Module 9 : Numerical Relaying II : DSP Perspective

Lecture 34 : Properties of Discrete Fourier Transform

Objectives

In this lecture, we will

- Discuss properties of DFT like:

  1) Linearity,

  2) Periodicity,

  3) DFT symmetry,

  4) DFT phase-shifting etc.

34.1 Linearity:

Let $\{x_0, x_1, \ldots\ldots x_{N-1}\}$ and $\{y_0, y_1, \ldots\ldots y_{N-1}\}$ be two sets of discrete samples with corresponding DFT's given by $X(m)$ and $Y(m)$. Then DFT of sample set $\{x_0 + y_0, x_1 + y_1, \ldots\ldots\ldots\ldots, x_{N-1} + y_{N-1}\}$ is given by $X(m) + Y(m)$

Proof: $X(m) = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$; $Y(m) = \sum_{n=0}^{N-1} y_n e^{\frac{-j2\pi nm}{N}}$

$$X(m) + Y(m) = \sum_{n=0}^{N-1} (x_n + y_n) e^{\frac{-j2\pi mn}{N}}$$

34.2 Periodicity :

We have evaluated DFT at $m = 0,1,\ldots\ldots\ldots, N-1$. There after, $(m \geq N)$ it shows periodicity. For example
$X(m) = X(N+m) = X(2N+m) = X(-N+m) = X(-2N+m) = X(kN+m)$
Where k is an integer.

Proof: $X(kN+m) = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi n}{N}(kN+m)}$

$$= \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}} e^{-j2\pi kn} \qquad (1)$$

Both k and n are integers. Hence $e^{-j2\pi kn} = 1$; Therefore from (1) we set

$$x(kN+m)=\sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}} = X(m)$$

34.3 DFT symmetry :

If the samples x_n are real, then extracting in frequency domain $X(0).......X(N-1)$ seems counter intuitive; because, from N bits of information in one domain (time), we are deriving 2N bits of information in frequency domain. This suggests that there is some redundancy in computation of $X(0)............X(N-1)$. As per DFT symmetry property, following relationship holds.

$X(N-m)=X^*(m) \quad m=0,1,.....,N-1$, where symbol $*$ indicates complex conjugate.

Proof:

$$X(m)=\sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$X(N-m)=\sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi n}{N}(N-m)}$$

$$= \sum_{n=0}^{N-1} x_n e^{\frac{j2\pi nm}{N}} e^{-j2\pi n}$$

Since $e^{-j2\pi n}=1$,

$$X(N-m)=\sum_{n=0}^{N-1} x_n e^{\frac{j2\pi nm}{N}}$$

$$=\sum_{n=0}^{N-1} (x_n e^{\frac{-j2\pi nm}{N}})^*$$

$$=\left[\sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}\right]^*$$

$$=(X(m))^* = X^*(m)$$

34.3 DFT symmetry : (contd..)

If the samples x_n are real; then they contain atmost N bits of information. On the otherhand, $X(m)$ is a complex number and hence contains 2 bits of information. Thus, from sequence $\{x_0, x_1,....., x_{N-1}\}$, if we derive $\{X(0), X(1),......., X(N-1)\}$, it implies that from N-bit of information, we are deriving $2N$ bits of information. This is counter intuitive. We should expect some relationship in the sequence $\{X(0), X(1),......., X(N-1)\}$

