Module 9 : Numerical Relaying II : DSP Perspective

Lecture 32 : Fourier Analysis

Objectives

In this lecture, we will show that

- Trignometric fourier series is nothing but LS approximate of a periodic signal over orthogonal basis of polynominals.
- Hence, we will extend fourier like method to functioning of other orthogonal functions like walsh, Harr etc.

32.1 Fourier series: A review

Let $f(t)$ be a real valued periodic function with time period T_0 that satisfies Dirchelet's conditions. They require that

1. f(t) is bounded and of period T_0
2. f(t) has finite number of maxima and minima in one period and
3. finite number of discontinuities.

Then, it can be expressed in either of the following equivalent forms.

a) $$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Where $\omega_0 = \frac{2\pi}{T_0}$ is called fundamental frequency and $n\omega_0$, n^{th} harmonic where n - is an integer.

b) $$f(t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{jn\omega_0 t}$$

In (a), the coefficients $(a_i$ and $b_i)$ are real valued, while in (b), coefficient c_n may be complex. Both forms (a) and (b) are equivalent. (b) can be obtained from (a) by following substitutions.

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

Thus, a_n, b_n and c_n are related as follows.

$$c_0 = a_0$$

$$\left.\begin{aligned} c_n &= \frac{a_n - jb_n}{2} \\ c_{-n} &= \frac{a_n + jb_n}{2} \end{aligned}\right\} n > 0$$

32.1 Fourier series: A review

Let $f(t)$ be a real valued periodic function with time period T_0 that satisfies following conditions [Dirchelet's conditions]. Then, it can be expressed in either of the following equivalent forms.

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32.1 Fourier series: A review (contd..)

In other words, $c_n = c_{-n}^*$, the usage of exponential form also introduces the concept of negative frequency. It can be interpreted in the following way.

- $e^{jn\omega_0 t}$ $(n > 0)$ represents a phasor of unit magnitude rotating in anti-clockwise direction at speed of $n\omega_0$ rad/s.
- $e^{-jn\omega_0 t}$ represents a phasor of unit magnitude rotating in clockwise direction at speed of $n\omega_0$ rad/s.

The coefficients a_n, b_n and c_n can be computed using the following expressions.

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t) \cos n\omega_0 t \, dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t) \cos n\omega_0 t \, d(\omega_0 t)$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin n\omega_0 t\ dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t)\sin n\omega_0 t\ d(\omega_0 t)$$

$$a_0 = \frac{1}{T_0}\int_0^{T} f(t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(\omega_0 t)d\omega_0 t$$

$$c_n = \frac{1}{T_0}\int_0^{T} f(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega_0 t}\ d\omega_0 t$$

32.1 Fourier series: A review (contd..)

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The coefficients a_n, b_n and c_n can be computed using the following expressions.

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos n\omega_0 t\ dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t)\cos n\omega_0 t\ d(\omega_0 t)$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin n\omega_0 t\ dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\omega_0 t)\sin n\omega_0 t\ d(\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_0^{T} f(t)\ dt = \frac{1}{2\pi}\int_0^{2\pi} f(\omega_0 t)\ d\omega_0 t$$

$$c_n = \frac{1}{T}\int_0^{T} f(t)\,e^{-jn\omega_0 t}\ dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega_0 t}\, d\omega_0 t$$

32.1 Fourier series: A review (contd..)

Since, many of the voltage and current phasor estimation methods are based upon least square estimation, it is a worthwhile intellectual exercise to show that Fourier series truncated at some ‘n’ can be interpreted through Least square estimation theory.

We define the concept of mean square error (MSE). MSE is defined as:

$$MSE = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f(t) - \sum_{i=1}^{r}\alpha_i\phi_i(t)\right]^2 dt$$

The interval $t_2 - t_1$ for periodic function should be time period T_0, while $\phi_i(t)$ should be sine/cosine function (or) exponential function of the Fourier series. Our job is to find the coefficient α_i's so as to minimize the mean square error. Thus, the optimization problem is given by,

$$\min MSE$$

Consider $f(t)$ approximated by $n-$ functions ϕ_i as follows:

$$f(t) \approx \alpha_0\phi_0(t) + \alpha_1\phi_1(t) + \alpha_2\phi_2(t) + ----+\alpha_r\phi_r(t) \qquad (1)$$

In truncated Fourier series, $\phi_i(t)$ belong to the set { $\cos m\omega_0 t, \sin m\omega_0 t; m$ integer} and $\phi_0(t) = 1$. To evaluate the quality of approximation,

