EECE 360
Lecture 14

Feedback Characteristics

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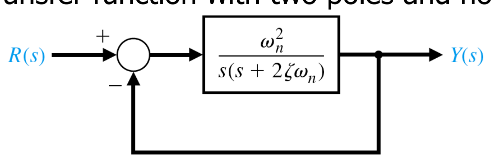
Chapter 5.4, 5.8

Outline

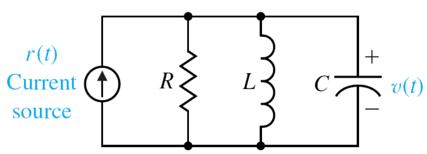
- Review
  - General form of second-order system
  - Natural frequency and critical damping
  - Generic input types
  - Approximations to second-order systems
- Second-order system feedback characteristics
  - Steady-state error
  - Type number

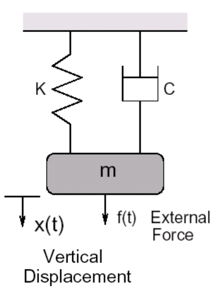
Review: Second-order systems

- Transfer function with two poles and no zeros



- Examples of 2nd order systems:

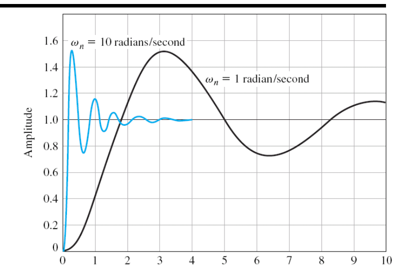


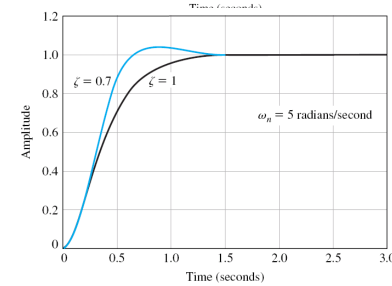


Review: Second-order systems

- Effect of ω_n
  - Frequency of oscillations
- Effect of ζ
  - Damping

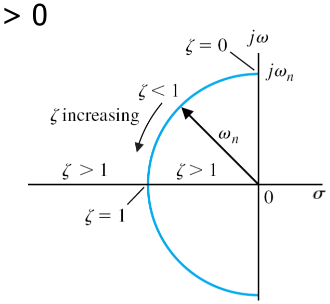
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



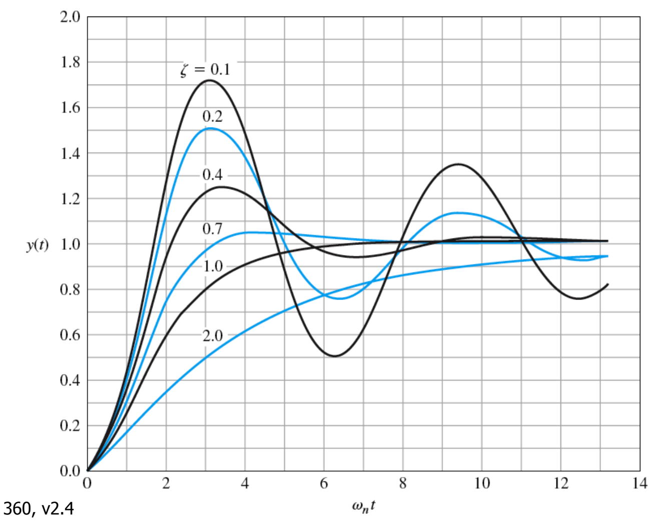


Review: Second-order systems

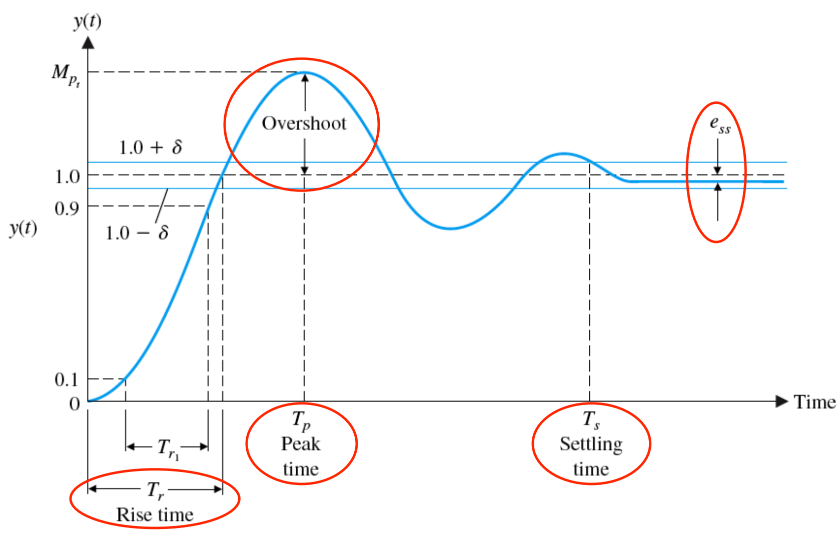
- Underdamped
  - Natural frequency $\omega_n > 0$
  - Damping ratio $1 > \zeta > 0$
- Critically damped
  - Damping ratio $\zeta = 1$
- Overdamped
  - Damping ratio $\zeta > 1$