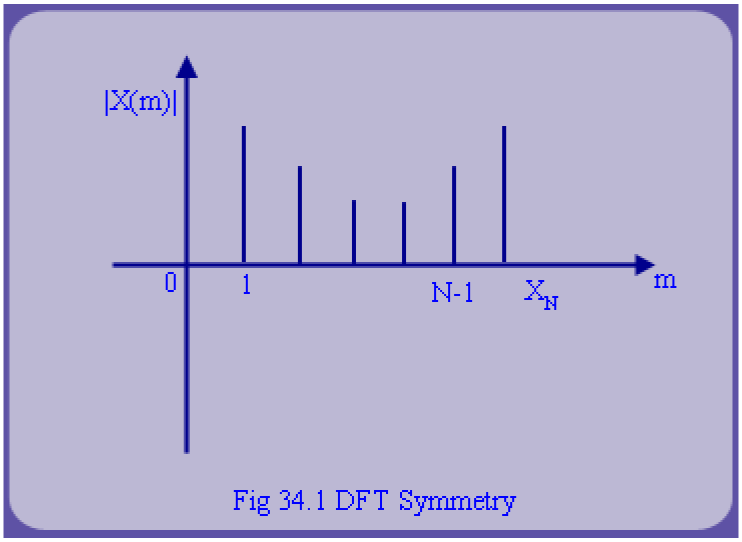

Fig 34.1 DFT Symmetry

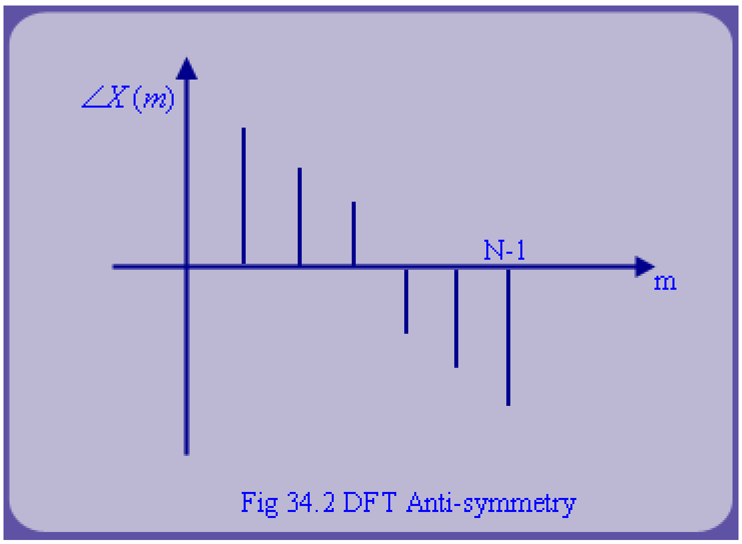

Fig 34.2 DFT Anti-symmetry

Thus, we conclude that

$\left|X(N-m)\right| = \left|X(m)\right|$ [Symmetry]

and $\angle X(N-m) = -\angle X(m)$ [Anti-symmetry]. DFT magnitude and phase plots appear as shown in fig 34.1 and 34.2.

34.4 DFT phase shifting :

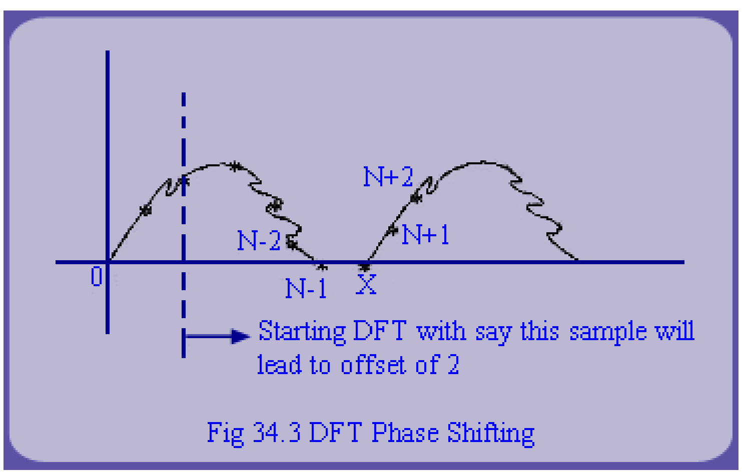

Fig 34.3 DFT Phase Shifting

DFT shifting property states that, for a periodic sequence with periodicity N i.e. $x(m) = x(m+lN)$, l an integer, an offset in sequence manifests itself as a phase shift in the frequency domain. In other words, if we decide to sample x(n) starting at n equal to some integer K, as opposed to n = 0, the DFT of those time shifted sequence, $\{x_K, x_{K+1}, \ldots, x_{n+K}, \ldots\}$ is

$$X_{\text{shifted}}(m) = e^{\frac{j2\pi Km}{N}} X(m)$$

Proof: By periodicity of samples, we have

$$x(N) = x(0)$$

$$x(N+1) = x(1)$$

$$x(N+K-1) = x(K-1)$$

$$X(m) = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi n}{N}m}$$

$$= \sum_{n=0}^{K-1} x_n e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$= \sum_{n=0}^{K-1} x_{N+n} e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$= \sum_{n=0}^{K-1} x_{n+N} e^{\frac{-j2\pi m(N+n)}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}} \qquad \left[\because e^{-j2\pi m\frac{N}{N}} = 1\right]$$

$$= \sum_{n=N}^{N+K-1} x_n e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

