EE102 - Practice Final Solutions

Problem 1 [15 pts]

In class we saw that the cascade of two linear time invariant (LTI) systems is also LTI.
Now we ask:

a) Is the cascade of two linear time varying (LTV) systems always LTV?

b) Is the cascade of two time invariant (TI, not necessarily linear) systems always TI?

c) Is the cascade of two nonlinear systems always nonlinear ?

For each case you must give either:

- a proof that the answer is affirmative.
- a counterexample showing it is not the case.

Solution

a) NO. For example, the systems $y(t) = e^t x(t)$ and $z(t) = e^{-t} y(t)$ are both time-varying, but the cascade $z(t) = x(t)$ is time invariant.

b) YES. Suppose $y(t) = T_1[x(t)]$ and $z(t) = T_2[y(t)]$ are time invariant. Then

$$T_2[T_1[x(t-\tau)]] = T_2[y(t-\tau)] = z(t-\tau)$$

so the cascade is time invariant.

c) NO. For example, the systems $y(t) = [x(t)]^3$ and $z(t) = [y(t)]^{1/3}$ are both nonlinear, but the cascade $z(t) = x(t)$ is linear.

Problem 2 [15 pts]

A linear, time invariant system has impulse response function given by

$$h(t) = a\ \delta(t) + b\ e^{-t}u(t) + ct\,e^{-t}u(t)$$

where a, b, c are constants. We are given the following information:

- When the input is $x(t) \equiv 1$ for $t \in (-\infty, \infty)$, the output is the same as the input.

- When the input is $x(t) = \cos(t)$ for $t \in (-\infty, \infty)$, the output is zero.

a) Find a, b, c.

b) Now let the input be $x(t) = \cos(t)u(t)$. Find the output.

Solution

a)

$$H(s) = a + \frac{b}{s+1} + \frac{c}{(s+1)^2},$$

with DOC $Re(s) > -1$, so $H(i\omega)$ is well defined. The response to the constant $x(t) \equiv 1$ is the output $y(t) \equiv H(0) = a + b + c$, so

$$a + b + c = 1. \tag{1}$$

The response to $\cos(t)$ is $y(t) = |H(i)| \cos(t + \theta_H(i))$. Therefore

$$H(i) = a + \frac{b}{2} - i\left(\frac{b}{2} + \frac{c}{2}\right) = 0$$

So we get

$$a + \frac{b}{2} = 0 \tag{2}$$

$$b + c = 0. \tag{3}$$

Solving the equations (1-3) we get

$$a = 1, \qquad b = -2, \qquad c = 2.$$

b) Substituting back leads to

$$H(s) = \frac{s^2+1}{(s+1)^2}.$$

Applying $X(s) = \frac{s}{s^2+1}$ we have

$$Y(s) = \frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Inverse transform gives

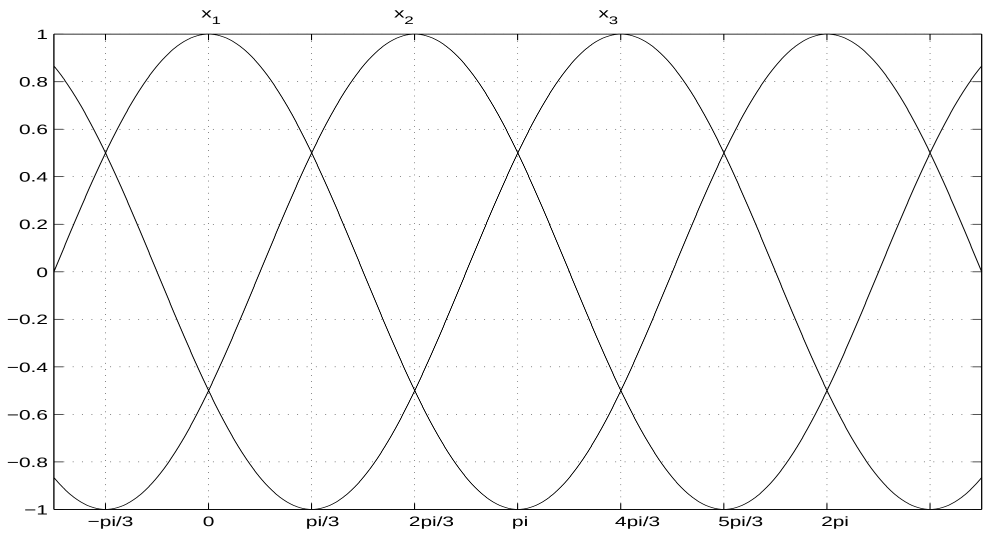
$$y(t) = u(t)e^{-t} - tu(t)e^{-t}.$$

Problem 3 [20 pts]

