# Module 9 : Numerical Relaying II : DSP Perspective

## Lecture 33 : Discrete Fourier Transform

### Objectives

In this lecture, we will

- Derive Discrete Fourier Transform (DFT) and its inverse.
- Different forms of DFT and IDFT will be derived.

### 33.1 Motivation

Consider a finite duration signal $g(t)$ of duration $T$ sampled at (fig 33.1) a uniform rate $t_s$ such that

$T = Nt_s$ where $N$ is an integer $N > 0$.

Then the Fourier transform of signal is given by:

$$G(f) = \int_0^T g(t) e^{-j2\pi ft}\, dt$$

If we now evaluate the above integral by trapezoidal rule of integration after padding two zeros (red dots in fig 33.1) at the extremity on either side [where the signal is zero], we obtain the following expressions.

$$G(f) = t_s \sum_{n=0}^{N-1} g(nt_s) e^{-j2\pi f n t_s} \qquad (1)$$

The corresponding inverse which is used to reconstruct the signal is given by:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df \qquad (2)$$

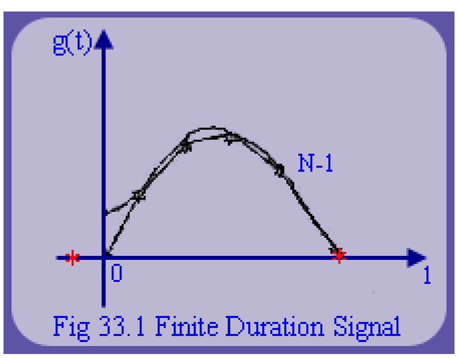

Fig 33.1 Finite Duration Signal

If from equation (1) we could compute complete frequency spectrum i.e. $G(f), -\infty \le f \le \infty$ then (2) would imply that we can obtain $g(t) \forall 0 \le t \le T$. The fallacy in the above statement is quite obvious as we have only finite samples and the curve connecting any 2-samples can be defined plausibly in infinitely many ways (see fig 33.2). This suggests that from (1), we should be able to derive only limited amount of frequency domain information.

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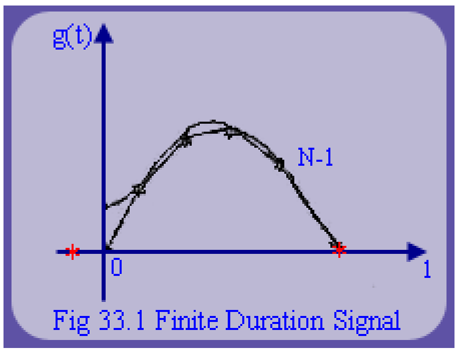


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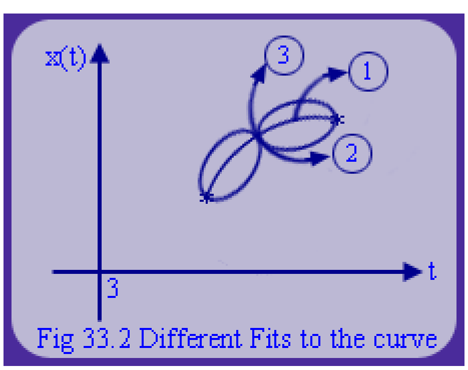
## 33.1 Motivation (contd..)

Since, we have N-data points [real] and $G(f)$ a complex number contains both magnitude and phase angle information in the frequency domain (2-units of information), it is reasonable to expect that we should be in a position to predict atmost $\frac{N}{2}$ transforms $G(f)$ for original signal.

