EECE 360
Lecture 9

State Equation Representation of Dynamic Systems (cont'd)

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Chapter 3.3-3.5

Review: Canonical Forms

- Transfer function to state-space

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$A = \left[\begin{array}{c}\begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix} \\ \boxed{\begin{matrix} -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{matrix}}\end{array}\right], \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \left[\ \boxed{b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_{n-1}}\ \right], \quad D = 0$$

Control canonical form

$$A = \left[\begin{matrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix} \ \boxed{\begin{matrix} -a_0 \\ -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \end{matrix}}\right], \quad B = \left[\ \boxed{\begin{matrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{matrix}}\ \right]$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad D = 0$$

Observer canonical form

State-space equations

- Last week
  - State-space to transfer function
  - Transfer function to state-space
    - Control canonical form
    - Observer canonical form
- Today
  - **Solution to state-space: *x(t)* = …**
  - More examples

The State Transition Matrix

- Consider the homogenous (i.e. zero-input) dynamics:

$$\dot{x} = Ax$$

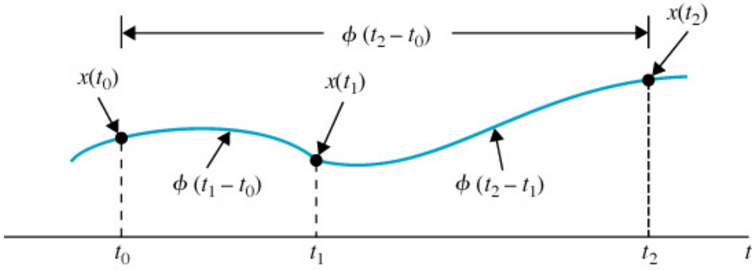
- The solution to this equation represents the evolution of the system's *free response* to non-zero initial conditions:

$$\boxed{x(t) = \Phi(t)x(0)}$$

State transition matrix

The State Transition Matrix

- Given an initial value, the state transition matrix predicts the state at any other time



- So what is the state transition matrix?

The State Transition Matrix

- Consider the homogenous (i.e. zero-input) dynamics:

$$\dot{x} = Ax$$

- A Taylor's series approximation taken about *t=0* provides the solution

$$\boxed{x(t) \quad = \quad e^{At}x(0) = \Phi(t)x(0)}$$

- In which the **matrix exponential** is defined as

$$\boxed{e^{At} = I + At + A^2\frac{t^k}{2!} + \ldots + A^k\frac{t^k}{k!} + \ldots = \sum_{k=1}^{\infty} A^k\frac{t^k}{k!}}$$

The Matrix Exponential

- Useful matrix exponential properties

$$\begin{aligned}
e^{A\cdot 0} &= I \\
e^{A(t_1+t_2)} &= e^{At_1}e^{At_2} = e^{At_2}e^{At_1} \\
(e^{At})^{-1} &= e^{-At} \\
e^{A^T t} &= (e^{At})^T \\
Ae^{At} &= e^{At}A \\
\frac{d}{dt}e^{At} &= Ae^{At}
\end{aligned}$$

- Makes computation of e^{At} easier for A with certain structure (e.g., diagonal, upper triangular, symmetric, otherse)

State Transition Matrix

- Instead of solving in the time domain, consider

$$\begin{aligned}
\dot{x}(t) &= Ax(t) \\
sX(s) - x(0) &= AX(s) \\
(sI - A)X(s) &= x(0) \\
X(s) &= (sI - A)^{-1}x(0) = \Phi(s)x(0)
\end{aligned}$$

- In the Laplace domain, the state transition matrix is

$$\boxed{\Phi(s) = (sI - A)^{-1}}$$

- therefore $\boxed{\Phi(t) = L^{-1}(\Phi(s)) = L^{-1}\left((sI - A)^{-1}\right)}$

Example 2

- Recall from last lecture

$$\dot{x} = Ax, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

- Since A is diagonal, the matrix exponential is

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

- The solution to $\dot{x} = Ax$ with *x(0) = [1 1]^T* is

$$x(t) = e^{At}x(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix}$$

Example 2

- Again consider the system

$$\dot{x} = Ax, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

with initial condition *x(0) = [1 1]^T.*

- Previously: Solved directly in time domain.
- Now: Solve in s-domain, then take inverse Laplace transform

$$x(t) = \Phi(t)x(0) = L^{-1}\left((sI - A)^{-1}\right)x(0)$$

Example 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

- Find the inverse matrix

$$\begin{aligned}(sI - A)^{-1} &= \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}^{-1} \\ &= \frac{1}{(s+1)(s+3)}\begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix}\end{aligned}$$

