EECE 360
Lecture 16

Stability and Routh-Hurwitz Criterion

Dr. Oishi

Electrical and Computer Engineering

University of British Columbia

http://courses.ece.ubc.ca/360

eece360.ubc@gmail.com

Chapter 6.1 - 6.5

Review: General Concept

- Two different notions of stability
  - Ability of the system to return to equilibrium after an arbitrary displacement away from the equilibrium **(internal stability)**
  - Ability of the system to produce a bounded output for any bounded input **(BIBO stability)**
- For linear systems, these two notions are closely related.

Review: BIBO Stability**

- Assuming there is no pole-zero cancellation in the transfer function, and $\lambda_i,\ i \in \{1,\cdots,n\}$ are roots of the n^{th} order characteristic equation:

Stability	Pole location	Description
BIBO stable	A λ_i, Re(λ_i)<0	All poles in open LHP
Marginally stable	∃λ_i, Re(λ_i)=0, $\lambda_i \neq \lambda_j$ ∧ ~∃λ_k, Re(λ_k)>0	Any simple poles on imaginary axis, and no poles in RHP
Unstable	∃λ_i, Re(λ_i)=0, $\lambda_i = \lambda_j$	Any repeated poles on imaginary axis, or
	∃λ_i, Re(λ_i)>0	Any poles in RHP

Internal vs. BIBO stability

- Internal stability implies BIBO stability
- Internal stability is stronger in some sense, because BIBO stability can "hide" unstable behaviors which don't appear in the output

  Consider the transfer function $G(s) = \dfrac{s-1}{(s-1)(s+3)}$
  - Zero at s=+1 cancels unstable pole
  - But is this really BIBO stable?
- With no pole-zero cancellation, same conditions exist for internal stability as for BIBO stability.

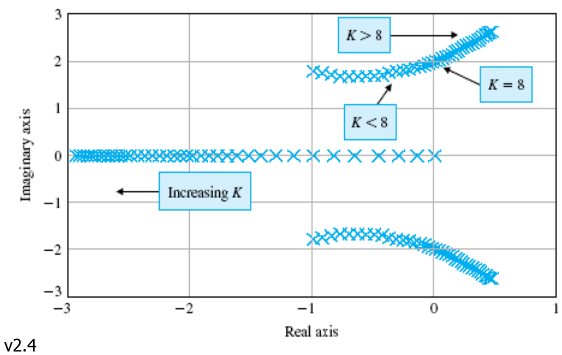
Review: Feedback & Stability**

- Feedback often improves stability.
- However, increasing the gain past a certain threshold can destabilize a system.
- **This threshold occurs when at least one root of the characteristic equation has real part equal to 0.**
- Increasing the gain can push poles from LHP to the RHP.

Review: Feedback & Stability

- For what values of K will the system with the following characteristic equation be stable?

$$0 = s^3 + 2s^2 + 4s + K$$



Today

- Review:
  - BIBO stability (all poles with negative real part)
  - Marginal stability (no repeated poles on the imaginary axis)
- Today
  - Routh-Hurwitz stability criterion
  - Introduction to Root Locus

Routh-Hurwitz Criterion

- Consider the polynomial characteristic equation

$$Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- Routh-Hurwitz stability criterion is a test to check for stability without computing the roots of characteristic equation. The test checks whether or not all roots of a polynomial have negative real part.
- It is presented here without proof. More details are in Dorf, Chapter 6.2.

Routh-Hurwitz Criterion

- Necessary and sufficient conditions for low-order systems:
  - First-order: All roots of $Q(s)=a_1s+a_0$ are in the LHP if **all coefficients are positive**.
  - Second-order: All roots of $Q(s)=a_2s^2+a_1s+a_0$ are in the LHP if **all coefficients are positive**.
  - Third-order: All roots of $Q(s)= a_3s^3+ a_2s^2+a_1s+a_0$ are in the LHP if **all coefficients are positive** and $a_1a_2-a_0a_3>0$.
- Positive coefficients for n^{th} order polynomials are **necessary but not sufficient** conditions for stability.