Now, let $f_0 = \frac{1}{T} = \frac{1}{Nt_s} = \frac{f_s}{N}$

and $f = mf_0 = \frac{m}{Nt_s} = \frac{mf_s}{N}$ (3)

then substituting (3) in (1), we get




$$G\left(\frac{mf_s}{N}\right) = t_s \sum_{n=0}^{N-1} g(nt_s) e^{-j2\pi \frac{m}{Nt_s} nt_s}$$

$$= t_s \sum_{n=0}^{N-1} g(nt_s)\, e^{\frac{-j2\pi mn}{N}}$$

Note that our choice of frequency is such that the exponential term in (1) is independent of $t_s$. The intuition for choosing such $f$ is that, in principles we are attempting a transform on discrete samples which may (or) may not have a corresponding analog 'parent' signal. This suggests to us the following discrete version of Fourier transform for a finite discrete sequence $\{x_0, x_1, \ldots\ldots, x_{N-1}\}$

## 33.1 Motivation (contd..)

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi mn}{N}} \quad (4)$$

Our next job should be to come up with an inverse transformation. If inverse transformation exists, then there is no loss of information from discrete (time) domain to frequency domain and vice-versa. Existence of inverse will establish, transform nature of (4). If (2) defines IFT in continuous domain, in the discrete domain, by analogy of (1) and (4) we can hypothesize following inverse transform.

$$x(n) = \frac{1}{K} \sum_{m=0}^{N-1} X(m)\, e^{\frac{j2\pi mn}{N}} \quad (5)$$

Where K is a suitable scaling factor.
Our next job is to verify that indeed (4) and (5) define a transformation pair. Substituting (4) in (5), we get following expression for right hand side of (5).

Right hand side $= \frac{1}{K} \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x(k) e^{\frac{-j2\pi mk}{N}} \right] e^{\frac{j2\pi mn}{N}}$ (6)

[Note the use of dummy subscript $k$ ]

Let us work this expression out in a long hand fashion; for compactness we use notation $x_n = x(n)$

$$RHS = \frac{1}{K}\begin{bmatrix} x_0 + x_1 + - - - - - - + x_{N-1} & \leftarrow \quad m=0 \\ + x_0 e^{\frac{j2\pi n}{N}} + x_1 e^{\frac{-j2\pi}{N}} e^{\frac{j2\pi n}{N}} + - - - + x_{N-1} e^{\frac{-j2\pi(N-1)}{N}} e^{\frac{j2\pi n}{N}} & \leftarrow \quad m=1 \\ + - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - + & \\ + x_0 e^{\frac{j2\pi(N-1)n}{N}} + x_1 e^{\frac{-j2\pi(N-1)}{N}} e^{\frac{j2\pi(N-1)n}{N}} + - - - - + x_{N-1} e^{\frac{-j2\pi(N-1)^2}{N}} e^{\frac{j2\pi(N-1)n}{N}} & \leftarrow \quad m=N-1 \end{bmatrix}$$

In the above expression, for the first row $m$ is set to zero, for the second row it is set to one and for the last row $m = N-1$.

## 33.1 Motivation (contd..)

Now, grouping terms column wise, we get

$$RHS = \frac{1}{K}\begin{bmatrix} x_0 (1 + e^{\frac{j2\pi n}{N}} + - - - - + e^{\frac{j2\pi(N-1)n}{N}}) + x_1 (1 + e^{\frac{j2\pi(n-1)}{N}} + - - - - + e^{\frac{j2\pi(N-1)(n-1)}{N}}) \\ + - - - - - - + x_{N-1} (1 + e^{\frac{j2\pi}{N}(n-\overline{N-1})} + - - - - e^{\frac{j2\pi(N-1)}{N}(n-\overline{N-1})}) \end{bmatrix}$$

$$= \frac{1}{K} \sum_{k=0}^{N-1} x_k \sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)}$$

Note that this jugglery shows that we can interchange the summation order. One order indicates row wise and another column wise summation

i.e. $RHS = \frac{1}{K}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x_k e^{\frac{j2\pi m}{N}(n-k)}$ (7)

Our primary task now is to evaluate the expression.

