EECE 360
Lecture 25

Nyquist Stability Criterion

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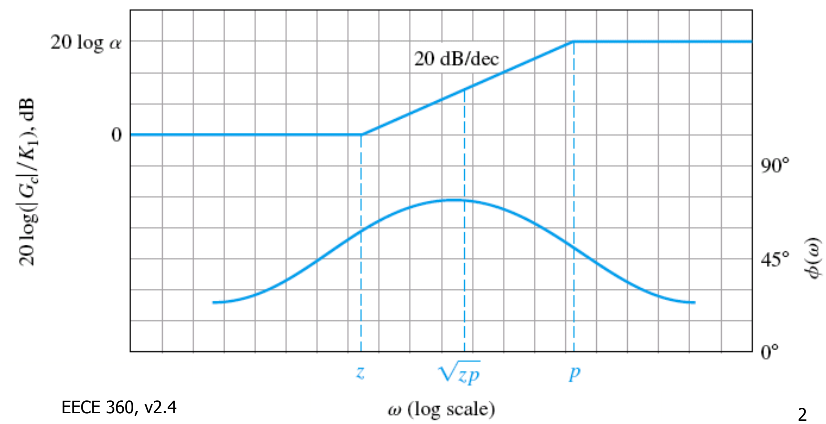
Chapters 9.2-9.4

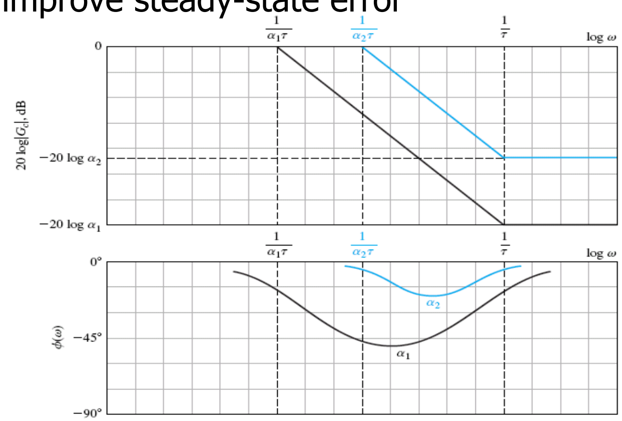
Review: Phase-Lead Controller

- Lead controllers **add** phase angle



Review: Phase-Lag Controller

- Lag controllers provide attenuation and improve steady-state error



Today's Lecture

- Review: Control design through Bode diagrams
  - Lead control design
  - Lag control design
- Today
  - Cauchy's theorem
  - Nyquist criterion

Harold Nyquist

- Born in 1889 in Sweden
- Died in 1976, USA
- Yale PhD, 1917
- Career at Bell Labs
- 138 patents
- Nyquist diagram, criterion, sampling theorem
- Laid the foundation for information theory, data transmission and negative feedback theory

The Nyquist Diagram

- *Polar plot* of the magnitude and phase of the open-loop system.
- Easily obtained from Bode diagrams of $G_c(s)G(s)$.
- Alternative way to analyze stability of a closed-loop system, based on analysis of the open-loop system.
- Procedure: Evaluate *Nyquist plot* (or *diagram)* according to **Nyquist criterion**
- Theory for Nyquist criterion based on Cauchy's theorem.

Mapping Contours in the s-Plane

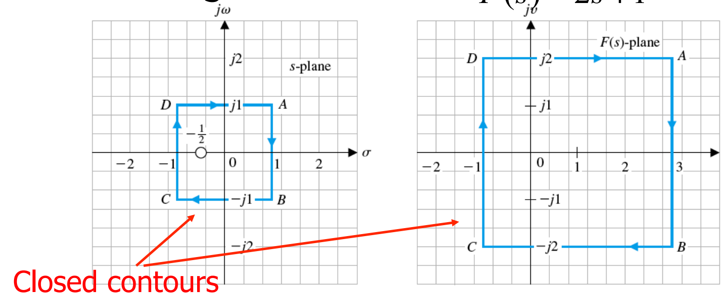
- Nyquist criterion based on Cauchy theorem on functions of a complex variable
- Mapping contours in the s-plane

Characteristic equation:

$$1 + L(s) = 0$$

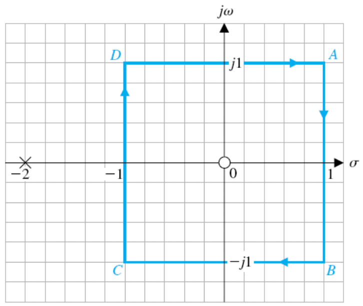
Mapping Contours in the s-Plane

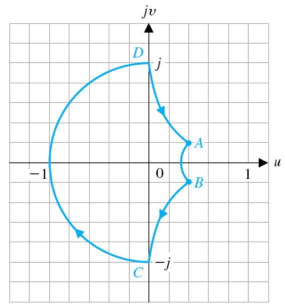
S $\qquad F(s) = 2s + 1$



- Use F(s) to map values of s, evaluated along a specific closed contour in the complex plane, to another closed contour in the complex plane.