Review: Second-order systems



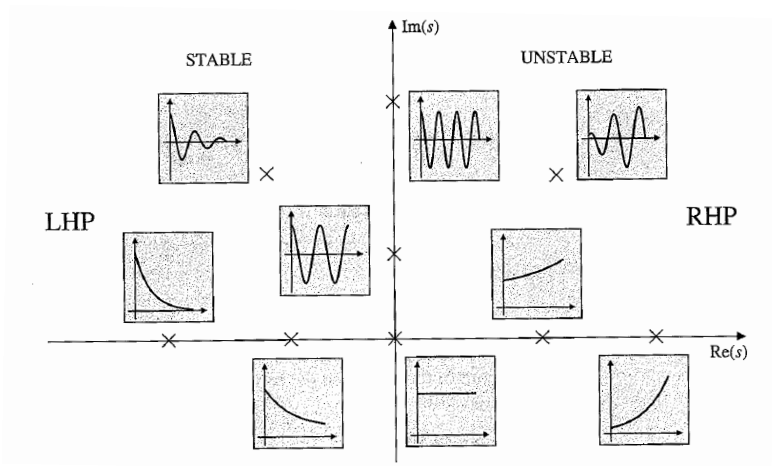
Review: Second-order systems



Review: Second-order systems

- Rise time $T_r \approx \dfrac{2.16\zeta + 0.60}{\omega_n}$
- Peak time $T_p = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}}$
- Overshoot $M_p = 1 + e^{-\zeta\pi / \sqrt{1-\zeta^2}}$
- Settling time $T_s \approx \dfrac{4}{\zeta\omega_n}$

s-Domain/Transient Response



Spring-Mass-Damper System

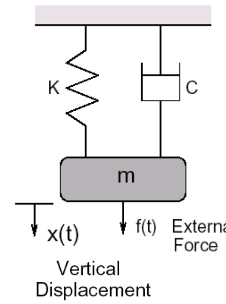
- Input *u=f(t)*, Output *y = x(t)*

$$u(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{a\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

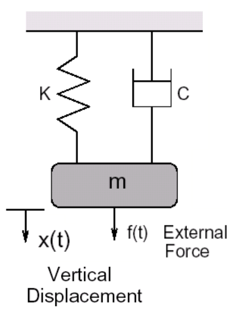


Spring-Mass-Damper System

- Input *u=f(t)*, Output *y = x(t)*

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{a\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

$$\omega_n = \sqrt{\tfrac{k}{m}}, \quad \zeta = \frac{b}{2\sqrt{mk}}, \quad a = m/k$$



- System response
  - Underdamped for $b < 2\sqrt{mk}$
  - Critically damped for $b = 2\sqrt{mk}$
  - Overdamped for $b > 2\sqrt{mk}$

Spring-Mass-Damper System

- Simulation platform ('Dynamic Systems Demonstrator' at

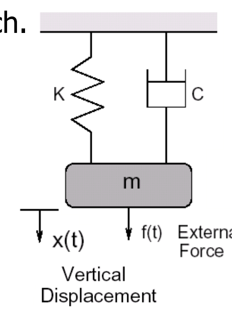
http://users.ece.gatech.edu/~bonnie/book1/applets/suspension/MSDdemo.htm

Developed by Prof. Heck, Georgia Tech.

- Exercise:

  Pick *b, k, m* in order to simulate:
  - 1) an underdamped response,
  - 2) a critically damped response, and
  - 3) an overdamped response

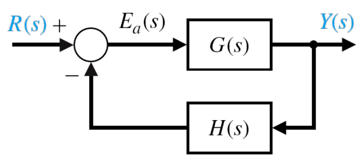
  with a step input.



