EE 102 # Homework #4 Solutions

Fall 2001 **Professor Paganini**

---

1. (a)

$$f(t) = u(t-2)e^{2t} = \begin{cases} 0 & \text{if } t < 2 \\ e^{-2t} & \text{if } t > 2. \end{cases}$$

$$\begin{aligned} F(s) &= \int_0^\infty u(t-2)e^{2t}e^{-st}dt = \int_2^\infty e^{-(s-2)t}dt \\ &= \left[-\frac{e^{(s-2)t}}{(s-2)} \right]_{t=2}^{t\to\infty} = \frac{e^4 e^{-2s}}{(s-2)} \text{with DOC Re}[s] > 2. \end{aligned}$$

(b)

$$\begin{aligned} f(t) &= u(t) - u(t-1) + u(t-2) - u(t-3) \\ &= \begin{cases} 1 & \text{if } 0 < t < 1 \text{ or } 2 < t < 3 \\ 0 & \text{Otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} F(s) &= \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt \\ &= \begin{cases} \left[-\frac{e^{-st}}{s} \right]_{t=0}^{t=1} + \left[-\frac{e^{-st}}{s} \right]_{t=2}^{t=3} & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \\ &= \begin{cases} (1 - e^{-s} + e^{-2s} - e^{-3s})/s & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \\ &= \begin{cases} (1 - e^{-s})(1 + e^{-2s})/s & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \end{aligned}$$

The DOC is the entire s-plane.

2. (a) $\mathbf{L}\left[e^{-at}u(t)\right] = \frac{1}{(s+a)}$ with DOC Re$[s] >$ Re$[a]$.

$$f(t) = e^t u(t) + e^{-2t}u(t)$$

$$F(s) = \frac{1}{s-1} + \frac{1}{s+2}$$

with DOC Re$[s] > 1$. Uses linearity.

(b)

$$f(t) = u(t-\pi)e^{(t-\pi)}\cos(t) = -e^{(t-\pi)}\cos(t-\pi)u(t-\pi)$$

since $\cos(t-\pi) = -\cos(t)$.

$$\mathbf{L}\left[e^t\cos(t)u(t)\right] = \frac{(s-1)}{(s-1)^2+1} \quad \text{with DOC Re}[s] > 1.$$

Hence, by property 6 (Delay property),

$$F(s) = -\frac{(s-1)e^{-s\pi}}{(s-1)^2+1} \quad \text{with DOC Re}[s] > 1.$$

(c)

$$f(t) = \int_0^t g(\sigma)d\sigma$$

where $g(t) = t^2e^{-t}$. By Property 4,

$$F(s) = \frac{G(s)}{s}$$

Now,

$$\mathbf{L}\left[e^{-t}\right] = \frac{1}{s+1}$$

with DOC Re$[s] > -1$. By Property 5,

$$\mathbf{L}\left[te^{-t}\right] = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

with DOC Re$[s] > -1$. Using Property 5 again,

$$\mathbf{L}\left[t^2e^{-t}\right] = -\frac{d}{ds}\left(\frac{1}{(s+1)^2}\right) = \frac{2}{(s+1)^3} = G(s)$$

Hence

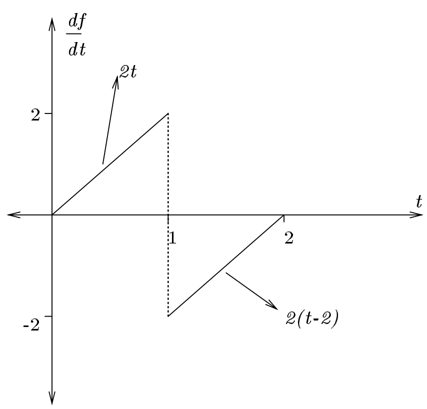
$$F(s) = \frac{2}{s(s+1)^3}$$

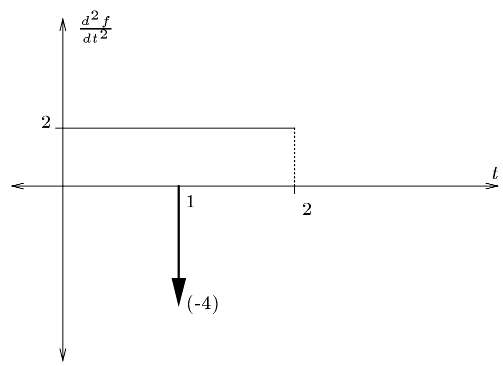
The DOC is now Re$[s] > 0$ (as also seen from the fact that f(t) is an increasing function of t and hence its integral doesn't exist).

