EECE360

Lecture 3

Laplace Transform, Transfer Functions

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Chapter 2.3-2.5

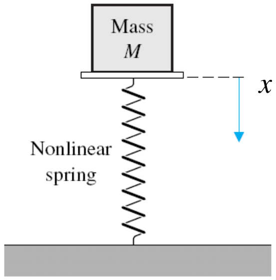
Topics

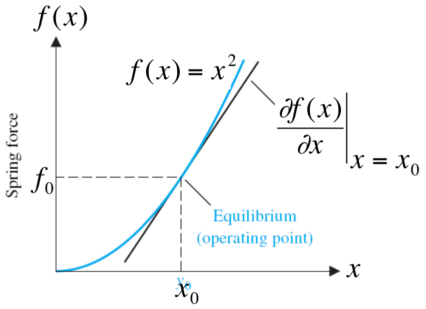
- ***Review***
  - Linearization through Taylor's Series approximation
- ***Today***
  - Physical systems:
    - Spring-mass-damper system
    - RLC circuits
    - DC motor
  - Laplace transform
  - Transfer functions

How to Linearize

$$\Delta y = \left.\frac{\partial f(x)}{\partial x}\right|_{x = x_0} \Delta x$$

- Identify an operating point
- Perform Taylor series expansion and keep only constant and 1st derivative terms





Physical Laws of Process

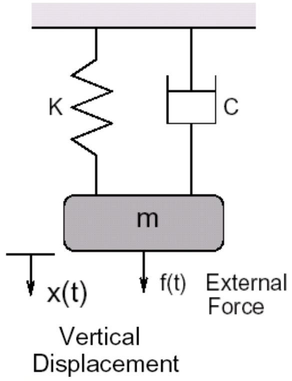
- For mechanical systems: Newton's laws
  - Newton's second law of motion: the relationship between an object's mass **m**, its acceleration **a**, and the applied force **F**

F = ma

- For electrical systems: Kirchhoff's laws
  - 1st law or the *junction rule (KCL)*: For a given junction or node in a circuit, the sum of the currents entering equals the sum of the currents leaving.
  - 2nd law or the *loop rule (KVL)*: Around any closed loop in a circuit, the sum of the potential differences across all elements is zero.

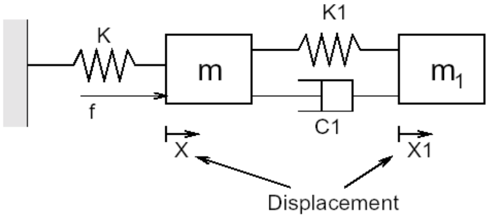
Spring-Mass-Damper System**

- *F(t)* : equivalent force on mass *m*
- The signs of *-kx* and *-cx* are negative because these forces oppose the motion of *f(t)*



$$m\ddot{x} = F(t) = -kx - c\dot{x} + f(t)$$

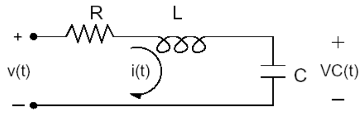
Two-Mass System**



$$\begin{cases} m\ddot{x} = f(t) - Kx + K_1(x_1 - x) + C_1(\dot{x}_1 - \dot{x}) \\ m_1\ddot{x}_1 = -K_1(x_1 - x) - C_1(\dot{x}_1 - \dot{x}) \end{cases}$$

Assume the motion directions: $x>0$; x_1-$x>0$

Series RLC Circuit**

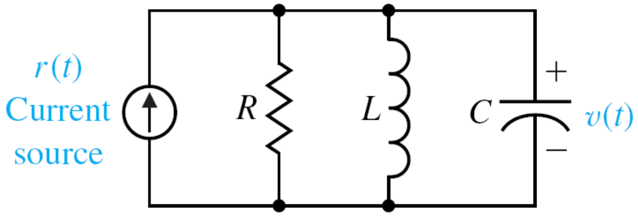


Applying Kirchhoff's voltage law yields an integro-differential model:

$$\begin{aligned} v(t) &= v_R(t) + v_L(t) + v_C(t) \\ &= Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)\,d\tau \end{aligned}$$

Input: *v(t);* Output: *i(t)*

Parallel RLC Circuit**



Applying Kirchhoff's current law:

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_0^t v(t) = r(t)$$

Input: *r(t);* Output: *v(t)*

The Laplace Transform

- The method of Laplace transforms converts a calculus problem (the linear differential equation) into an algebra problem.
- The solution of the algebra problem is then fed backwards through a the *Inverse Laplace Transform* and the solution to the differential equation is obtained.