Routh-Hurwitz Table

- Create a table based on the coefficients of the characteristic equation
- The first two rows are taken directly from Q(s)
- The remaining rows are computed from these two rows

s^n	1	a_{n-2}	a_{n-4}	☒
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	☒
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	
⋮				
s^0	h_{n-1}			

Routh-Hurwitz Table

- The number of roots of *Q(s)* with **positive real part** is equal to the **number of sign changes in the first column** of the Routh-Hurwitz table.

$Q(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0$

s^n	1	a_{n-2}	a_{n-4}	☒
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	☒
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	
⋮				
s^0	h_{n-1}			

Routh-Hurwitz Criterion

s^4	$a_4 = 1$	a_2	a_0	0
s^3	a_3	a_1	0	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$	
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$		
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$			

One row is calculated from the two rows directly above it.

The number of roots in the right half plane equals the number of sign changes in the first column.

Example 1

$$\phi(s) = s^3 + 2\rho s^2 + s, \quad \rho \neq 0$$

s^3	1	1	0
s^2	2ρ	0	0
s^1	1	0	
s^0	0		

$\Rightarrow$ for $\rho > 0$, there are 0 roots in RHP

for $\rho < 0$, there are 2 roots in RHP

Example 2: Robotic Arm

$$G_{cl} = \frac{20K(s+0.1)}{s^5 + 15s^4 + 74.25s^3 + 121s^2 + 20Ks + 2K}$$

s^5	1	74	$20K$
s^4	15	121	$2K$
s^3	65.9	$19.86K$	
s^2	$121-4.52K$	$2K$	
s^1	$\frac{2271K-89.76K^2}{121-4.52K}$		
s^0	$2K$		

$121 - 4.52K > 0,\ 2271K - 89.76K^2 > 0,\ K > 0$

$$K < \frac{121}{4.52} = 26.769$$

$$K < \frac{2271}{89.76} = 25.308$$

Root Locus

- Performance of a control system is described in terms of the location of the roots of the characteristic equation in the *s*-plane.
- A desired response of a closed-loop control system can be achieved by adjusting one or more system parameters (control gains).
- Root locus is a method for analysis and design of control system
- The root locus plot is a graph of the locus of roots as one system parameter is varied

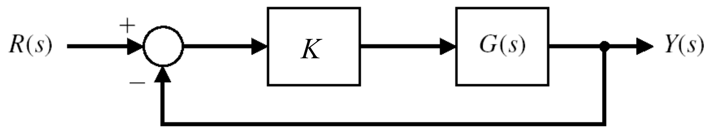
Root Locus

- Developed by W. Evans while a graduate student at UCLA
- Use the poles and zeros of the open-loop system to determine the closed-loop poles when ***one*** parameter is changing

Walter R. Evans, 1920-1999

Root Locus Method

- Consider the unit feedback system with a scalar control gain K



$$1 + KG(s) = 0$$

- The root locus is the path of the roots of the characteristic equation in the s-plane as the gain is varied (from 0 to infinity)

Root Locus Method

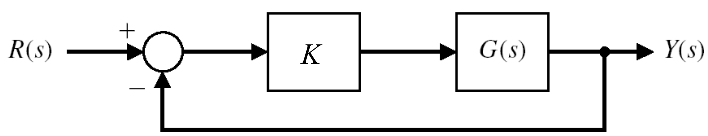
- Consider the unity feedback system with
  G(s) = 1/(s(s+2))
- The characteristic equation is

$$0 = 1 + KG(s) = 1 + K\frac{1}{s(s+2)}$$
$$= s^2 + 2s + K$$

- Start by examining K=0: The poles are s = 0, -2.
- For 0<K<1, the system is overdamped with poles at s = $-1 \pm \sqrt{1-K}$
- For K=1, the system is critically damped with poles at s = -1, -1.
- For K>1, the system is underdamped, with poles at s= $-1 \pm j\sqrt{K-1}$

Root Locus Method

- Consider the unit feedback system with a scalar control gain K



$$1 + KG(s) = 0$$

- The root locus **originates at the poles of G(s)** and **terminates on the zeros of G(s).**