3.

$$f(t) = t^2[u(t) - u(t-1)] + (t-2)^2[u(t-1) - u(t-2)]$$
$$f'(t) = 2t[u(t) - u(t-1)] + 2(t-2)[u(t-1) - u(t-2)]$$

$$\begin{aligned} f''(t) &= 2[u(t) - u(t-1)] - 4\delta(t-1) + 2[u(t-1) - u(t-2)] \\ &= 2[u(t) - u(t-2)] - 4\delta(t-1) \end{aligned}$$





$$\begin{aligned} \mathbf{L}[f''(t)] &= 2\mathbf{L}[u(t)] - 2\mathbf{L}[u(t-2)] - 4\mathbf{L}[\delta(t-1)] \\ &= \frac{2}{s} - \frac{2e^{-2s}}{s} - 4e^{-s} = \frac{2(1-e^{-2s})}{s} - 4e^{-s} \end{aligned}$$

Actually, the above equations are valid only for $s \neq 0$. For $s = 0$, using the definition,

$$\mathbf{L}[f''(t)] = \int_0^\infty (2[u(t) - u(t-2)] - 4\delta(t-1))dt = 0$$

Hence, the DOC is the entire complex plane. Now,

$$\mathbf{L}[f''(t)] = s\mathbf{L}[f'(t)] - f'(0-)$$

Since $f'(0-) = 0$ (note it's the limit from the *left*):

$$\mathbf{L}[f'(t)] = \left(\frac{1}{s}\right)\mathbf{L}[f''(t)] = \frac{2(1-e^{-2s})}{s^2} - \frac{4e^{-s}}{s}, \text{if } s \neq 0.$$

If $s = 0$,

$$\mathbf{L}[f'(t)] = \int_0^\infty f'(t)dt = 0$$

The DOC is again the entire complex plane.

Since $f(0-) = 0$,

$$\mathbf{L}[f(t)] = \left(\frac{1}{s}\right)\mathbf{L}[f'(t)] = \frac{2(1-e^{-2s})}{s^3} - \frac{4e^{-s}}{s^2}, \text{for } s \neq 0.$$

$$\text{For } s = 0, \quad \mathbf{L}[f(t)] = \int_0^\infty f(t)dt = \frac{2}{3}$$

The DOC is again the entire complex plane.

4. We will use the expansion of $F(s)$ into partial fractions to get f(t).

(a)

$$F(s) = \frac{s+11}{s^2-3s+4} = \frac{s+11}{(s+1)(s-4)} = \frac{A}{(s+1)} + \frac{B}{(s-4)}$$

$$A = \left[\frac{s+11}{s-4}\right]_{s=-1} = \frac{-1+11}{-1-4} = -2; \qquad B = \left[\frac{s+11}{s+1}\right]_{s=4} = \frac{4+11}{4+1} = 3$$

$$F(s) = \frac{-2}{(s+1)} + \frac{3}{(s-4)} \implies f(t) = (-2e^{-t} + 3e^{4t})u(t)$$

(b)

$$F(s) = \frac{4s+10}{s^3+6s^2+10s} = \frac{4s+10}{s[(s+3)^2+1]} = \frac{A}{s} + \frac{B(s+3)+C}{(s+3)^2+1}$$

We have

$$4s + 10 = A[(s+3)^2 + 1] + s(Bs + 3B + C)$$

Comparing coefficients of different powers of s on both sides, we get

$$\begin{aligned} 0 &= A + B \\ 4 &= 6A + 3B + C \\ 10 &= 10A \end{aligned}$$

which are easily solved to give $A = 1$, $B = -1$, $C = 1$. So

$$F(s) = \frac{1}{s} + \frac{(-1)(s+3)+1}{(s+3)^2+1} \implies f(t) = u(t) + e^{-3t}[-\cos(t) + \sin(t)]u(t)$$

