Module 8 : Numerical Relaying I : Fundamentals

Lecture 30 : Least Square Method for Estimation of Phasors - II

Objectives

In this lecture, we will learn 3-sample estimation

- Method for phasor estimation.
- Linear Least Squares (LS) probem will be formulated and solved.
- Improvement, in the accuracy of estimation due to redundancy in measurements will be demonstrated.

30.1 Three sample technique

We now present a 3-sample technique for measurement of current and voltage which has better noise elimination characteristic than the 2-sample technique discussed earlier. Three sample technique has a redundancy of 1.5. In other words, we plan to show that standard deviation of the estimate reduces with the use of a larger data window. For the voltage signal, consider three most recent samples given as follows.

$$v_k = V_m \sin(\omega t_k + \phi_v)$$

$$v_{k-1} = V_m \sin(\omega t_{k-1} + \phi_v)$$

$$v_{k-2} = V_m \sin(\omega t_{k-2} + \phi_v)$$

After doing the necessary trigonometric expansions, we get

$$v_k = V_m \sin\theta_k \cos\phi_v + V_m \cos\theta_k \sin\phi_v$$

$$v_{k-1} = V_m \sin\theta_{k-1} \cos\phi_v + V_m \cos\theta_{k-1} \sin\phi_v$$

$$v_{k-2} = V_m \sin\theta_{k-2} \cos\phi_v + V_m \cos\theta_{k-2} \sin\phi_v$$

Where $\theta_k = \omega t_k$. Arranging the equations in the matrix format, we get

$$\begin{bmatrix} v_k \\ v_{k-1} \\ v_{k-2} \end{bmatrix} = \begin{bmatrix} \sin\theta_k & \cos\theta_k \\ \sin\theta_{k-1} & \cos\theta_{k-1} \\ \sin\theta_{k-2} & \cos\theta_{k-2} \end{bmatrix} \begin{bmatrix} V_m \cos\varphi_v \\ V_m \sin\phi_v \end{bmatrix} \qquad (1)$$

Since, the real life signal has also noise in it, hence a more appropriate system model is given by

$$\begin{bmatrix} v_k \\ v_{k-1} \\ v_{k-2} \end{bmatrix} = \begin{bmatrix} \sin\theta_k & \cos\theta_k \\ \sin\theta_{k-1} & \cos\theta_{k-1} \\ \sin\theta_{k-2} & \cos\theta_{k-2} \end{bmatrix} \begin{bmatrix} V_m \cos\phi_v \\ V_m \sin\phi_v \end{bmatrix} + \begin{bmatrix} e_k \\ e_{k-1} \\ e_{k-2} \end{bmatrix} \qquad (2)$$

The above equations can be written in simple matrix format $b = Ax + e$

It should be obvious by now that our job of estimating, unknown $V_m \cos\phi_v$ and $V_m \sin\phi_v$ is no-longer as

simple as solving the linear system equations in the 2-sample method.
Therefore, we now introduce a widely used technique for estimating the unknowns in presence of redundant measurements. It is known as least square estimation.

30.2 Least Square Estimation

Consider the problem of solving the following linear system of equations.

$$3x_1 + 2x_2 = 1 \quad (3)$$

$$2x_1 + 3x_2 = 3 \quad (4)$$

$$5x_1 + 5x_2 = 5 \quad (5)$$

30.2 Least Square Estimation (contd..)

It is quite easy to verify that the system of equations is inconsistent because left hand side of (5) is the sum of left hand side of (3) and (4) but right hand side of (5) is not the sum of RHS of (3) and (4). If we view linear system of equations in matrix algebra format $Ax = b$; then, we know that the system of equations is consistent if and only if

rank (A) = rank (A, b).

In this case

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 5 & 5 \end{bmatrix}; \; b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

rank (A) = 2 and rank (A, b) = 3

Some of you may be wondering as to what is the connection of this example with the problem at our hand. Suppose that noise contribution in b(1) is 0.1, in b(2) is -0.1. Then the noise in the third measurement is 1. It is now apparent that inconsistency in the linear system of equations is happening because of noise. Thus, our job should be to estimate the noise and eliminate it. This can not be done within the frame work of linear system of equations. It is for this purpose the method of least squares is used.