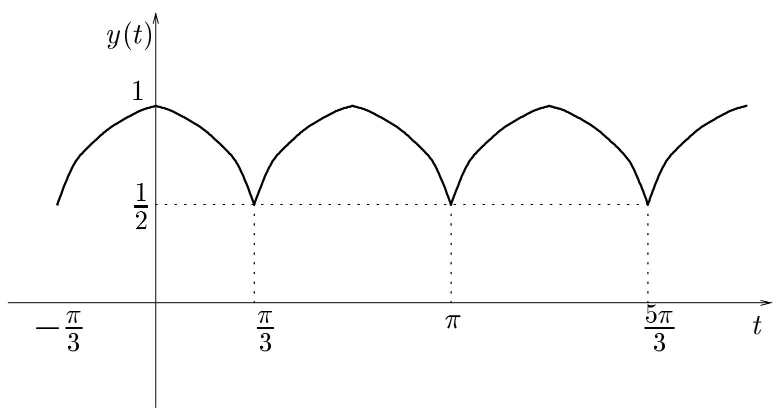
Consider the three signals

$$x_1(t) = \cos(t); \qquad x_2(t) = \cos\left(t - \frac{2\pi}{3}\right); \qquad x_3(t) = \cos\left(t + \frac{2\pi}{3}\right).$$

a) Sketch the signals in the plot below; $x_1(t)$ has been provided for your convenience; you should specify the coordinates in the t-axis.



b) Now let $y(t) = \max\{x_1(t), x_2(t), x_3(t)\}$; in other words $y(t)$ takes the maximum of the three signals at each instant in time.
Sketch $y(t)$ in a separate plot. What is the period of $y(t)$?



The period of $y(t)$ is $T = \frac{2\pi}{3}$.

c) Find the mean square error $\overline{\epsilon_0^2}$ resulting from approximating $y(t)$ by a constant function.

$$
\begin{aligned}
\overline{\epsilon_0^2} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y^2(t)dt - Y_o^2 \\
&= \frac{3}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2(t)dt - \Big(\frac{3}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos(t)dt\Big)^2 \\
&= \frac{3}{\pi}\int_{0}^{\frac{\pi}{3}} \cos^2(t)dt - \Big(\frac{3}{\pi}\int_{0}^{\frac{\pi}{3}} \cos(t)dt\Big)^2 \\
&= \frac{3}{2\pi}\Big(t + \frac{1}{2}\sin(2t)\Big)\Big|_0^{\frac{\pi}{3}} - \Big(\frac{3}{\pi}\sin(t)\Big)\Big|_0^{\frac{\pi}{3}}\Big)^2 \\
&= \frac{1}{2} + \frac{3\sqrt{3}}{8\pi} - \Big(\frac{3\sqrt{3}}{2\pi}\Big)^2 \approx 0.0228
\end{aligned}
$$

Problem 4 [15 pts]

We are given an LTI system with impulse response function

$$h(t) = \begin{cases} 1 - \frac{|t|}{\pi} & \text{if } |t| < \pi \\ 0 & \text{otherwise} \end{cases}$$

a) Find the frequency response function $H(i\omega)$.

b) We now apply to this system a periodic input, with period T. Discuss whether the following is true or false:
There exists a value of T such that for every periodic input of this period, the output is a constant function of time.
You should either show it's true and find an appropriate T, or show no such T exists.

