EECE 360
Lecture 11

Properties of Feedback

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Chapter 4.1-4.4, 12.1-12.2

Last Class

- Solution to general state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{Forced response}}$$

- Characteristic equation
  - Poles of transfer function = Eigenvalues of state matrix *A*
  - Eigenvalues and eigenvectors

Review: State Trans. Matrix

- Solve $\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$
for x(t) in either the Laplace- or time-domain

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$
$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{Natural response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{Forced response}}$$

- Often easier to use the Laplace domain, then take the inverse Laplace transform of the result.

Review: Characteristic equation

- Recall that for a transfer function *G(s)=N(s)/D(s)*
  - The **characteristic equation** is *D(s)=0*
  - The roots of the characteristic equation are the **poles of *G(s)*.**
- Recall that the denominator of the transfer function of a state-space representation is *det(sI-A)*
  - The **characteristic equation** is *det(sI-A)=0*
  - The roots of the characteristic equation are the **eigenvalues of the matrix *A*.**

Review: Linear Algebra Rev. 2

- Eigenvalues of *A*
  Find λ_i such that $$0 = \det(\lambda_i I - A)$$
- Eigenvectors of *A*
  Find v_i such that $$Av_i = \lambda_i v_i$$ $$0 = (\lambda_i I - A)v_i$$
- To numerically compute eigenvalues and eigenvectors in Matlab, use `[V,D]=eig(A)`

Review: Putting it all together

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$
$$\Phi(t) = e^{At}$$

Inverse Laplace transform ↑ ↓ Laplace transform

$$X(s) = \Phi(s)x(0) + \Phi(s)BU(s)$$
$$\Phi(s) = (sI - A)^{-1}$$

Review: Putting it all together

n^{th}-order or integro-differential equations
F=ma, KVL, KCL, etc.

→ 1. Choose state variables
2. Write as set of 1^{st} order diff. eq

State-space differential equations
dx/dt=Ax+Bu
y=Cx+Du
Char. eqn 0=|sI-A|

→ Laplace transform

Transfer function
G(s)=Y(s)/U(s)
Char. eqn 0=D(s)

Control- or observer-canonical form (Transfer function → State-space)

Y(s)/U(s)= C(sI-A)$^{-1}$B+D (State-space → Transfer function)

Today

Moving on:

- Feedback characteristics
  - Open-loop vs. closed-loop
  - Sensitivity and complementary sensitivity functions
  - Disturbance signals in feedback

Basic Idea of State Feedback

- Consider the state feedback controller where K is constant feedback gain matrix

$$\dot{x} = Ax + Bu, \quad u = -Kx$$

- Then one can write

$$\dot{x} = Ax + B(-Kx) \\ = (A - BK)x$$

- Whereas the poles of the open-loop system are given by the eigenvalues of A, the poles of the closed-loop system are given by the eigenvalues of ***(A-BK)***

State Feedback Requirements

- The poles of the closed-loop system can be arbitrarily assigned if and only if the system is controllable
- Not all states may be measurable.
- With observers, the state can be reconstructed from the output vector. The reconstructed state is used instead of the true state to generate the feedback signal.
- Observers can only be designed for systems which are *observable*.

Outline

- Open-loop control vs. Closed-loop control
- Sensitivity of Feedback
  - Sensitivity function
  - Complementary sensitivity function
- Disturbance propagation in the feedback loop

Desired characteristics of a closed-loop system:

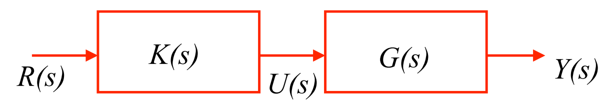
• Stability

• Insensitivity to variations in process parameters

• Disturbance rejection; insensitivity to noise

• Steady-state accuracy

Open-Loop Control

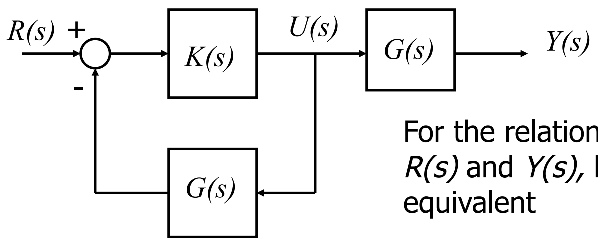


- Perfect servo control could be achieved if *Y(s)=R(s),* e.g. if

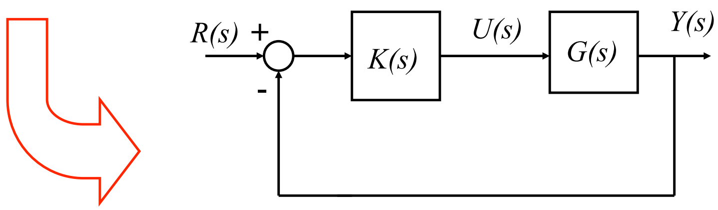
$$K(s)G(s) = 1, \text{ or } K(s) = \frac{1}{G(s)}$$

- Perfect control requires open-loop inversion of the plant (if the model is perfect and there are no disturbances).
- True inversion is never achieved in practice.
- Various ways to approximate it.