Method of least squares is used for point estimation, curve fitting etc. It can be shown that even Fourier series solves a least square estimation problem. So what is least square estimation problem? In some sense, we want to find a vector x such that Ax is nearest to vector b. For this purpose, we define a residual vector r as $b - Ax$ i.e. $r = b - Ax$. The Eucledian length of the vector $\sqrt{r_1^2 + r_2^2 + \dots\dots\dots + r_m^2}$ gives a measure of closeness of Ax to b. Our aim now is to minimize the length of the residual vector.

min $\sqrt{r_1^2 + r_2^2 + \dots\dots\dots + r_m^2}$

Minimization of length of residual vector or square of length are equivalent in the sense, the minimum is reached at same value of x^*. However, working with squares eliminates the problem of square roots in calculating derivatives. Hence, a least square problem is defined as

$$\phi(x) = \min \frac{1}{2} r^T r = \frac{r_1^2 + r_2^2 + \dots\dots + r_m^2}{2} = \frac{1}{2}(b - Ax)^T (b - Ax) = \frac{1}{2} x^T A^T Ax - b^T Ax + \frac{b^T b}{2} \quad (6)$$

The scalar $\frac{1}{2}$ is introduced only for the sake of convenience. Again it does not affect the optimal x^*. Constant in the estimation problem can be neglected without affecting x^*. Let us now work out the least square solution for the example problem under consideration.

Substituting the values of A and b in (6), we get following least square problem.

$$\min \frac{1}{2}\left[x^T\right]\begin{bmatrix} 38 & 37 \\ 37 & 38 \end{bmatrix}[x]-\left[34 \;\; 36\right][x]+17.5$$

i.e. min $\frac{1}{2}\left(38x_1^2+38x_2^2+2\times 37\times x_1\times x_2\right)-\left(34x_1+36x_2\right)$

i.e. min $\left(19x_1^2+19x_2^2+37x_1x_2-34x_1-36x_2\right)$

Thus, $\phi(x)=19x_1^2+19x_2^2+37x_1x_2-34x_1-36x_2$

30.2 Least Square Estimation (contd..)

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min $\sqrt{r_1^2+r_2^2+\ldots\ldots\ldots+r_m^2}$

Minimization of length of residual vector or square of length are equivalent in the sense, the minimum is reached at same value of x^*. However, working with squares eliminates the problem of square roots in calculating derivatives. Hence, a least square problem is defined as

$$\phi(x)=\min\frac{1}{2}r^Tr=\frac{r_1^2+r_2^2+\ldots\ldots+r_m^2}{2}=\frac{1}{2}(b-Ax)^T(b-Ax)=\frac{1}{2}x^TA^TAx-b^TAx+b^Tb \quad (6)$$

The scalar $\frac{1}{2}$ is introduced only for the sake of convenience. Again it does not affect the optimal x^*. Constant in the estimation problem can be neglected without affecting x^*. Let us now work out the least square solution for the example problem under consideration.

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i.e. min $\frac{1}{2}\left(38x_1^2+38x_2^2+2\times 37\times x_1\times x_2\right)-\left(34x_1+36x_2\right)$

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Thus, $\phi(x)=19x_1^2+19x_2^2+37x_1x_2-34x_1-36x_2$

30.2 Least Square Estimation (contd..)

At the minimum, partial derivative of $\phi(x)$ with x_1 and x_2 should be zero. i.e.

$$\frac{\partial \phi}{\partial x_1}=38x_1^{*}+37x_2^{*}-34=0$$

$$\frac{\partial \phi}{\partial x_1}=38x_2^{*}+37x_1^{*}-36=0$$

Hence,

$$\begin{bmatrix} 38 & 37 \\ 37 & 38 \end{bmatrix}\begin{bmatrix} x_1^{*} \\ x_2^{*} \end{bmatrix}=\begin{bmatrix} 34 \\ 36 \end{bmatrix}$$

$\Rightarrow x_1^{*}=-0.5333$ and $x_2^{*}=1.4667$

We can now generalize the method of least squares.
Consider system of equations given by

$$b=Ax+e$$

The number of unknowns (n) is less than number of knowns (m) i.e. redundancy $\frac{m}{n}$ >1. In other words, b is a $m\times 1$ vector, A is a $m\times n$ matrix. We know from the basics of the optimization theory that at the minimum or maximum, the gradient of the objective function is zero. The gradient of the objective function

$$\phi(x)=\frac{1}{2}(b-Ax)^T(b-Ax)$$

is a $n\times 1$ vector given by

$$\nabla\phi(x)=A^TAx-A^Tb$$

Setting this gradient to zero amounts to solving the following linear system of equations.