Now,

$$MSE = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f(t) - \alpha_0\phi_0(t) - \alpha_1\phi_1(t) - \alpha_2\phi_2(t) ----\alpha_r\phi_r(t)\right]^2 dt$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\begin{aligned}&[f^2(t) + \alpha_0^2\phi_0^2(t) + \alpha_1^2\phi_1^2(t) + \alpha_2^2\phi_2^2(t) + ---++\alpha_r^2\phi_r^2(t)\\ &-2\alpha_0 f(t)\phi_0(t) - 2\alpha_1 f(t)\phi_1(t) - 2\alpha_2 f(t)\phi_2(t) ----2\alpha_r f(t)\phi_r(t)]\,dt\end{aligned}$$

$$+\frac{1}{t_2 - t_1}\sum_{j=0}^{r}\sum_{i=0}^{r}\int_{t_1}^{t_2}\alpha_j\alpha_i\phi_i(t)\phi_j(t)\,dt \quad i \neq j \qquad (2)$$

32.1 Fourier series: A review (contd..)

Now, if we choose $\phi_i(t)$ to be either $\cos n\omega_0 t$ or $\sin n\omega_0 t$ from the trigonometric form of Fourier series, then the expression for the MSE simplifies drastically. It can be verified that with $\phi_n(t) = \cos n\omega_0 t$ and $\phi_m(t) = \cos m\omega_0 t \;\; (m \neq n)$.

$$\int_0^{T_0} \phi_n(t)\phi_m(t)\,dt$$

$$= \int_0^{T_0} \cos n\omega_0 t \, \cos m\omega_0 t \, dt$$

$$= \frac{1}{2}\int_0^{T_0} [\cos \overline{n-m}\,\omega_0 t + \cos \overline{n+m}\,\omega_0 t]\,dt$$

$= 0$;

In fact, this is analogous to the statement that we will have zero average power exchange over a time-interval (T_0) if the voltages and currents belong to different frequencies of the harmonic spectrum.

Similarly, with $\phi_n(t) = \cos n\omega_0 t$ and $\phi_m = \sin m\omega_0 t$, it can be verified that,

$$\int_0^{T_0} \phi_n(t)\phi_m(t)\,dt = \int_0^{T_0} \cos n\omega_0 t \, \sin m\omega_0 t \, dt$$

$$= \frac{1}{2}\int_0^{T_0} [\sin \overline{n-m}\,\omega_0 t + \sin \overline{n+m}\,\omega_0 t]\,dt = 0; \qquad (3)$$

Equation (3) holds true even when $n = m$. When $n = m$, we have the analogy of interaction between voltage and current phasors separated by 90°, which leads to zero average power.
The concept of zero average power interaction at (1) sines and cosines harmonic at different frequencies and (2) sines and cosines at same frequency can be generalized by the concept of orthogonal functions.

32.2 Orthogonal functions:

A set of complex valued functions $\{\phi_i(t), \phi_j(t)\}$, $t \in [a, b]$, are said to be orthogonal if,

$$\int_a^b \phi_i(t)\phi_j^*(t) = 0 \qquad i \neq j$$

The definition obviously applies to real valued functions where in $\phi_j^* = \phi_j$. Clearly, the functions used in Fourier series (both in trigonometric and complex exponential form) are orthogonal. Many more set of orthogonal functions like Walsh, Harr etc and corresponding approximate series can be found in the literature of signal processing. They also have applications in power quality. However, for our use, Fourier series suffices.

From the discussion so far, we conclude that in case of Fourier series over a time period T_0, the second term in (2) vanishes. Hence,

$$MSE = \frac{1}{T_0}\int_0^{T_0}[f^2(t)+\alpha_0^2\phi_0^2(t)+\alpha_1^2\phi_1^2(t)+\alpha_2^2\phi_2^2(t)+---++\alpha_n^2\phi_n^2(t)$$
$$-2\alpha_0 f(t)\phi_0(t)-2\alpha_1 f(t)\phi_1(t)-2\alpha_2 f(t)\phi_2(t)----2\alpha_n f(t)\phi_n(t)]\,dt$$

For the dc signal $\phi_0(t)=1$, it can be shown that,

$$\frac{1}{T_0}\int_0^{T_0}\phi_0^2(t)\,dt = 1 \qquad (4)$$

Further, for $\phi(t)$ to be $\sin m\omega_0 t$, or $\cos n\omega_0 t$, it can be verified that,

$$\frac{1}{T_0}\int_0^{T_0}\sin^2 n\omega_0 t\,dt = \frac{1}{T_0}\int_0^{T_0}\frac{1-\cos 2n\omega_0 t}{2}\,dt = 1/2$$

