EECE 360
Lecture 17

Root Locus

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Chapter 7.1 – 7.2

Today's class

- Review
  - Role of feedback on stability
  - Routh-Hurwitz criterion
- Root locus
  - Introduction
  - Gain and phase criterion
  - Steps to sketch a root locus

Review: Proving stability

- Goal: Show that all poles have negative real part. (Recall def'n of stability)
- Several ways to do this:
  - Compute roots of characteristic equation
  - Use Routh-Hurwitz Criterion (when computing the roots is prohibitively complex)
  - Evaluate how poles move in complex plane when one parameter (usually gain K) is varied from 0 to infinity.

Review: Routh-Hurwitz criteria

- A stable linear system requires that all poles of the transfer function (roots of the characteristic equation) have negative real part.
- Routh-Hurwitz stability criterion is a test to ascertain without computing the roots, whether or not all roots of a polynomial have negative real part.

Review: Routh-Hurwitz criteria

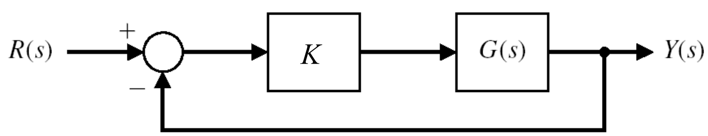
- Necessary and sufficient conditions for low-order systems:
  - First-order: All roots of $Q(s)=a_1s+a_0$ are in the LHP if **all coefficients are positive**.
  - Second-order: All roots of $Q(s)=a_2s^2+a_1s+a_0$ are in the LHP if **all coefficients are positive**.
  - Third-order: All roots of $Q(s)= a_3s^3+ a_2s^2+a_1s+a_0$ are in the LHP if **all coefficients are positive** and $a_1a_2-a_0a_3>0$.
- Positive coefficients for n^{th} order polynomials are **necessary but not sufficient** conditions for stability.

Root Locus

- Performance of a control system is described in terms of the location of the roots of the characteristic equation in the s-plane.
- A desired response of a closed-loop control system can be achieved by adjusting one or more system parameters (control gains).
- Root locus is a method for analysis and design of control system
- The root locus plot is a graph of the locus of roots as one system parameter is varied

Root Locus Method

- Consider the unit feedback system with a scalar control gain K



- Poles of the **closed loop system** solve

$$1+KG(s)=0$$

- The root locus originates at the poles of G(s) and terminates on the zeros of G(s).

Example 1

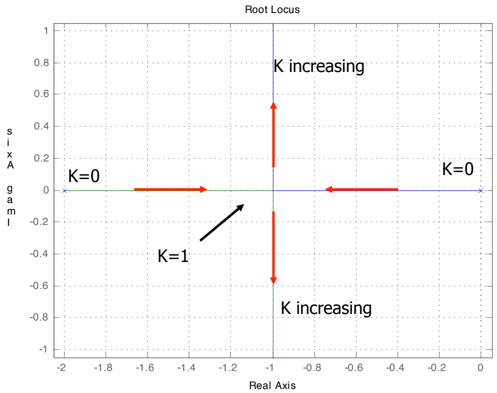
- Consider the unity feedback system with G(s) = 1/(s (s+2))
- The characteristic equation is

$$0=1+KG(s)=1+K\frac{1}{s(s+2)}$$
$$=s^2+2s+K$$

- Start by examining K=0: The poles are s = 0, -2.
- For 0<K<1, the system is overdamped with poles at s = $-1\pm\sqrt{1-K}$
- For K=1, the system is critically damped with poles at s = -1, -1.
- For K>1, the system is underdamped, with poles at s= $-1\pm j\sqrt{K-1}$

Example 1: Root Locus

$$0 = 1 + KG(s) = 1 + K\frac{1}{s(s+2)}$$
$$= s^2 + 2s + K$$



Root Locus Method

$$1 + KG(s) = 0$$
$$1 + K\frac{N(s)}{D(s)} = 0$$
$$D(s) + KN(s) = 0$$

The root locus originates on the poles of G(s)H(s) (for K=0) and terminates on the zeros of G(s)H(s), including those at infinity.