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)}$$

We now claim that

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi}{N}(n-k)m} = \left.\begin{matrix} = N \\ = 0 \end{matrix}\right\} \begin{matrix} if\ k = n; \\ if\ k \neq n; \end{matrix}$$

*Proof:* For $k = n$, $e^{\frac{j2\pi}{N}(n-k)m} = e^{j.0.m} = 1 \ \forall m$

Hence, the first case is obvious.
Now, if $k \neq n$, let $k - n = k^1$

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)} = e^{\frac{j2\pi}{N}(n-k)0} + e^{\frac{j2\pi}{N}(n-k)1} + - - - - + e^{\frac{j2\pi}{N}(n-k)(N-1)}$$

## 33.1 Motivation (contd..)

Now, $e^{\frac{j2\pi}{N}k^1 m} = \left(e^{\frac{j2\pi}{N}k^1}\right)^m = \left(a^{k^1}\right)^m$

where $a = e^{\frac{j2\pi}{N}}$

$$\therefore \quad \sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)} = \sum_{m=0}^{N-1}\left(e^{j\frac{2\pi}{N}k^1}\right)^m = \frac{1 - e^{\left(\frac{j2\pi}{N}k^1\right)N}}{1 - e^{\frac{j2\pi}{N}k^1}} = \frac{1 - e^{j2\pi k^1}}{1 - e^{\frac{j2\pi}{N}k^1}} = 0$$

AS $k^1$ is integer, then $e^{j2\pi k^1} = 1$

Note that we have used the following geometric series expression $a + ar + ........ + ar^{n-1} = \frac{a(1-r^n)}{1-r}$

Thus, RHS in (6) is equal to $\frac{N}{K}x_n$

We see that equation (6) defines the inverse transformation if we choose $K = N$;
Thus, N-point DFT and IDFT for samples $[x_0, x_1, .....x_{N-1}]$ are defined as follows.

where m = 0, ........., N - 1

$$X(m)=\sum_{n=0}^{N-1}x_n e^{\frac{-j2\pi nm}{N}}$$

$$x(n) = \frac{1}{N}\sum_{m=0}^{N-1}X(m)\, e^{\frac{j2\pi mn}{N}}$$

## 33.1 Motivation (contd..)

Now, $e^{\frac{j2\pi}{N}k^{\prime}m} = \left(e^{\frac{j2\pi}{N}k^{\prime}}\right)^{m} = \left(a^{k^{\prime}}\right)^{m}$

where $a = e^{\frac{j2\pi}{N}}$

$$\therefore \quad \sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)} = \sum_{m=0}^{N-1}\left(e^{j\frac{2\pi}{N}k^{\prime}}\right)^{m} = \frac{1-e^{\left(\frac{j2\pi}{N}k^{\prime}\right)N}}{1-e^{\frac{j2\pi}{N}k^{\prime}}} = \frac{1-e^{j2\pi k^{\prime}}}{1-e^{\frac{j2\pi}{N}k^{\prime}}} = 0$$

$\because$ AS $k^1$ is integer, then $e^{j2\pi k^{\prime}}=1$

Note that we have used the following geometric series expression $a+ar+\ldots\ldots+ar^{n-1}=\frac{a(1-r)^{n}}{1-r}$

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$$X(m)=\sum_{n=0}^{N-1}x_n e^{\frac{-j2\pi nm}{N}}$$ where m = 0, ........., N - 1

$$x(n) = \frac{1}{N}\sum_{m=0}^{N-1}X(m)\, e^{\frac{j2\pi mn}{N}}$$

## 33.1 Motivation (contd..)

Note that in general DFT and inverse DFT can be defined in many ways, each only differing in choice of constant $C_1$ and $C_2$. Thus, the generic form of DFT and IDFT is as follows:

DFT

i.e. $X(m)=C_1\sum_{n=0}^{N-1}x_n e^{\frac{-j2\pi nm}{N}}$

IDFT

$x_n = C_2\sum_{m=0}^{N-1}X(m)e^{\frac{j2\pi nm}{N}}$

The constraint in choosing the constraints is that product $C_1C_2 = {}^{1}\!/\!_{N}$

For example, when

$$C_1 = 1, \qquad C_2 = \frac{1}{N}$$

$$C_1 = \frac{2}{N} \qquad C_2 = \frac{1}{2}$$

$$C_1 = \frac{1}{\sqrt{N}} \qquad C_2 = \frac{1}{\sqrt{N}}$$

Choice of $C_1 = \frac{2}{N}$ is commonly used in relaying because it simplifies phasor estimation.

### Review Questions

1. Repeat the DFT and IDFT derivation yourself.

### Recap

In this lecture we have learnt the following:

Derived DFT and IDFT.