Example 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

- Using Laplace transform tables (App. D.1)

$$\boxed{\frac{1}{(s+a)^n} \Leftrightarrow \frac{t^{n-1}e^{-at}}{(n-1)!}}$$

$$\begin{aligned}x(t) &= L^{-1}\left((sI - A)^{-1}\right)x(0) \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix}\end{aligned}$$

- This is the **same result** as we got from solving directly for *e^{At}*.

State Transition Matrix

- The matrix exponential can be easily solved for some forms of *A* (diagonal, upper triangular, and others)
- **But for general *A*, an easier way to solve for the state transition matrix is to find its **Laplace transform**.
- Can be computed in Matlab using '**expm'** for specific *A* and *t*

```
>>A=[0 -2; 1 -3]; dt=0.2; Phi=expm(A*dt)

Phi =

   0.9671   -0.2968
   0.1484    0.5219
```

State transition matrix for a Δt of 0.2 second

Example 3

- Given $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$,
- The state transition matrix is

$$\begin{aligned} \Phi(s) &= (sI - A)^{-1} \\ &= \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}^{-1} \\ &= \frac{1}{\Delta(s)} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}, \text{ where} \\ \Delta(s) &= s^2 + 3s + 2 = (s+1)(s+2) \end{aligned}$$

- The time-domain state transition matrix can be obtained using the inverse Laplace transform

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Example 3

- Find $\begin{aligned} \Phi(t) &= \mathcal{L}^{-1}(\Phi(s)) \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix} \end{aligned}$
- And using known Inverse Laplace Transforms (Table D.1, Dorf and Bishop),

$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(e^{-at} - e^{-bt})$
$\frac{s+\alpha}{(s+a)(s+b)}$	$\frac{1}{(b-a)}[(\alpha - a)e^{-at} - (\alpha - b)e^{-bt}]$

$$\Phi(t) = \begin{bmatrix} e^{-t} & -e^{-t} + e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

Example 3

- State transition matrix (time domain)

$$\Phi(t) = \begin{bmatrix} e^{-t} & -e^{-t} + e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

- With initial conditions $x_0 = [1\ 1]^T$, the free (unforced) response is

$$\begin{aligned} x(t) &= \Phi(t)x(0) \\ &= \begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix} \end{aligned}$$

State Transition Matrix

- For the homogeneous system $\dot{x}(t) = Ax(t)$ we examined two ways to solve for *x(t):*

$$x(t) = \Phi(t)x(0), \qquad \Phi(t) = e^{At}$$
$$x(t) = L^{-1}\big(\Phi(s)\big)x(0), \quad \Phi(s) = (sI - A)^{-1}$$

- Now, for the inhomogeneous system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(e.g. with a non-zero input (forcing function)), **what is the solution *x(t)*?**

State Transition Matrix

- In the time-domain:

$$\dot{x} = Ax + Bu$$
$$e^{-At}(\dot{x} - Ax) = e^{-At}Bu$$
$$\frac{d}{dt}(e^{-At}x) = e^{-At}Bu$$
$$\int_0^t \frac{d}{d\tau}(e^{-A\tau}x)d\tau = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$
$$e^{-At}x(t) - e^{-A\cdot 0}x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

State Transition Matrix

- Rearranging,

$$x(t) = e^{At}x(0) + \int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau$$

- Recall that $\Phi(t) = e^{At}$
- Therefore the solution is

$$x(t) = \underbrace{\Phi(t)x(0)}_{\textbf{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\textbf{Forced response}}$$

State Transition Matrix

- Now examine in the Laplace domain

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

- Recall that

$$\Phi(s) = (sI - A)^{-1}$$

- Therefore the solution in the Laplace domain is

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$

State Transition Matrix

- This solution matches the time-domain solution

$$X(s) = \underbrace{\Phi(s)x(0)}_{} + \underbrace{\Phi(s)BU(s)}_{}$$

$$x(t) = \underbrace{\Phi(t)x(0)}_{\textbf{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\textbf{Forced response}}$$

- To solve for *x(t)* it is often easier to use the Laplace domain, then take the inverse Laplace transform of the result.