When $K = 0$, this collapses to $D(s) = 0$.
Since the roots of $D(s) = 0$ are the poles of $G(s)$, those are the closed-loop poles for $K = 0$.

When K is large, $D(s) + KN(s) = \frac{1}{K} + \frac{N(s)}{D(s)} = 0$ tends to $\frac{N(s)}{D(s)} = 0$
thus the closed-loop poles tend to the roots of $N(s) = 0$, i.e. the open-loop zeros, and also to infinity if $\frac{N}{D}$ is strictly proper.

Example 2

- Consider the system with transfer function
$$G(s) = \frac{s+1}{s^2+s+1}$$
- The characteristic equation is:
- Values of K for KG/(1+KG) to have real roots:
- Values of K for KG/(1+KG) to have imaginary roots:

Example 2

Suppose

$$1 + KG(s) = 1 + K\frac{s+1}{s^2+s+1} = 0$$

$$s^2 + s + 1 + Ks + K = 0$$ (Char. Equation)

$$s^2 + (K+1)s + (K+1) = 0$$

$$\Delta = (K+1)^2 - 4(K+1) = K^2 - 2K - 3$$

For $\Delta > 0$, $s^*(K) = -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{K^2 - 2K - 3}$

For $\Delta < 0$, $s^*(K) = -\frac{1}{2}(K+1) \pm j\frac{1}{2}\sqrt{|K^2 - 2K - 3|}$

(this occurs for $-1 < K < 3$) (roots)

Example 2

For $K=0$, $s^*(0) = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ ← Poles of $G(s)$

For $K=3$, $s^*(3) = -\frac{1}{2}(3+1) = -2$

For large K,

$$s^*(K) = -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{K^2 - 2K + 1 - 4}$$

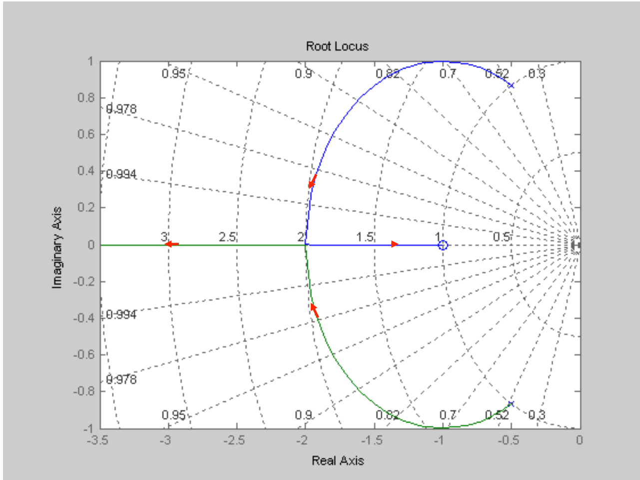
$$= -\frac{1}{2}(K+1) \pm \frac{1}{2}\sqrt{(K-1)^2 - 4}$$

$$\approx -\frac{1}{2}(K+1) \pm \frac{1}{2}(K-1) = -1, -K$$

Zero of $G(s)$ (→ -1)

Tends to $-\infty$ (→ $-K$)

Example 2: Root Locus



Sketching Root Locus

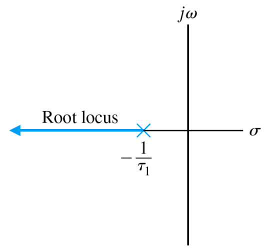
- There is an easier way to find points on the root locus
- A simple sketch can be made, just based on the **open-loop transfer function G(s)**
- Important features on the root locus:
  - Where the locus crosses the imaginary axis
  - Where the locus is centered
  - Where the locus breaks away from the real axis
  - Which asymptotes the loci follow as K --> infinity.
- *These features can be computed from the zeros and poles of G(s)*

Plotting Root Locus

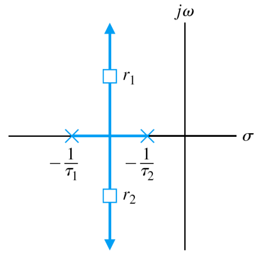
- Control engineers should be able to sketch root locus in a 'back of the envelope' way in practice
- Tools in Matlab for precise root locus plots
  - 'rlocus'
  - 'pzmap'
  - 'damp'
- **Goal: Relate location of the poles in the complex plane as a function of a single parameter (control gain) K.**