Solution

a) Two versions of the Fourier transform (both correct):

$$H(i\omega) = \frac{2(1 - \cos(\omega\pi))}{\pi\omega^2} = \pi\left(\frac{\sin(\omega\pi/2)}{\omega\pi/2}\right)^2.$$

Note that $H(0) = \pi$.

b) For a periodic input

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{in\omega_0 t},$$

the output is

$$y(t) = \sum_{n=-\infty}^{\infty} H(in\omega_0) X_n e^{in\omega_0 t}.$$

For this to be a constant function, no matter what the X_n's are you need

$$H(in\omega_0) = 0 \quad \forall n \neq 0.$$

Choose $\omega_0 = 2$. Then

$$H(in\omega_0) = \frac{2(1 - \cos(2n\pi))}{4\pi n^2} = 0, \quad \forall n \neq 0.$$

So the choice $T = \pi$ has the required property.

Problem 5 [20 pts]

Given a periodic function $f(t)$ with Fourier series expansion $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}$.

a) Differentiate the above to derive a formula for the Fourier series of $\frac{df}{dt}(t)$.

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{in\omega_o t} \\ \frac{df(t)}{dt} &= \sum_{n=-\infty}^{\infty} F_n \frac{d}{dt} e^{in\omega_o t} \\ &= \sum_{n=-\infty}^{\infty} (in\omega_o) F_n e^{in\omega_o t} \end{aligned}$$

therefore,

$$F_n' = (in\omega_o) F_n$$

b) Now consider $f(t)$ of period $T = 2$ and such that

$$f(t) = \begin{cases} (t+1)^2 & \text{for } t \in [-1, 0] \\ (t-1)^2 & \text{for } t \in [0, 1] \end{cases}$$

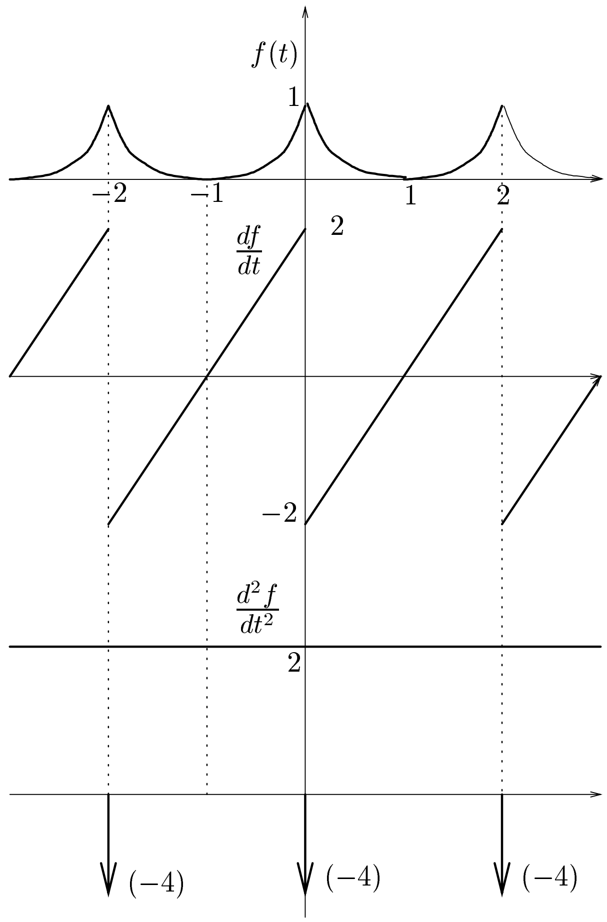
Sketch $f(t)$, $\frac{df}{dt}(t)$ and $\frac{d^2 f}{dt^2}(t)$.

You can sketch $f(t)$ in $[-1, 1]$, then repeat periodically. See next page.