Root Locus Method

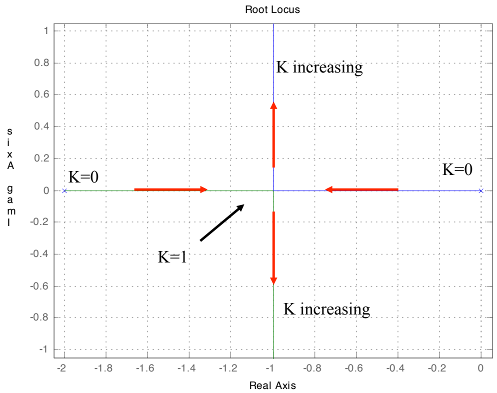
$$1 + KG(s) = 0$$
$$1 + K\frac{N(s)}{D(s)} = 0$$
$$D(s) + KN(s) = 0$$

When $K = 0$, this collapses to $D(s) = 0$.
Since the roots of $D(s) = 0$ are the poles of $G(s)$, those are the closed-loop poles for $K = 0$.

When K is large, $D(s) + KN(s) = \frac{1}{K} + \frac{N(s)}{D(s)} = 0$ tends to $\frac{N(s)}{D(s)} = 0$
thus the closed-loop poles tend to the roots of $N(s) = 0$, i.e. the open-loop zeros, and also to infinity if $\frac{N}{D}$ is strictly proper.

Example 3: Root Locus

$$0 = 1 + KG(s) = 1 + K\frac{1}{s(s+2)}$$
$$= s^2 + 2s + K$$



Example 4

- Consider the system with transfer function
$$G(s) = \frac{s+1}{s^2+s+1}$$
- The characteristic equation is:
- Values of K for KG/(1+KG) to have real roots:
- Values of K for KG/(1+KG) to have imaginary roots:

Example 4

Suppose

$$1 + KG(s) = 1 + K\frac{s+1}{s^2+s+1} = 0$$

$$s^2 + s + 1 + Ks + K = 0 \qquad \text{(Char. Equation)}$$

$$s^2 + (K+1)s + (K+1) = 0$$

$$\Delta = (K+1)^2 - 4(K+1) = K^2 - 2K - 3$$

$$\text{For } \Delta > 0,\ s^*(K) = -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{K^2 - 2K - 3}$$

$$\text{For } \Delta < 0,\ s^*(K) = -\frac{1}{2}(K+1) \pm j\frac{1}{2}\sqrt{|K^2 - 2K - 3|}$$

(this occurs for $-1 < K < 3$) (roots)

Example 4

$$\text{For } K = 0,\ s^*(0) = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$ ← Poles of $G(s)$

$$\text{For } K = 3,\ s^*(3) = -\frac{1}{2}(3+1) = -2$$

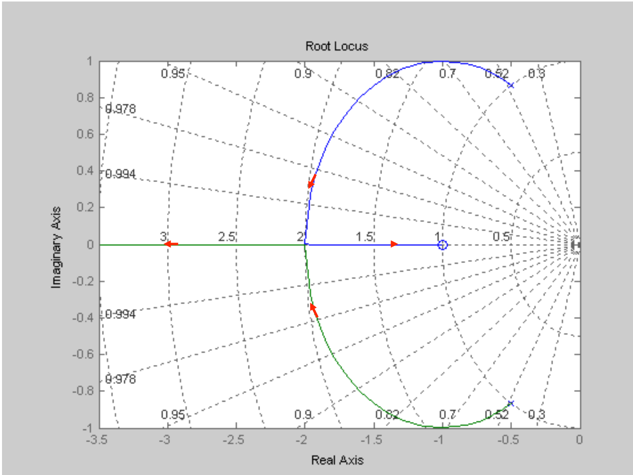
For large K,

$$s^*(K) = -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{K^2 - 2K + 1 - 4}$$
$$= -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{(K-1)^2 - 4}$$
$$\approx -\frac{1}{2}(K+1) \pm \frac{1}{2}(K-1) = -1, -K$$

Zero of $G(s)$

Tends to $-\infty$

Example 4: Root Locus



Summary

- Feedback and stability
- Routh-Hurwitz stability criterion
  - Check for stability without computing roots of characteristic equation
- Root Locus
  - Poles and zeros of the open-loop system can determine the closed-loop poles as gain K increases from 0 to infinity
  - Starts at poles of open-loop system, ends at zeros of open-loop system, or at infinity