation of Noisy Signal

28.2 Sampling

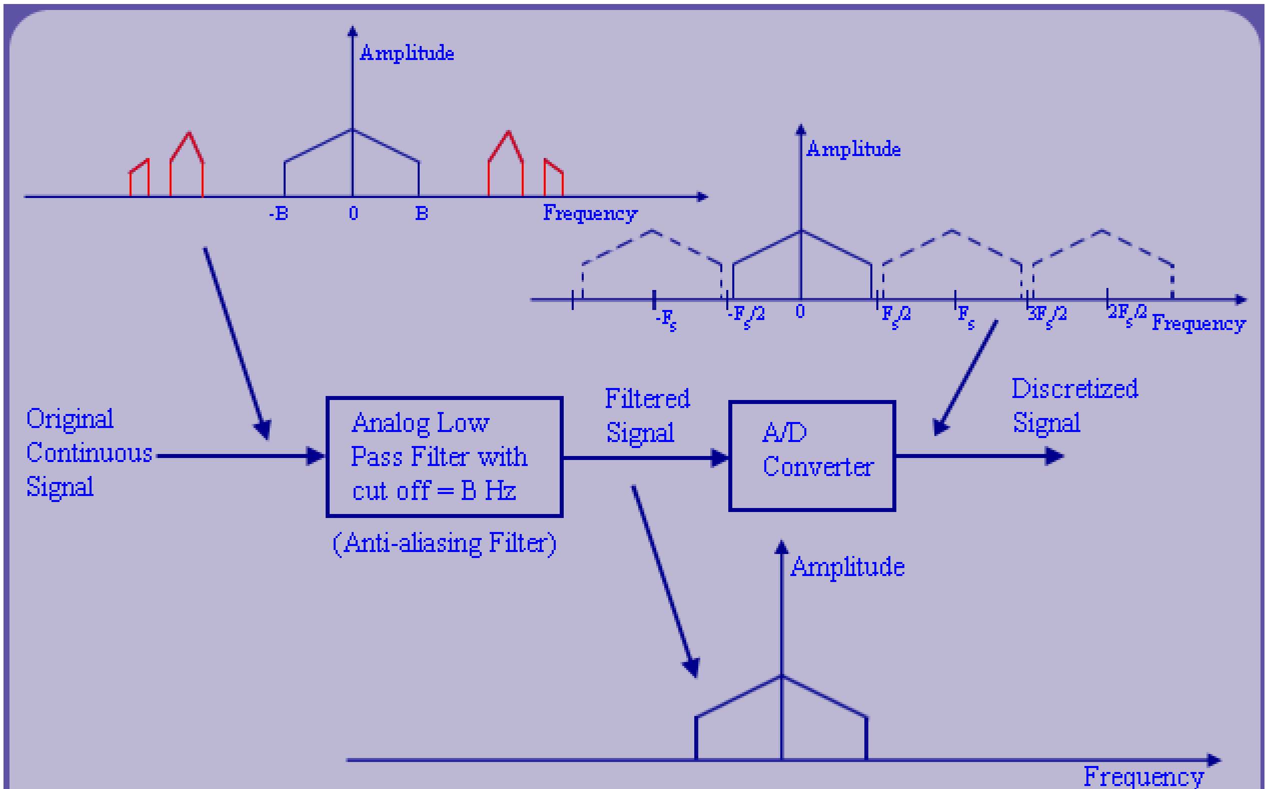

Fig 28.9 Role of Low-pass Analog (Anti-aliasing) Filtering

Review Questions

1. Derive the sampling theorem.
2. A 40 kHz signal is sampled at 49 kHz. What is the minimum frequency to which this signal will be aliased.
3. For the signal in fig 28.2, suggest appropriate sampling frequency.

Recap

In this lecture we have learnt the following:

- Role of DSP in relaying.
- Sampling theorem.
- Role of anti-aliasing filter.