Analytical formulas:

$$\frac{df(t)}{dt} = \begin{cases} 2(t+1) & \text{for } t \in [-1, 0] \\ 2(t-1) & \text{for } t \in [0, 1] \end{cases},$$

$$\frac{d^2 f(t)}{dt^2} = 2 - 4\delta(t) \text{ in } [-1, 1].$$



c) Find the Fourier series expansions of $f(t)$, $\frac{df}{dt}(t)$ and $\frac{d^2f}{dt^2}(t)$ in b).
Start with,

$$\frac{d^2f(t)}{dt^2} = \sum_{n=-\infty}^{\infty} F_n'' e^{in\omega_o t}$$

where $\omega_o = \frac{2\pi}{2} = \pi$.

$$\begin{aligned} F_n'' &= \frac{1}{2}\int_{-1}^{1} \frac{d^2f(t)}{dt^2} e^{-in\pi t} dt \\ &= \frac{1}{2}\int_{-1}^{1} (2 - 4\delta(t)) e^{-in\pi t} dt \\ &= \begin{cases} 0 & n = 0 \\ -2 & n \neq 0 \end{cases} \end{aligned}$$

Now, $F''_n = (in\pi)F'_n$, so for $n \neq 0$ we obtain $F'_n = \frac{-2}{in\pi}$. It remains to find F'_0:

$$F'_0 = \frac{1}{2}\int_{-1}^{1} \frac{df(t)}{dt} dt = 0$$

Summarizing,

$$F'_n = \begin{cases} 0 & n = 0 \\ \frac{-2}{in\pi} & n \neq 0 \end{cases}$$

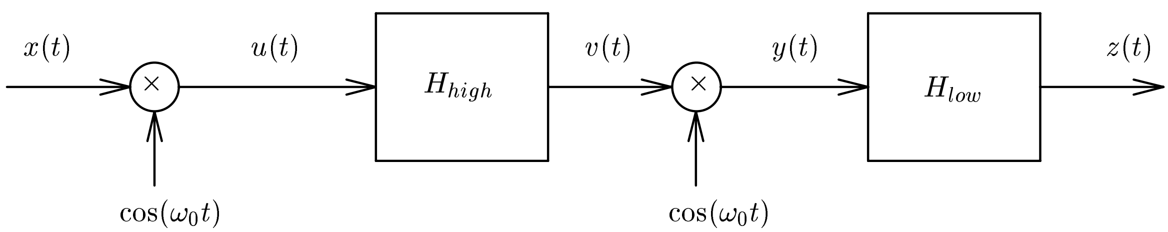
We repeat the method for calculating $F_n = \frac{F'_n}{in\pi}$, for $n \neq 0$, and

$$F_0 = \frac{1}{2}\int_{-1}^{1} f(t)dt = \int_{0}^{1} (t-1)^2 dt = \frac{1}{3}.$$

Therefore,

$$F_n = \begin{cases} \frac{1}{3} & n = 0 \\ \frac{2}{n^2\pi^2} & n \neq 0 \end{cases}$$

Problem 6 [15 pts]



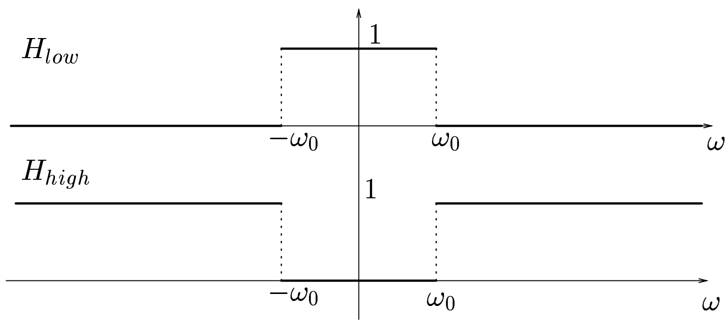
In the above system

- H_{high} is an ideal high-pass filter with cutoff frequency ω_0.
- H_{low} is an ideal low-pass filter, also with cutoff frequency ω_0.
- $u(t) = x(t)\cos(\omega_0 t)$ and $y(t) = v(t)\cos(\omega_0 t)$.
- $x(t)$ is band-limited to $[-B, B]$, as depicted in the figure below. $X(0) = A$.
- $\omega_0 > 2B$.

Sketch the Fourier transforms $U(i\omega)$, $V(i\omega)$, $Y(i\omega)$ and $Z(i\omega)$, and relate $z(t)$ to $x(t)$. Justify your answer.