The Laplace Transform

- Pierre-Simon Laplace (1749-1827)
- Laplace proved the stability of the solar system. He also put the theory of mathematical probability on a sound footing
- *"All the effects of Nature are only the mathematical consequences of a small number of immutable laws."*
- Studied, but did not fully developed the Laplace transform

The Laplace Transform

- Like the Fourier transform, the Laplace transform is an *integral transform*

$$\boxed{\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt}$$

- Alternately the Laplace variable ***s*** can be considered to be the differential operator:

$$\boxed{s = \frac{d(.)}{dt}}$$

Laplace Transform Properties

- Linearity:

$$\begin{aligned}\mathcal{L}[kf(t)] &= kF(s)\\ \mathcal{L}[f_1(t)+f_2(t)] &= F_1(s)+F_2(s)\end{aligned}$$

- Differentiation:

$$\begin{aligned}\mathcal{L}[\dot{f}(t)] &= sF(s)-f(0^-)\\ \mathcal{L}[\ddot{f}(t)] &= s^2F(s)-s\dot{f}(0^-)-f(0^-)\\ \mathcal{L}[f^{(n)}(t)] &= s^nF(s)-s^{n-1}f(0^-)-\cdots-f^{(n-1)}(0^-)\end{aligned}$$

- **Final value theorem**: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
- Initial value theorem: $f(0^+) = \lim_{s\to\infty} sF(s)$

Important Laplace Tr. Pairs

- Impulse function
- Step function
- Exponential decay
- Sine and cosine

$F(s)$	$f(t)$
1	$\delta(t)$
$\frac{1}{s}$	$\mathbf{1}(t)$
$\frac{1}{s+a}$	e^{-at}
$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$

See Dorf and Bishop, *Table 2.3*, p. 47 and *Table D.1* (App. D) online.

Important Laplace Tr. Pairs

- Damped oscillations

$F(s)$	$f(t)$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin(\omega t)$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos(\omega t)$
$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2 t})$

See Dorf and Bishop, *Table 2.3*, p. 47 and *Table D.1* (App. D) online.

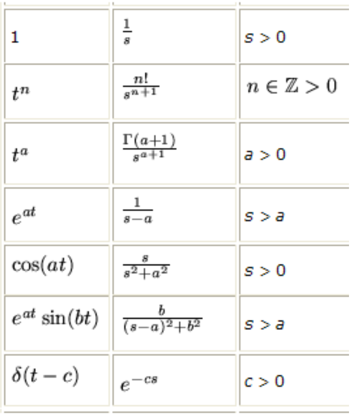
Laplace Transforms

- Region of convergence:
  - Where the transform integral converges
  - **The Laplace transform exists only when the integral converges!**
- If $|f(t)| < Me^{at}$ for all positive t, then

$$\int_{0-}^{\infty} |f(t)|e^{-st}dt < \infty$$

will converge for *s > a.*

f(t) *F(s)* *R.o.C.*



f(t)	*F(s)*	*R.o.C.*
1	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$n \in \mathbb{Z} > 0$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$a > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$\delta(t-c)$	e^{-cs}	$c > 0$

Example

- Consider the spring-mass-damper system

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

- Assuming the system is initially at rest (all initial conditions at zero), then

$$R(S) = Ms^2Y(s) + bsY(s) + kY(s)$$

output

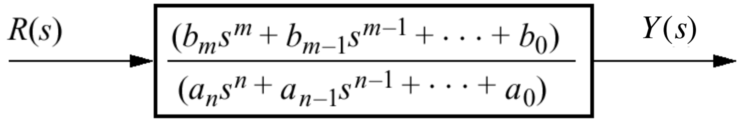
input

$$\boxed{\frac{Y(s)}{R(s)}} = \frac{1}{Ms^2 + bs + k}$$

Transfer function

Transfer Function

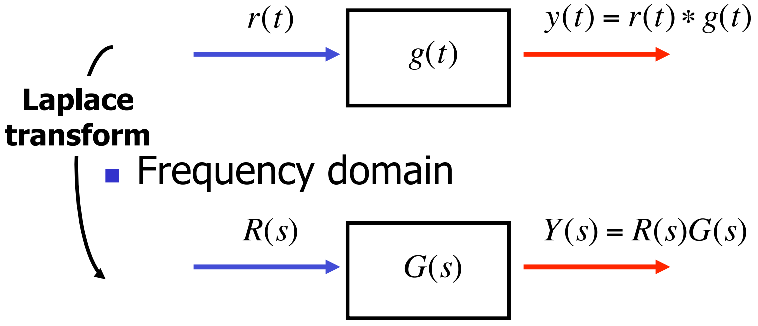
- Defined as the ratio of the Laplace transform of the output to that of the input
- Describes dynamics of a LTI system