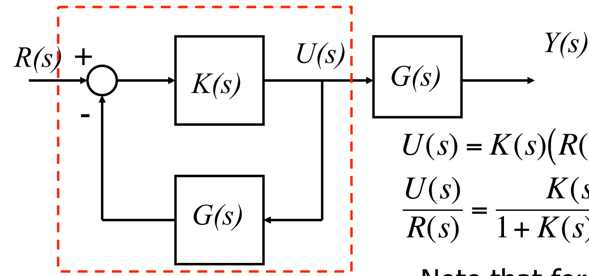
From Open Loop to Closed Loop



For the relationship between *R(s)* and *Y(s)*, both are equivalent



High Gain Feedback and Inversion



$$U(s) = K(s)\big(R(s) - G(s)U(s)\big)$$

$$\frac{U(s)}{R(s)} = \frac{K(s)}{1 + K(s)G(s)}$$

Note that for K(s)>>1

$$\frac{U(s)}{R(s)} \cong \frac{K(s)}{K(s)G(s)} \cong \frac{1}{G(s)}$$

High gain feedback implicitly generates the inverse of G(s) without having to actually carry out the inversion!

Why This is Not Practical

- Relies on very accurate model
- Requires the plant and its inverse to be stable
- Poor at rejecting disturbances

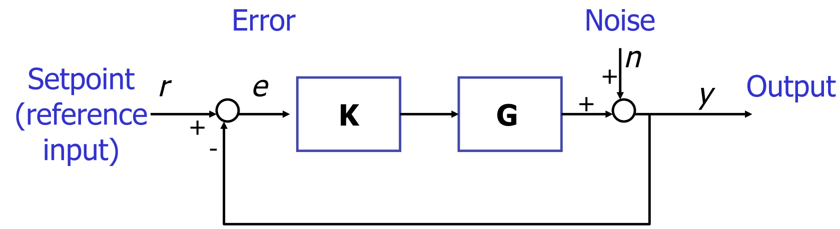
From Open Loop to Closed Loop

- In open loop, the controller has internal feedback
- In closed-loop, the feedback depends on what actually happens, since it is based on the output of the plant
- This will bring two benefits:
  - De-sensitized to modeling errors
  - De-sensitized to disturbances and noise

Trade-offs

- Although it seems that all is needed is high gain feedback, there is a cost attached to the use of high-gain feedback
  - Results in very large control actions
  - Increases the risk of instability
  - Increases the sensitivity to measurement noise
- High gain increases performance, but decreases robustness to noise
- **This is the essence of control design**

Sensitivity to Noise



$$Y(s) = N(s) + GKE(s)$$

$$Y(s) = N(s) + GK(R(s) - Y(s))$$

$$Y(s)(1+GK) = N(s) + GKR(s)$$

$$\boxed{Y(s) = \frac{1}{1+GK}N(s) + \frac{GK}{1+GK}R(s)}$$

Sensitivity to Noise

$$Y(s) = \frac{1}{1+GK}N(s) + \frac{GK}{1+GK}R(s)$$

- Sensitivity function $S = \frac{1}{1+GK}$
  - Effect of noise on the output
- Complementary sensitivity function $T = \frac{GK}{1+GK}$
  - Effect of reference input on the output
- Note that $S + T = \frac{1}{1+GK} + \frac{GK}{1+GK} = 1$

Sensitivity Function

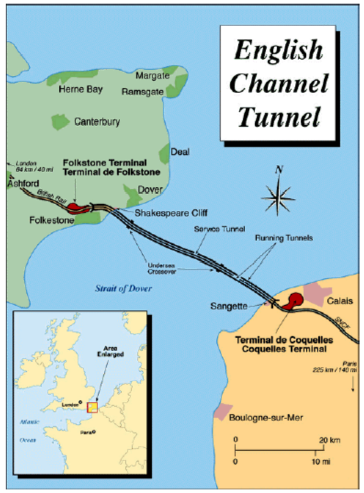
- S is a function of *s*. If we replace *s* by $j\omega$, we have sensitivity as a frequency response.
- Typically GK is large at low frequencies and small at high frequencies, hence

$$S(0) \approx 0 \text{ while } S(\infty) = 1$$

- This implies

$$T(0) \approx 1 \text{ while } T(\infty) = 0$$

Example: Eurotunnel



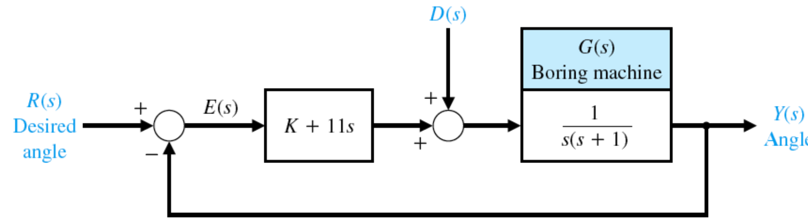
From France to Great Britain
1987 – 1992

23.5 miles long, bored 200 feet below sea level

Total cost $14 billion

Example: Eurotunnel

- Consider the following control system



- How can we select K to maintain performance while rejecting disturbances?