State Transition Matrix

- Note that the system response has two components:
  - Natural response – "zero input response" due to initial conditions
  - Forced response – "zero state response" due to input
- Overall response is the sum of the two

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$\Phi(t) = e^{At}$$

Natural response

Forced response

Example 1B

- Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

with initial condition *x(0) = [1 1]^T* and input u(t)=**1**(t).

- What is the state at *t=1* ? At *t=5* ?
- Solution: Find

$$x(t) = \Phi(t)x(0) + \boxed{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}$$

Example 1B

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Solve by using the Laplace domain representation

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$

$$\Phi(s) = (sI - A)^{-1}$$

- From Example 2, Lecture 9, we know

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix}$$

- Therefore

$$\Phi(s)x(0) = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 1B

$$A=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}, B=\begin{bmatrix}1\\2\end{bmatrix}$$

$$X(s)=\Phi(s)x(0)+\Phi(s)BU(s)$$

- Natural response:

$$\Phi(s)x(0)=\begin{bmatrix}\frac{1}{s+1}\\ \frac{1}{s+3}\end{bmatrix}$$

- Forced response:

$$\Phi(s)BU(s)=\frac{1}{(s+1)(s+3)}\begin{bmatrix}s+3 & 0\\ 0 & s+1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}U(s)$$

$$=\begin{bmatrix}\frac{1}{s+1}\\ \frac{2}{s+3}\end{bmatrix}U(s),\quad U(s)=\frac{1}{s}$$

$$=\begin{bmatrix}\frac{1}{s(s+1)}\\ \frac{2}{s(s+3)}\end{bmatrix}$$

Example 1B

$$A=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}, B=\begin{bmatrix}1\\2\end{bmatrix}$$

- Total response (Laplace domain)

$$X(s)=\Phi(s)x(0)+\Phi(s)BU(s)$$

$$=\begin{bmatrix}\frac{1}{s+1}\\ \frac{1}{s+3}\end{bmatrix}+\begin{bmatrix}\frac{1}{s(s+1)}\\ \frac{2}{s(s+3)}\end{bmatrix}=\begin{bmatrix}\frac{1}{s}\\ \frac{s+2}{s(s+3)}\end{bmatrix}$$

Laplace transform pairs

$$\frac{1}{(s+a)^n}\Leftrightarrow\frac{t^{n-1}e^{-at}}{(n-1)!}$$

$$\frac{1}{(s+a)(s+b)}\Leftrightarrow\frac{e^{-at}-e^{-bt}}{b-a}$$

- Total response (time-domain)

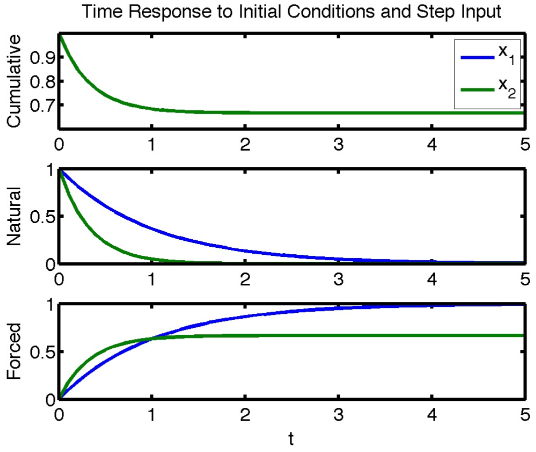
$$x(t)=L^{-1}\left(\Phi(s)x(0)\right)+L^{-1}\left(\Phi(s)BU(s)\right)$$

$$=\begin{bmatrix}e^{-t}\\ e^{-3t}\end{bmatrix}+\begin{bmatrix}1-e^{-t}\\ \frac{2}{3}(1-e^{-3t})\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{3}(2+e^{-3t})\end{bmatrix}$$

Ex. 1B

$$x(t)=\begin{bmatrix}e^{-t}\\ e^{-3t}\end{bmatrix}+\begin{bmatrix}1-e^{-t}\\ \frac{2}{3}(1-e^{-3t})\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{3}(2+e^{-3t})\end{bmatrix}$$