Steady-State Error

- Reduction or elimination of steady-state error is a fundamental reason for using feedback
- With closed-loop transfer function

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$



the error

$$E(s) = R(s) - Y(s)$$
$$= \frac{1}{1 + G(s)H(s)} R(s)$$

has steady-state error (**with *H(s)*=1**)

$$\lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

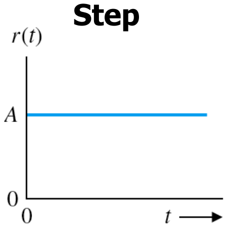
Steady-State Error

- Steady-state error is another performance metric (e.g. settling time, overshoot, etc.)
- We can classify systems according to their ability to track specific **input types**.
- This classification is known as a **type number**. (e.g. a system of type number 0, 1, 2, etc.…)
- The type number indicates the number of integrators a system has in *G(s).*

Question: How many integrators must *G(s)* contain to be able to exactly track a step? A ramp? A parabola?

Test Input Signals

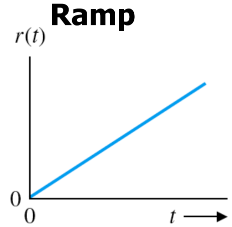
Step



$$r(t) = \begin{cases} A & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(s) = \frac{A}{s}$$

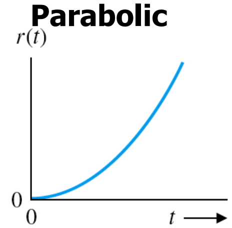
Ramp



$$r(t) = \begin{cases} At & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(s) = \frac{A}{s^2}$$

Parabolic



$$r(t) = \begin{cases} \frac{1}{2}At^2 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(s) = \frac{A}{s^3}$$

- "Base-case" used to evaluate system response.

Type Number

- The number of integrators in *G(s)* indicates its **type number**
- Step input, R(s) = 1/s:

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \cdot (1/s)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{1 + G(s)}$$

- Ramp input, R(s) = 1/s^2

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \cdot (1/s^2)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \lim_{s \to 0} \frac{1}{sG(s)}$$

- Parabolic input, R(s) = 1/s^3

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \cdot (1/s^3)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2G(s)} = \lim_{s \to 0} \frac{1}{s^2G(s)}$$

Type Number

Define the constants

$$K_s = \lim_{s\to 0} G(s), \quad K_v = \lim_{s\to 0} sG(s), \quad K_a = \lim_{s\to 0} s^2 G(s)$$

Then the steady-state error for

- Step input, R(s) = 1/s:

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+K_s}$$

- Ramp input, R(s) = $1/s^2$

$$e_{ss} = \lim_{s\to 0}\frac{1}{sG(s)} = \frac{1}{K_v}$$

- Parabolic input, R(s) = $1/s^3$

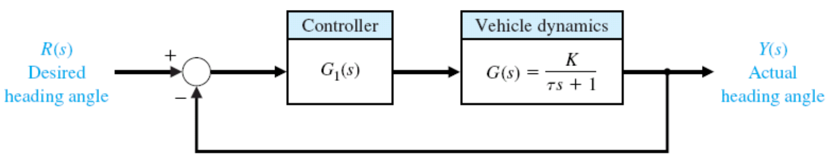
$$e_{ss} = \lim_{s\to 0}\frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

Type Number

$$K_s = \lim_{s\to 0} G(s), \quad K_v = \lim_{s\to 0} sG(s), \quad K_a = \lim_{s\to 0} s^2 G(s)$$

Type Number	Unit Step	Unit Ramp	Unit Parabola
0	$e_{ss} = \frac{1}{1+K_p}$	Infinite	Infinite
1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_v}$	Infinite
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_a}$

Mobile Steering Robot



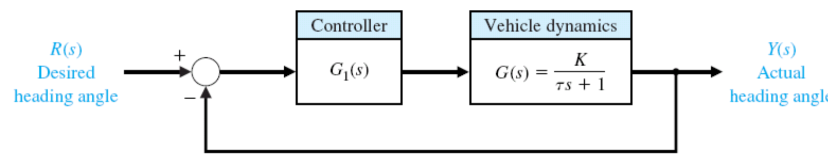
- Case 1: $G_1(s)=K_1$
  - Open-loop transfer function $G_1(s)G(s)$ is type 0.

$$G_1(s)G(s) = \frac{K_1 K}{\tau s + 1}, \quad K_s = \lim_{s\to 0} G_1(s)G(s) = K_1 K$$

  - Therefore with a step input, the closed-loop system will have steady-state error