32.2 Orthogonal functions: (contd..)

Hence,

$$MSE = \frac{1}{T_0}\left[\int_0^{T} f^2(t)\,dt\right]+\frac{\alpha_1^2+\alpha_2^2+---+\alpha_n^2}{2}+\alpha_0^2-\frac{2\alpha_0}{T_0}\int_0^{T_0}f(t)dt$$
$$-\frac{2\alpha_1}{T_0}\int_0^{T_0}f(t)\phi_1(t)\,dt---\frac{2\alpha_n}{T_0}\int_0^{T_0}f(t)\phi_n(t)\,dt$$

$$=\frac{1}{T_0}\int_0^{T}f^2(t)\,dt+\left[\frac{\alpha_1^2}{2}-\frac{2\alpha_1}{T_0}\int_0^{T_0}f(t)\phi_1(t)\,dt\right]+---+\left[\frac{\alpha_n^2}{2}-\frac{2\alpha_n}{T_0}\int_0^{T_0}f(t)\,\phi_n(t)\,dt\right]+\left[\alpha_0^2-\frac{2\alpha_0}{T_0}\int_0^{T_0}f(t)dt\right]$$

At the minimum, the partial derivative of MSE with variable α_i should be zero. Now,

$$\frac{\partial MSE}{\partial\alpha_i}=\mathbf{0} \;\Rightarrow\; \alpha_i-\frac{2}{T_0}\int_0^{T_0}f(t)\,\phi_i(t)\,dt=0 \text{ For } i\neq 0$$

$$\Rightarrow \alpha_i=\frac{2}{T_0}\int_0^{T_0}f(t)\,\phi_i(t)\,dt$$

Thus, when we substitute $\phi_i(t)=\cos n\omega_0 t$, we get coefficient a_n, similarly by substituting $\phi_i(t)=\sin n\omega_0 t$, we get coefficient b_n. Finally, for dc signal, which is defined by $\phi_i(t)=1$, it can be shown that

For $\dfrac{\partial MSE}{\partial\alpha_0}=0$

$$\alpha_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

These coefficients of LS approximation are nothing but the coefficients of Fourier series. This leads us to a very important conceptual result viz. coefficients in Fourier series minimize the MSE. This establishes the linkage between Fourier series and Least square methods. Similar series can be developed for other set of orthogonal functions. [See Q:2]

Review Questions

1. For the square periodic wave form as shown in fig 32.1.

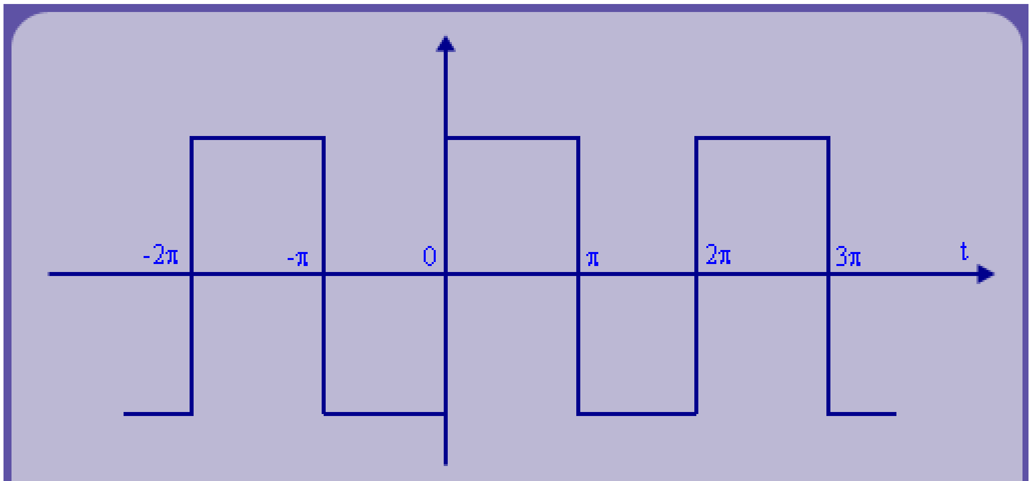

Fig 32.1 Square Periodic Waveform

Show that $C_n = \left.\begin{matrix} \frac{4}{\pi n} \\ 0 \end{matrix}\right\}\begin{matrix} \text{n - odd} \\ \text{n - even} \end{matrix}$

Interpret you answers in terms of a_n and b_n.