- In Matlab, we can plot this result



Ex. 1B

$$x(t)=\begin{bmatrix}e^{-t}\\ e^{-3t}\end{bmatrix}+\begin{bmatrix}1-e^{-t}\\ \frac{2}{3}(1-e^{-3t})\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{3}(2+e^{-3t})\end{bmatrix}$$

- Defining the variables

```
>> t = 0:.1:5;
>> x = [ones(size(t)); 1/3*
   (2+exp(-3*t))];
>> xN = [exp(-t);
        exp(-3*t)];
>> xF = [1-exp(-t);
        2/3*(1-exp(-3*t))];
```

- Plotting the top graph

```
>> subplot(311);
>> plot(t,x);
>> ylabel('Cumulative');
>> legend('x_1','x_2');
>> title('Time Response to
   Initial Conditions and
   Step Input')
```

Plotting the middle graph

```
>> subplot(312);
>> plot(t,xN);
>> ylabel('Natural');
```

- Plotting the bottom graph

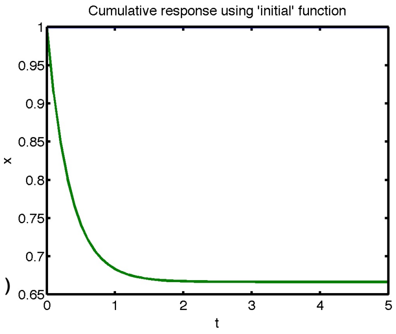
```
>> subplot(313);
>> plot(t,xF);
>> xlabel('t')
>> ylabel('Forced');
```

Ex. 1B

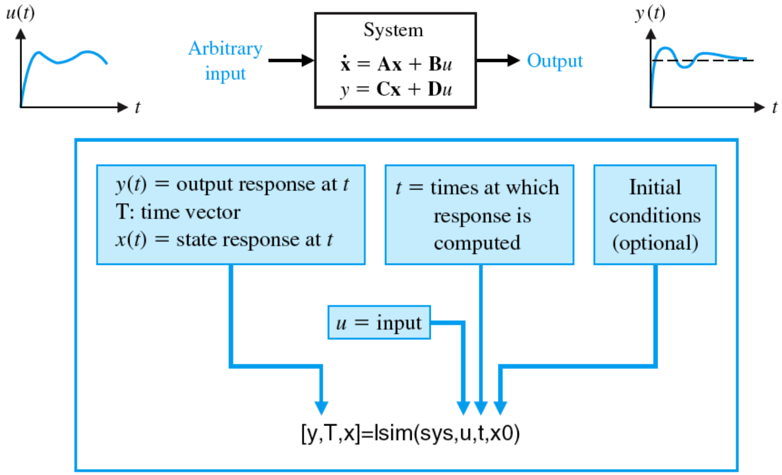
$$x(t)=\begin{bmatrix}e^{-t}\\ e^{-3t}\end{bmatrix}+\begin{bmatrix}1-e^{-t}\\ \frac{2}{3}(1-e^{-3t})\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{3}(2+e^{-3t})\end{bmatrix}$$

- Can also obtain the cumulative response with `'lsim'`

```
>> A = [-1 0; 0 -3];
>> B = [1 2]';
>> C = eye(2,2);
>> D = 0;
>> sys = ss(A,B,C,D);
>> t = 0:.1:5;
>> u = ones(size(t));
>> x0 = [1 1]';
>> [y,t,x]=lsim(sys,u,t,x0)
>> plot(t,x);
```



Using Matlab: `'lsim'`



$y(t)$ = output response at t
T: time vector
$x(t)$ = state response at t

t = times at which response is computed

Initial conditions (optional)

u = input

[y,T,x]=lsim(sys,u,t,x0)

Summary

- Canonical forms
  - Control canonical
  - Observer canonical
- State transition matrix $\Phi(t)$
- Matrix exponential e^{At}
- State transition equation

$$\boxed{\begin{aligned}x(t)&=\Phi(t)x(0), & \Phi(t)&=e^{At}\\ x(t)&=L^{-1}\big(\Phi(s)\big)x(0), & \Phi(s)&=(sI-A)^{-1}\end{aligned}}$$