Example: Eurotunnel

- Output $Y(s) = T(s)R(s) + T_d(s)D(s)$

$$= \frac{K + 11s}{s^2 + 12s + K} R(s) + \frac{1}{s^2 + 12s + K} D(s)$$

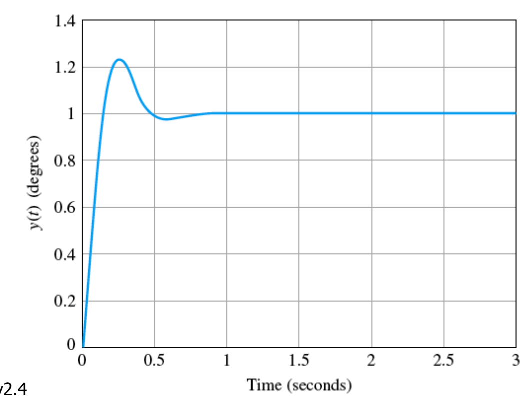
- Sensitivity function
  - Should be small for low freq., 1 for high freq.

$$S(s) = \frac{1}{s^2 + 12s + K}$$

  - High values of K can achieve this, but create performance issues

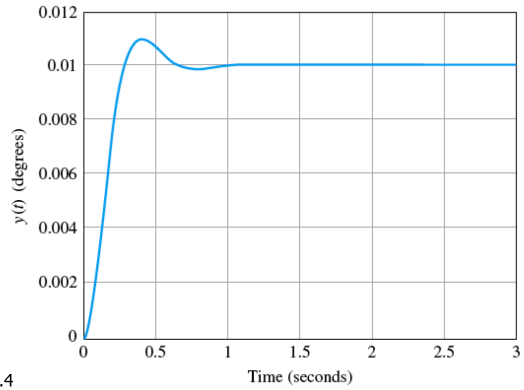
Example: Eurotunnel

- Output response to step input $r(t)$ shows significant 'overshoot'



Example: Eurotunnel

- Output response to step disturbance input *d(t)* is of minimal magnitude

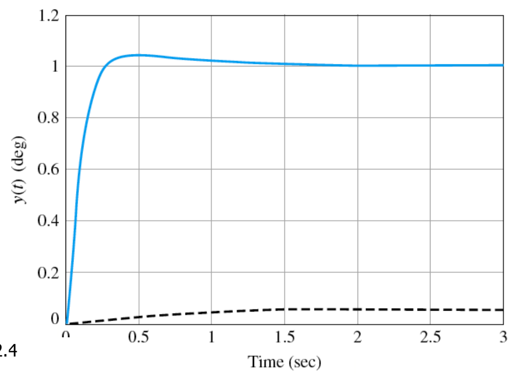


Example: Eurotunnel

- While *K=100* provides good disturbance rejection, performance is poor due to excessive overshoot.
- Reducing K will improve performance by reducing overshoot.
- Reducing K will also worsen disturbance rejection.
- Try *K=20* to achieve a response which is a compromise of the above goals.

Example: Eurotunnel

- Output response with controller *K=20*
  - to unit step input *r(t)* (blue, solid)
  - to unit step disturbance input *d(t)* (black, dotted)



Example: Eurotunnel

- Controller with *K=20* significantly improves short-term (transient) performance.
- This inevitably means that disturbance rejection worsens.
- An appropriate choice of controller will take into account other restrictions or goals (e.g. quantified short-term (transient) or long-term (steady-state) performance specifications).

Robustness: System sensitivity

- Sensitivity of the closed-loop transfer function *T* to variations in the process parameters
- We want to relate *dT/T* to *dG/G*
- Since $T = \frac{GK}{1+GK}$

$$\frac{dT}{dG} = \frac{K}{1+GK} - \frac{GK^2}{(1+GK)^2}$$

$$= \frac{K(1+GK) - GK^2}{(1+GK)^2}$$

$$= \frac{K}{(1+GK)^2}$$

Robustness: System sensitivity

$$dT = \frac{K}{(1+GK)^2} dG$$

$$= \frac{GK}{(1+GK)} \frac{1}{(1+GK)} \frac{dG}{G}$$

$$\frac{dT}{T} = \frac{1}{1+GK} \frac{dG}{G}$$

$$\frac{dT}{T} = S \frac{dG}{G}$$

Sensitivity : $S = \frac{\Delta T / T}{\Delta G / G}$

Evaluated in the limit as $\Delta G \to 0$:

$$S = \lim_{\Delta G \to 0} \frac{\Delta T / T}{\Delta G / G} = \frac{dT}{dG} \frac{G}{T}$$

Summary

- Inversion as essence of control
- Can be achieved through high gain feedback
- High gain increases performance but decreases robustness
- All control design involves a trade off between performance and robustness (S+T = 1)