Typical Root Locus Plots

- First-order systems with no zeros



- Second-order systems with no zeros



Typical Root Locus Plots

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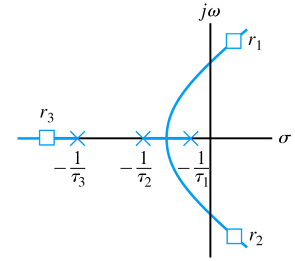
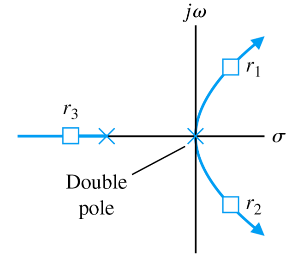
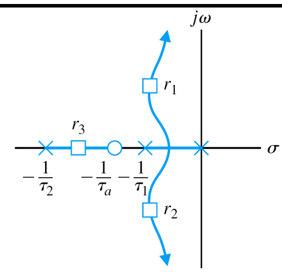


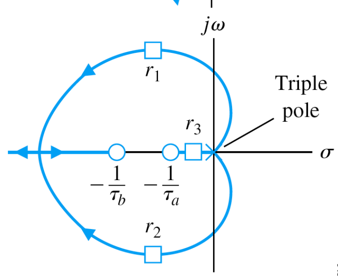


Typical Root Locus Plots

- Third-order system with one zeros

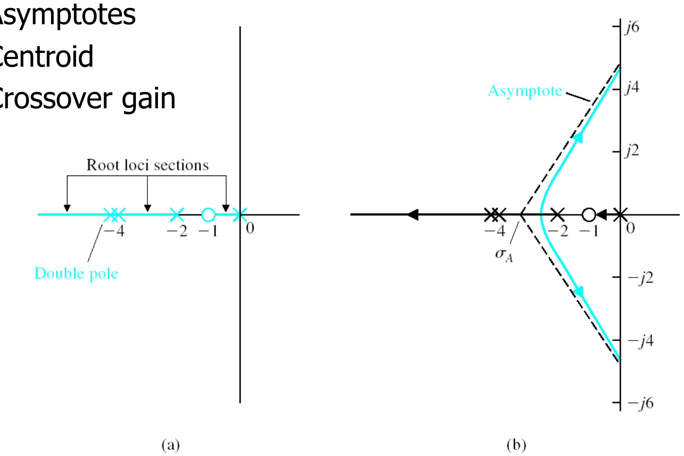


- Third-order system with two zeros



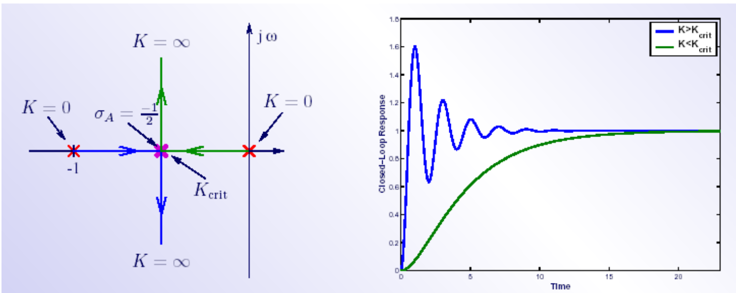
Important Root Locus features

- Asymptotes
- Centroid
- Crossover gain



Important Root Locus features

- Breakaway point



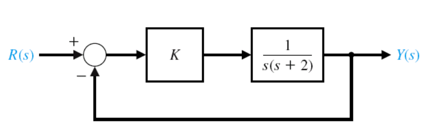
- Each breakaway points is a point where a double (or higher order) root exists for some value of K.