$R(s)$ → $\boxed{\dfrac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}}$ → $Y(s)$

Transfer Function

- Time domain



$r(t)$ → $\boxed{g(t)}$ → $y(t) = r(t) * g(t)$

Laplace transform

- Frequency domain

$R(s)$ → $\boxed{G(s)}$ → $Y(s) = R(s)G(s)$

Transfer Function

- Differential equation replaced by algebraic relation *Y(s)=G(s)R(s)*
- Note that if *R(s)=1* then *Y(s)=G(s)* is the impulse response of the system
- Note that if *R(s)=1/s,* the unit step function, then *Y(s)=G(s)/s* is the step response
- The magnitude and phase shift of the response to a sinusoid at frequency ω is given by the magnitude and phase of the complex number $G(j\omega)$ (see Chapter 8)

Transfer Function

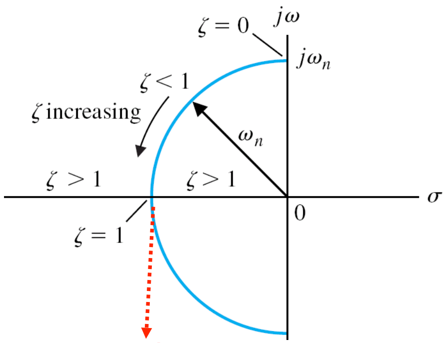
- Using the Final Value Theorem, the static or D.C. gain of a transfer function *G(s)* is given by *G(0):*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sG(s)$$

- Let *G(s) = N(s)/D(s)*, then
  - **Zeros** of *G(s)* are the roots of *N(s)=0*
  - **Poles** of *G(s)* are the roots of *D(s)=0*

Second Order System Poles **

$$G(s) = \frac{1/M}{s^2 + (b/M)s + (k/M)} = \frac{1/M}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Critical damping

✖ Damping ratio

ω_n Natural frequency

In the **underdamped case** (complex roots due to the quadratic):

$$\boxed{\begin{array}{l} s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ with \quad \zeta < 1 \end{array}}$$

Second Order System Poles **

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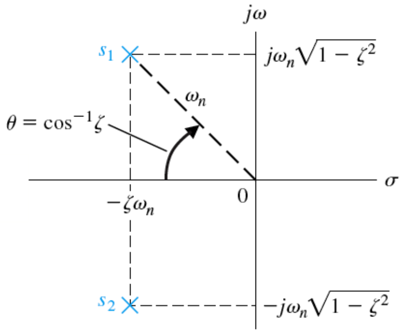

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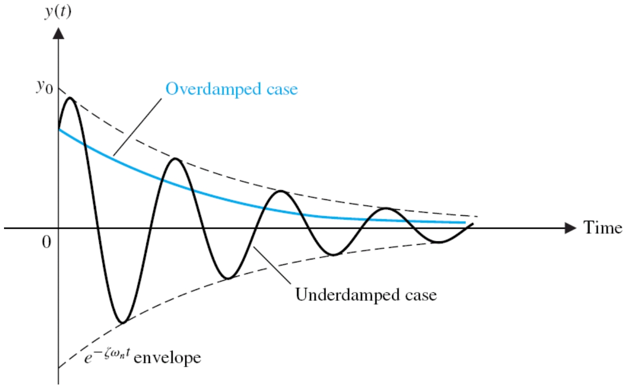
$$\boxed{\begin{array}{l} s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ with \quad \zeta < 1 \end{array}}$$

Natural Response (no input)

$$\frac{w_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Leftrightarrow \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2} t), \zeta < 1$$

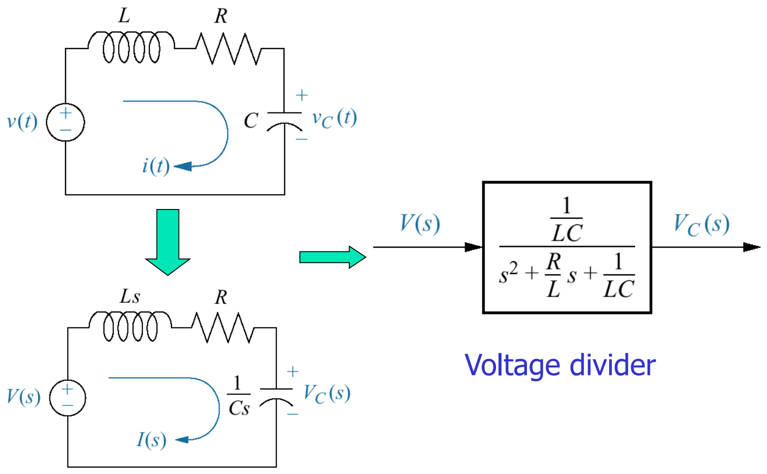


Transfer Functions

- A transfer function is
  - **strictly proper** when the degree of the denominator is greater than that of numerator
  - **proper** if those degrees are equal
  - **improper** if the degree of the numerator is greater than than of the denominator