34.4 DFT phase shifting :

DFT shifting property states that, for a periodic sequence with periodicity N i.e. $x(m) = x(m + lN)$, l an integer, an offset in sequence manifests itself as a phase shift in the frequency domain. In other words, if we decide to sample x(n) starting at n equal to some integer K, as opposed to n = 0, the DFT of those time shifted samples.

$$X_{\text{shifted}}(m) = e^{\frac{j2\pi Km}{N}} X(m)$$

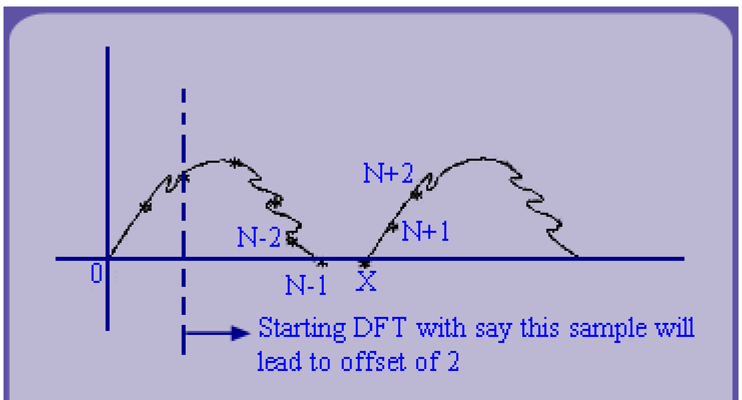

Fig 34.3 DFT Phase Shifting

Proof: By periodicity of samples, we have

$$x(N) = x(0)$$

$$x(N+1) = x(1)$$

$$x(N+K-1) = x(K-1)$$

$$X(m) = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi n}{N}m}$$

$$= \sum_{n=0}^{K-1} x_n e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$= \sum_{n=0}^{K-1} x_{N+n} e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$= \sum_{n=0}^{K-1} x_{n+N} e^{\frac{-j2\pi m(N+n)}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}} \qquad \left[\because e^{-j2\pi m\frac{N}{N}} = 1\right]$$

$$= \sum_{n=N}^{N+K-1} x_n e^{\frac{-j2\pi nm}{N}} + \sum_{n=K}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

34.4 DFT phase shifting: (contd..)

$$X(m) = \sum_{n=K}^{N+K-1} x_n e^{\frac{-j2\pi n}{N}m} \quad (2)$$

Now to compute $X_{shifted}$, let us map the samples $x_K, x_{K+1}, \ldots\ldots, x_{N+K-1}$ to $y_0, y_1 \ldots\ldots, y_{N-1}$. Apply DFT to sequence y.

$$X_{shifted}(m) = \sum_{n=0}^{N-1} y_n e^{\frac{-j2\pi nm}{N}}$$

$$= \sum_{n=0}^{N-1} x_{K+n} e^{\frac{-j2\pi n}{N}m}$$

$$= e^{\frac{j2\pi mK}{N}} \sum_{n=0}^{N-1} x_{K+n} e^{\frac{-j2\pi m}{N}(n+K)}$$

$$= e^{\frac{j2\pi mK}{N}} \sum_{n=K}^{N+K-1} x_n e^{\frac{-j2\pi m}{N}N}$$

$$= e^{\frac{j2\pi nK}{N}} X_n(m) \text{ (from (2))}$$

Review Questions

1. Compute 8 - pt. DFT $(m = 0, 7)$ of the following sequence.

   $x(0) = 0.35$, $x(1) = 0.33$, $x(2) = 0.68$, $x(3) = 1.07$, $x(4) = 0.40$, $x(5) = -1.12$, $x(6) = -1.35$, $x(7) = -0.35$. Hence, illustrate the various DFT properties discussed in this lecture.

2. By using inverse DFT, show that discrete samples can be recovered with knowledge of $x(0), \ldots\ldots x(7)$

3. Calculate the N pt. DFT of rectangular function given by, $x_0 = x_1 \ldots\ldots\ldots = x_{N-1} = 1$. Verify the various DFT properties

   for this signal.

Recap

In this lecture we have learnt the following:

- The properties of DFT like:

  1) Linearity,

  2) Symmetry,

  3) DFT symmetry,

4) DFT phase-shifting etc.