(c)

$$F(s) = \frac{2s^2 - s - 5}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$B = \left[\frac{2s^2 - s - 5}{s+3}\right]_{s=1} = -1; \qquad C = \left[\frac{2s^2 - s - 5}{(s-1)^2}\right]_{s=-3} = 1$$

To find A, we let $s = 0$ in the partial fraction expansion and get

$$-\frac{5}{3} = -A + (-1) + \frac{1}{3} \implies A = 1$$

$$F(s) = \frac{1}{s-1} + \frac{(-1)}{(s-1)^2} + \frac{1}{s+3} \implies f(t) = e^t u(t) - te^t u(t) + e^{-3t}u(t)$$

5.

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0^-) = s[F(s)]$$
$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0^-) = s^2[F(s)]$$

Taking Laplace transform on both sides of the differential equation,

$$s^2F(s) + \alpha sF(s) + F(s) = \frac{1}{s}$$

$$\implies F(s) = \frac{1}{s(s^2 + \alpha s + 1)}$$

(a) By the Initial Value Theorem,

$$\lim_{t \to 0^+} f(t) = \lim_{s \to +\infty} [sF(s)] = \lim_{s \to +\infty} \frac{1}{(s^2 + \alpha s + 1)} = 0.$$

which is independent of α.

(b) The Final Value Theorem states that

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0^+} [sF(s)]$$

if the poles of $F(s)$ lie at 0 or strictly on the left half of the complex plane. The poles of F(s) are at $s = 0$ and at

$$s_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

$s_{1,2}$ are real for $\alpha \geq 2$ and $\alpha \leq -2$.
For $\alpha \geq 2$, $\sqrt{\alpha^2 - 4} < \alpha$, so $s_{1,2} < 0$.
For $\alpha \leq -2$, $\sqrt{\alpha^2 - 4} < -\alpha$, so $s_{1,2} > 0$.
Hence, the Final value theorem is valid if $\alpha \geq 2$ and not for $\alpha \leq -2$.

$s_{1,2}$ are complex for $-2 < \alpha < 2$.
For $0 < \alpha < 2$, $\text{Re}[s_{1,2}] = -\frac{\alpha}{2} < 0$.
For $-2 < \alpha < 0$, $\text{Re}[s_{1,2}] = -\frac{\alpha}{2} > 0$.
For $\alpha = 0$, $s_{1,2} = \pm i$. Hence, the Final value theorem is valid for $0 < \alpha < 2$ and not for $-2 < \alpha \leq 0$.

So, if $\alpha > 0$, the Final Value Theorem is valid, and

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0^+} \frac{1}{(s^2 + \alpha s + 1)} = 1$$

If $\alpha < 0$, then the poles satisfy $Re[s_{1,2}] > 0$. Using the partial fraction expansion

$$F(s) = \frac{A}{s} + \frac{B}{s - s_1} + \frac{C}{s - s_2}$$

we see that $f(t)$ will be of the form $f(t) = Au(t) + Be^{s_1 t}u(t) + Ce^{s_2 t}u(t)$. The magnitude of the complex exponential is $e^{Re[s_i]t}$ and goes to infinity as $t \to +\infty$.
If $\alpha = 0$, then $f(t)$ contains $\sin(t)$ or $\cos(t)$, and does not have a limit as $t \to +\infty$.

(c) If $\alpha = 1$,

$$F(s) = \frac{1}{s(s^2 + s + 1)} = \frac{1}{s[(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2]} = \frac{A}{s} + \frac{Ms + N}{s^2 + s + 1}.$$

Multiply by s, evaluate at $s = 0$: $\Longrightarrow A = 1$.
Multiply by s, limit as $s \to \infty$: $\Longrightarrow 0 = A + M \Longrightarrow M = -1$.
Evaluate at $s = -1$: $\Longrightarrow -1 = -A - M + N \Longrightarrow N = -1$.
Use formula given in class ($\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$, $M = N = -1$), to get

$$F(s) = \frac{1}{s} + \frac{-s - 1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \quad \Longrightarrow \quad f(t) = u(t) + e^{-\frac{t}{2}} \left[-\cos(\frac{\sqrt{3}}{2}t) - \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t) \right] u(t)$$