$$A^TAx=A^Tb \qquad (7)$$

The solution of the above system of equations gives the optimal x^{*}. It can be verified from the positive definiteness property of that is the minimum. (see review questions)

$A^T A$ x^*

30.3 3-sample technique continued

Thus applying the least square estimation methodology to eq(2), we can compute the estimate of $V_m \cos \phi_v$ and $V_m \sin \phi_v$ by solving the following system of equations.

$$\begin{bmatrix} \sum_{j=k-2}^{k} \sin^2 \theta_j & \sum_{j=k-2}^{k} \sin \theta_j \cos \theta_j \\ \sum_{j=k-2}^{k} \sin \theta_j \cos \theta_j & \sum_{j=k-2}^{k} \cos^2 \theta_j \end{bmatrix} \begin{bmatrix} V_m \cos \phi_v \\ V_m \sin \phi_v \end{bmatrix} = \begin{bmatrix} \sum_{i=k-2}^{k} \sin \theta_i . v_i \\ \sum_{i=k-2}^{k} \cos \theta_i . v_i \end{bmatrix} \quad (8)$$

Example : 1

We now repeat the previous example with a 3-sample window. The analogous results to example 1 are shown in table 1. It can be seen that as a consequence of using a larger data window, the accuracy of voltage estimation improves significantly.

Table 1 : 3 - Sample Estimation

Randn multiplier(E)	Mean	Standard deviation
0.1	10.0061	0.0927
0.2	10.0130	0.1855
0.3	10.0207	0.2783
0.4	10.0293	0.3712
0.5	10.0387	0.4641
0.6	10.0490	0.5570
0.7	10.0601	0.6499
0.8	10.0719	0.7428
0.9	10.0846	0.8358
1.0	10.0982	0.9287
1.1	10.1125	1.0216
1.2	10.1277	1.1144
1.3	10.1436	1.2072
1.4	10.1604	1.3000
1.5	10.1780	1.3927
1.6	10.1964	1.4853
1.7	10.2156	1.5778
1.8	10.2356	1.6702
1.9	10.2564	1.7625
2.0	10.2781	1.8547
2.1	10.3005	1.9468
2.2	10.3238	2.0386
2.3	10.3479	2.1303
2.4	10.3727	2.2218
2.5	10.3985	2.3130
2.6	10.4250	2.4040
2.7	10.4524	2.4947
2.8	10.4807	2.5849
2.9	10.5099	2.6747
3.0	10.5400	2.7638

For example with E = 0.1, 2-sample technique had a σ of 0.1596 while with the 3-sample window, the standard deviation reduces to 0.0927. Similarly with E = 3, standard deviation with 2-sample approaches 4.3830, which reduces to 2.7638 with 3-sample window. This shows that redundancy helps in filtering out noise. However estimation spans a large data window.

Review Questions

1. Repeat Example 1 yourself.
2. A real symmetric square $n \times n$ matrix Q is said to be positive definite. If $xTQx \geq 0$ for all $x \in R^n \neq 0$. Now consider
$Q = A^T A$ where A is a full-column, rank matrix. Show that $A^T A$ is positive definite.
3. Show that at the minimum of a function f(x), the matrix of second derivatives $H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$ is at least positive semi
definite. Also, show that if H is positive definite, then we have a strict local minimum.
4. A symmetric real $n \times n$ matrix Q is said to be positive semi definite, if $xTQx \geq 0$ for all $x \in R^n \neq 0$. Hence, show
that if A is not at all full-column rank matrix, then $A^T A$ is positive semi definite.
5. Show that a real symmetric $n \times n$ matrix is positive definite, if and only if all its Eigen values are real and greater than
zero. Hence, comment on the positive definiteness of following matrices.
a) I_n ($n \times n$ - identity matrix).
b) $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Recap

In this lecture, we have formulated

- The phasor estimation problem as a least squares estimation problem.
- A 3-sample methodology for phasor estimation was discussed.
- Improvements over 2-sample technique was demonstrated.