Solution

The filter frequency responses are depicted below:



From the modulation property of the Fourier transform, we have

$$U(i\omega) = \mathcal{F}[x(t)\cos(\omega_0 t)] = \frac{X(i(\omega-\omega_0)) + X(i(\omega+\omega_0))}{2}$$

By highpass filtering, we have

$$V(i\omega) = \begin{cases} U(i\omega) & \text{for } |\omega| \geq \omega_0 \\ 0 & \text{for } |\omega| < \omega_0 \end{cases}$$

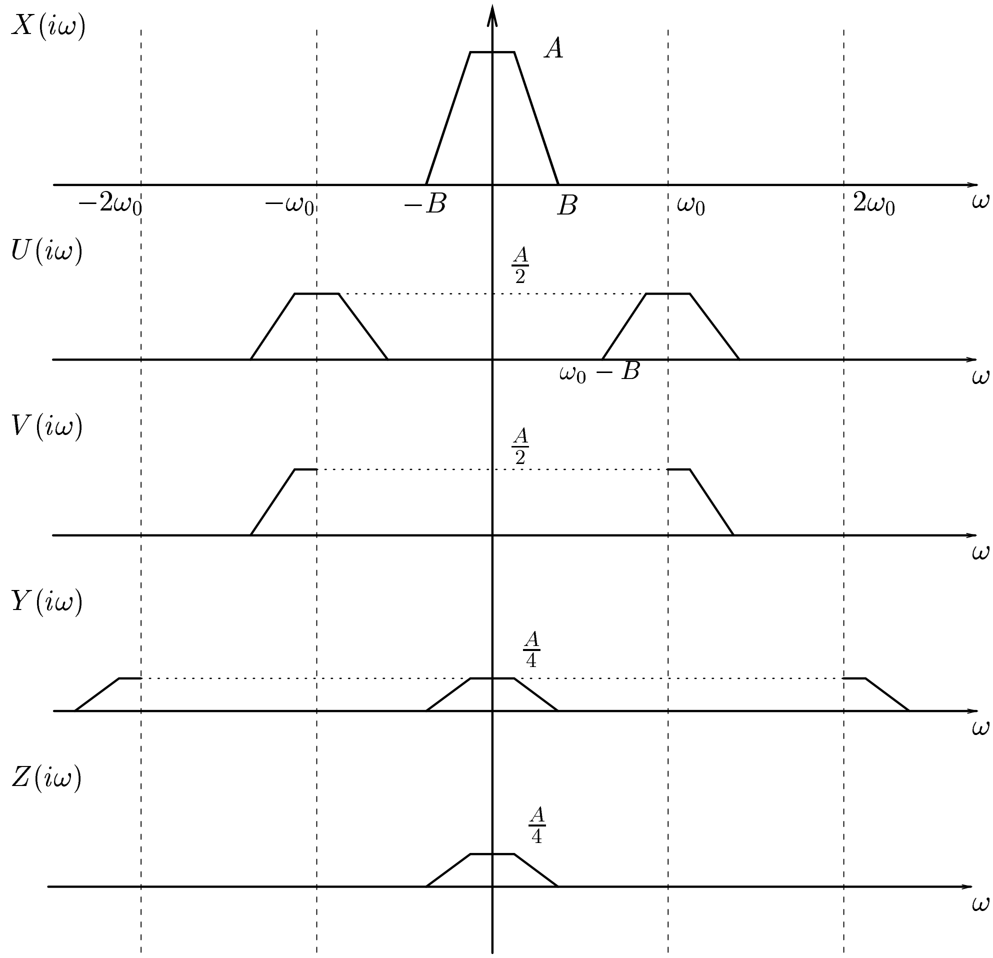
Once again the modulation property gives

$$Y(i\omega) = \frac{V(i(\omega-\omega_0)) + V(i(\omega+\omega_0))}{2}$$

Finally, lowpass filtering gives

$$Z(i\omega) = \begin{cases} Y(i\omega) & \text{for } |\omega| \leq \omega_0 \\ 0 & \text{for } |\omega| > \omega_0 \end{cases}$$

Using the conditions above ($X(i\omega)$ band-limited to $[-B, B]$, and $\omega_0 > 2B$) we obtain the following plots:



From the plots we see that

$$Z(i\omega) = \frac{1}{4}X(i\omega)$$

Therefore by inverse transform,

$$z(t) = \frac{1}{4}x(t).$$