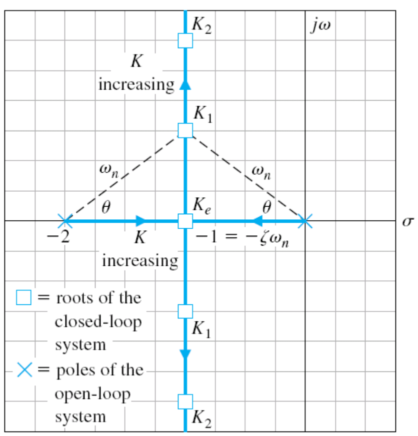
Finding Points on the Locus**

- A location s* is on the locus if
  $1 + K\,G(s^*) = 0$
  which is equivalent to
  $G(s^*) = -1/K$
- Recall that *s* is a complex number (therefore it has a *magnitude* and a *phase*) and assume K > 0

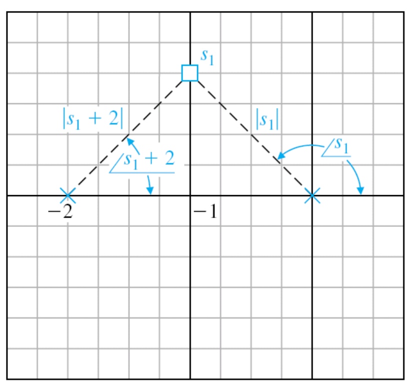
- Phase condition
  $\text{angle}[G(s^*)]=180^\circ \pm 360^\circ n$
  determines which points are on the locus
- The magnitude condition
  $|G(s^*)|=1/K$
  determines the value of K at s*

Root Locus Concept





Root locus for a 2nd-order system



Evaluation of angle and gain

The 7 Steps to the Root Locus

- Procedure to facilitate rapid sketching of the root locus
- Locate roots of the characteristic equation in a graphical manner in the *s*-plane
- Roots exist on the locus for various values of the single parameter, K.

Steps to Root Locus

- Step 1: Prepare the root locus sketch
  - Step 1.1: Find the char. equation 1+KG(s)=0
  - Step 1.2: Find the *m* zeros z_i and *n* poles p_i of G(s)

$$G(s^*) = \frac{k(s^*+z_1)(s^*+z_2)...(s^*+z_m)}{(s^*+p_1)(s^*+p_2)...(s^*+p_n)}$$

  - Step 1.3: Draw the poles and zeros on the s-plane
  - Step 1.4: Identify number of loci (=*n*)
  - Step 1.5: Exploit symmetry across the real axis

Steps to Root Locus

- Step 2: Locate loci segments on the *real axis*.
  - Locus lie in sections of the real axis **left of an odd number of poles and zeros**
- Step 3: Find asymptotes
  - Total of *(n-m)* asymptotes
  - **Angle** of asymptotes is

$$\phi_A = \frac{2q+1}{n-m}\cdot 180, \quad q = 0,1,...,(n-m-1)$$

  - **Center** (intersection) of asymptotes is

$$\sigma_A = \frac{\sum(-p_i) - \sum(-z_i)}{n-m}$$

Steps to Root Locus

- Step 4: Determine the points at which the root locus crosses the imaginary axis
  - Evaluate using the Routh-Hurwitz criterion
- Step 5: Find **breakaway points**
  - Loci converge or diverge on the locus at

$$\frac{dp(s)}{ds} = 0, \quad p(s) = -\frac{1}{G(s)} \text{ (unity feedback)}$$

  - Loci approach/diverge at angles **spaced equally** about the breakaway point (and with symmetry about the real axis).

Steps to Root Locus

- Step 6: Determine the angle of departure (from poles) and the angle of arrival (at zeros) using the **phase criterion** (e.g.,

$$180 = \sum_k \theta_{z_k} - \sum_k \theta_{p_k}$$

$$-\theta_{p_i} = 180 + \sum_{j \neq i} \theta_{p_j} - \sum_k \theta_{z_k}$$

$$\theta_{z_i} = 180 + \sum_k \theta_{p_k} - \sum_{j \neq i} \theta_{z_j}$$

- Step 7: Complete the root locus sketch.

Summary

- Root locus shows evolution of closed-loop poles as **one** parameter changes.
- A simple set of rules allow the loci to be sketched.
- If details are needed, use Matlab to plot the root locus.
- To develop insight should be able to determine the main features of root locus without a computer.