- Physical systems are proper or strictly proper

RLC Network

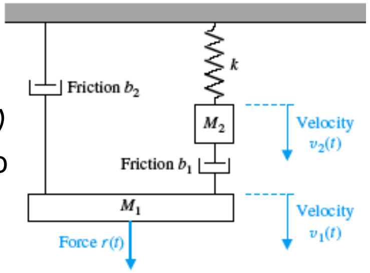


$$\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Voltage divider

Example: Spring-mass-damper

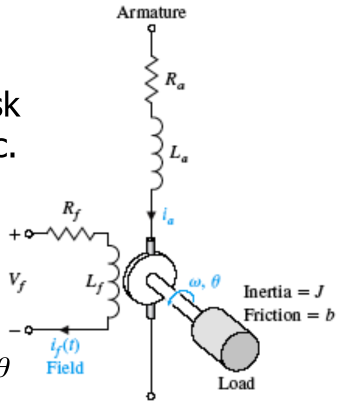
- Find transfer function *G(s)* between the position $x_1(t)$ and the forcing function *r(t)*
- Key assumption: IC are zero
  - $x_1(t)=x_2(t)=0$
  - $dx_1/dx=dx_2/dt=0$
- Note that $x_2(t)$ does not appear explicitly. (Why?)



Dorf and Bishop, Example 2.4, p. 56

Example: DC Motor

- Power actuator
- Robotic manipulators, disk drives, machine tools, etc.
- Armature-controlled
  - Input: voltage applied to armature V_a $(=E_a)$
  - Output: angle of rotation θ
  - Control variable: current i_a



Example: DC Motor

$i_a(t) =$ armature current

$R_a =$ armature resistance

$e_b(t) =$ back emf

$T_L(t) =$ load torque

$T_m(t) =$ motor torque

$\theta_m(t) =$ rotor displacement

$K_i =$ torque constant

$L_a =$ armature inductance

$e_a(t) =$ applied voltage

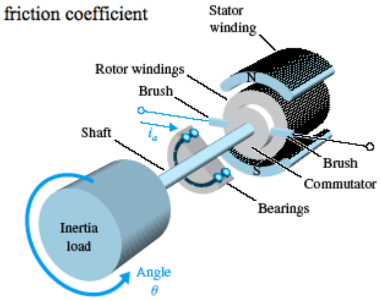
$K_b =$ back-emf constant

$\phi =$ magnetic flux in the air gap

$\omega_m(t) =$ rotor angular velocity

$J_m =$ rotor inertia

$B_m =$ viscous friction coefficient



Example: DC Motor

- Armature circuit: $e_a(t) - e_b(t) = R_a i_a + L_a \frac{di_a}{dt}$
- Motor relations: $$T_m(t) = K_i i_a(t)$$ $$e_b(t) = K_b \omega_m(t)$$
- Mechanical Response: $J_m \dot{\omega}_m(t) = T_m(t) - T_d(t) - B_m \omega_m(t)$

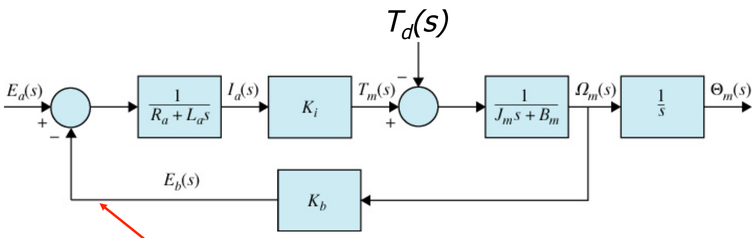
Example: DC Motor

- Armature circuit:
$$E_a(s) - E_b(s) = (R_a + sL_a) I_a(s)$$
- Motor relations:
$$\begin{cases} T_m(s) = K_i I_a(s) \\ E_b(s) = K_b \Omega_m(s) \end{cases}$$
- Mechanical Response:

$$J_m s \Omega_m(s) = T_m(s) - T_d(s) - B_m \Omega_m(s)$$

Example: DC Motor

Using Laplace transform, we can represent this DC motor by the block diagram below



Notice inherent feedback in the model

$$\text{Transfer function}: H(s) = \frac{\theta_m(s)}{E_a(s)}$$

Summary

- ***Today***
  - Laplace transform
  - Final Value Theorem
  - Transfer functions
- ***Next***
  - Operational Amplifiers
  - Block diagram models