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+K_s} = \frac{1}{1+K_1 K}$$

Mobile Steering Robot



- Case 1: $G_1(s)=K_2/s$
  - Open-loop transfer function $G_1(s)G(s)$ is type 1.

$$G_1(s)G(s) = \frac{K_2 K}{s(\tau s + 1)}, \quad K_v = \lim_{s\to 0} sG_1(s)G(s) = K_2 K$$

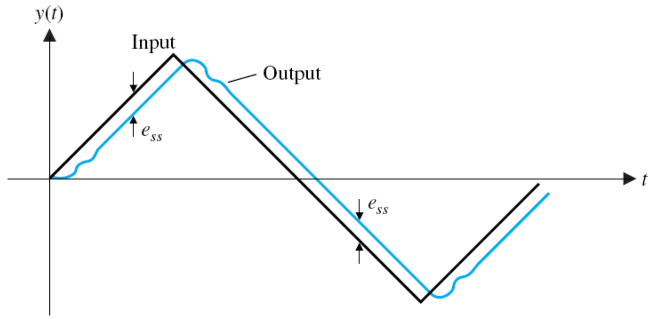
  - Therefore with a step input, the closed-loop system will have steady-state error

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G_1(s)G(s)} = \lim_{s\to 0}\frac{s(\tau s+1)}{s(\tau s+1)+K_2 K} = 0$$

Mobile Steering Robot

- With a ramp input, the steady-state error is

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G_1(s)G(s)} = \lim_{s\to 0}\frac{1}{K_v} = \frac{1}{K_2 K}$$

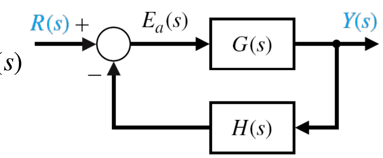


Error for Non-Unity Feedback

- Now consider the case when H(s)≠1
- With error defined as

$$E(s) = R(s) - Y(s)$$

$$= \frac{1+G(s)H(s)-G(s)}{1+G(s)H(s)}R(s)$$



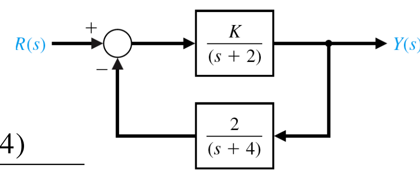
the steady-state error is

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1+G(s)H(s)-G(s)}{1+G(s)H(s)}sR(s)$$

- To find the steady-state error for specific input types, use the same derivation as with unity feedback, but with a different transfer function.

Example 5.5

- The error is



$$E(s) = R(s) - Y(s)$$

$$= 1 - \frac{K(s+4)}{(s+2)(s+4)+2K}$$

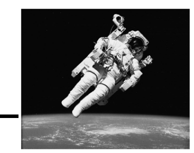
$$= \frac{(s+2)(s+4-K)}{s^2+6s+8+2K}$$

- Therefore with a step input, the closed-loop system will have steady-state error

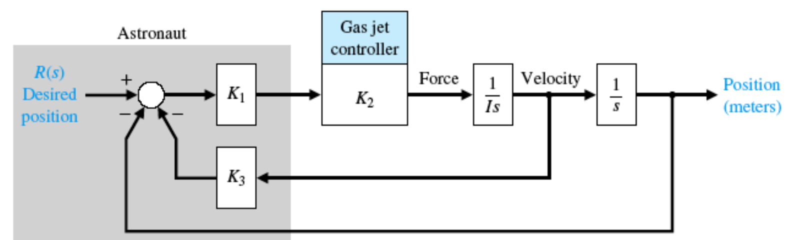
$$e_{ss} = \lim_{s\to 0} sE(s)\cdot(1/s) = \frac{8-2K}{8+2K} = \frac{4-K}{4+K}$$

- What is the steady-state error with a ramp input?

Example: Astronaut



- Consider an astronaut in space. The astronaut controls position with gas jet propulsion.
- What values of K_1, K_2, K_3 will provide an acceptable system response? The settling time should be 0.3 seconds, the overshoot less than 5%, and steady-state error to a ramp input less than 0.5 m.



Summary

- Second-order system feedback characteristics
  - Natural frequency, damping ratio
  - Relating poles (s-domain) to transient response
  - Transient response requirements (settling time, overshoot, rise time, peak time)
  - Steady-state error
  - Type number
  - Potential conflicts in designing controllers for multiple transient/steady-state requirements

Next week

- Stability
  - BIBO stability
  - Asymptotic (exponential) stability
  - Conditional stability
  - Relative stability
- Root locus
  - Routh-Hurwitz criterion
  - Root-locus plots
  - Controller design through root-locus