2. A class of function known as Walsh function are defined as follows:

$$\phi_0(t) = 1, \quad 0 \le t \le 1$$

$$\phi_1(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ -1 & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\phi_2^{(1)}(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{4}, \quad \frac{3}{4} < t \le 1 \\ -1 & \frac{1}{4} \le t \le \frac{3}{4} \end{cases}$$

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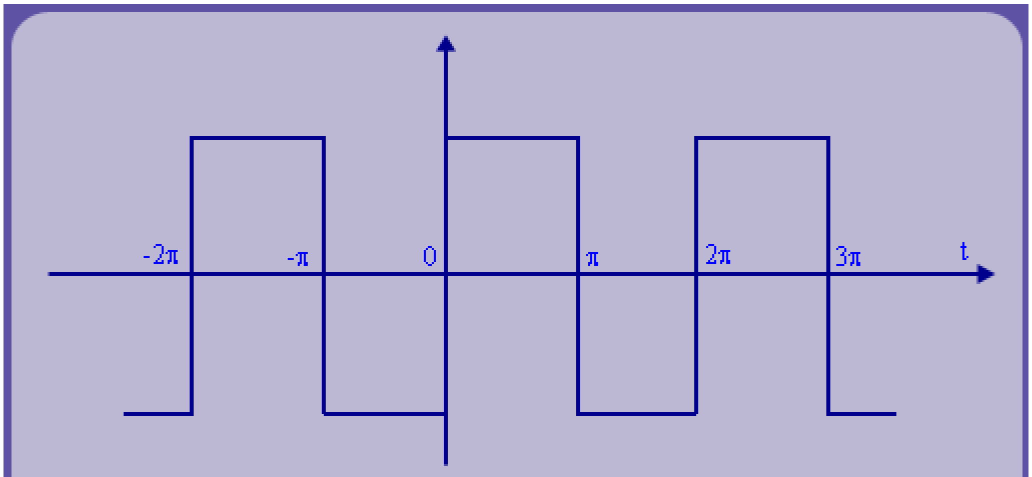

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Interpret you answers in terms of a_n and b_n.

2. A class of function known as Walsh function are defined as follows:

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Review Questions (contd..)

2. $$\phi_2^{(2)}(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{4}, \quad \frac{1}{2} < t < \frac{3}{4} \\ -1 & \frac{1}{4} < t \le \frac{1}{2}, \quad \frac{3}{4} < t \le 1 \end{cases}$$

$$\phi_{m+1}^{(2k-1)}(t) = \begin{cases} \phi_m^k(2t) & 0 \le t < \frac{1}{2} \\ (-1)^{k+1}\,\phi_m^{(k)}(2t-1) & \frac{1}{2} < t \le 1 \end{cases} \quad \begin{matrix} m = 1, 2, 3, \ldots \\ k = 1, 2, \ldots, 2^{m-1} \end{matrix}$$

$$\phi_{m+1}^{(2k)}(t) = \begin{cases} \phi_m^k(2t) & 0 \le t < \frac{1}{2} \\ (-1)^k \phi_m^{(k)}(2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

a) Plot the functions.

b) Show that they are orthogonal.

c) What advantages can they have in DSP.

3. For the square pulse shown below with duty cycle d/T, compute the harmonic spectrum. Evaluate its behaviour in the limit S $\Rightarrow \infty$.

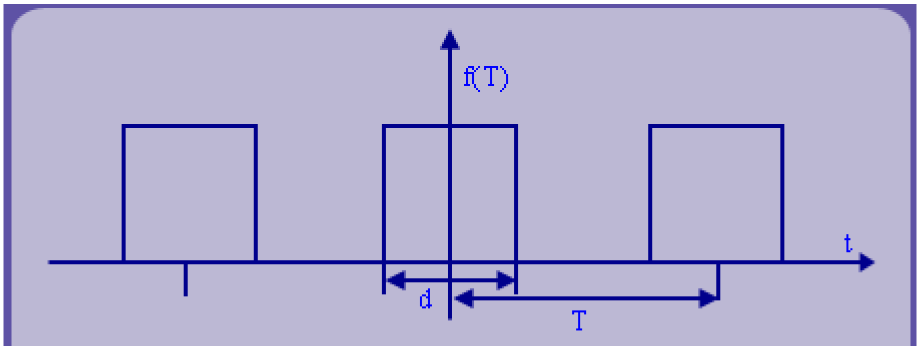

Fig 32.2 Problem 32.3

4. State the conditions under which periodic signal can be represented by Fourier Series.

Recap

In this lecture we have learnt the following:

- Trignometric Fourier series is nothing but LS approximate of a periodic signal over orthogonal basis of polynominals.

- Hence, we can extend Fourier like method to functioning of other orthogonal functions like walsh, Harr etc.