Mapping Contours in the s-Plane

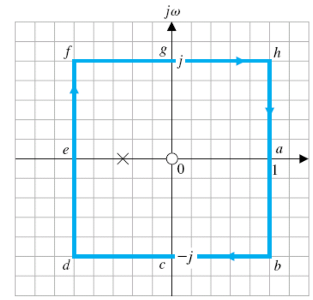


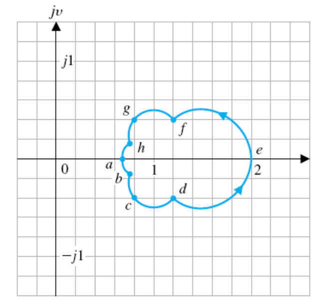


s

$$F(s) = \frac{s}{s+2}$$

Mapping Contours in the s-plane





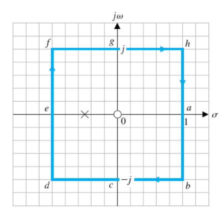
s

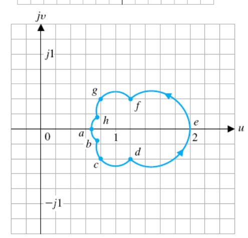
$$F(s) = s/(s+1/2)$$

Cauchy's Theorem

"Principle of the Argument"

- If a contour Γ_s in the s-plane
  - encircles Z zeros of F(s) and P poles of F(s)
  - does not pass through any poles of zeros of F(s)
  - is traversed clockwise
- Then the corresponding contour Γ_F in the F(s)-plane encircles the origin of the F (s)-plane N=Z-P times in the clockwise direction.

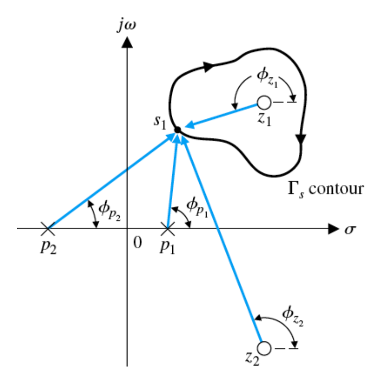


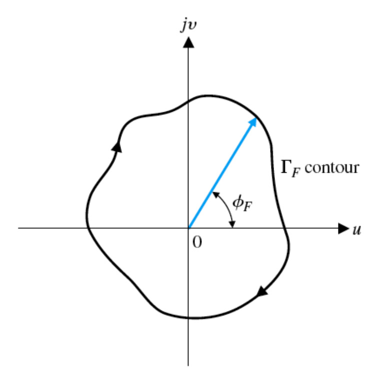


$$F(s) = s/(s+1/2)$$

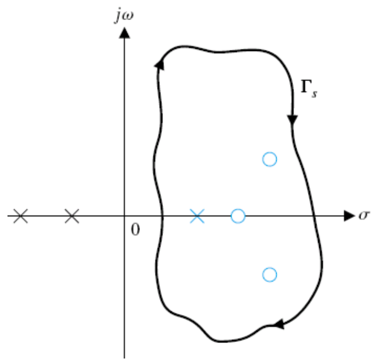
Cauchy's Theorem

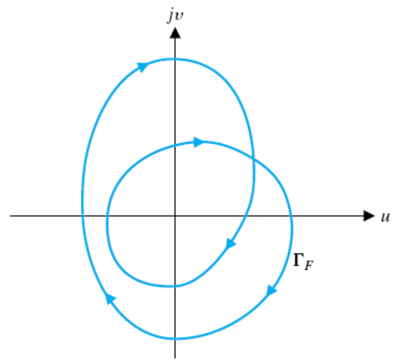
- An encirclement of 0 arises when a zero or pole of F(s) lies within the contour





Cauchy's Theorem

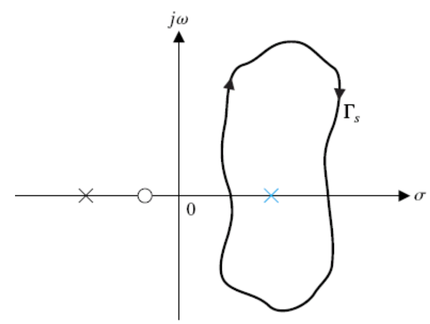


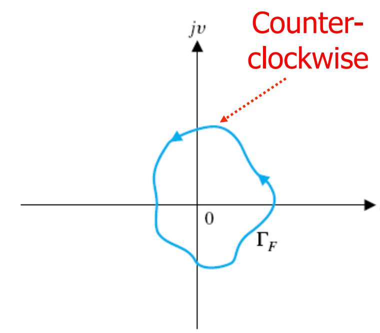


Z=3, P=1

N=3-1=2

Cauchy's Theorem



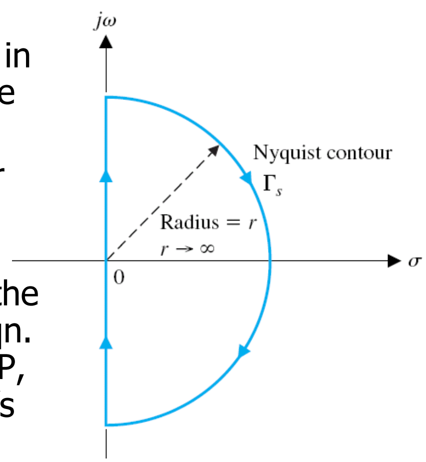


Z=0, P=1

N=0-1=-1

The Nyquist Criterion

- Choose the contour in the s-plane to be the **Nyquist contour**
- The Nyquist contour encircles the entire right-half plane
- Goal: Find roots of the closed-loop char. eqn. which are in the RHP, by applying Cauchy's Theorem.



The Nyquist Criterion**

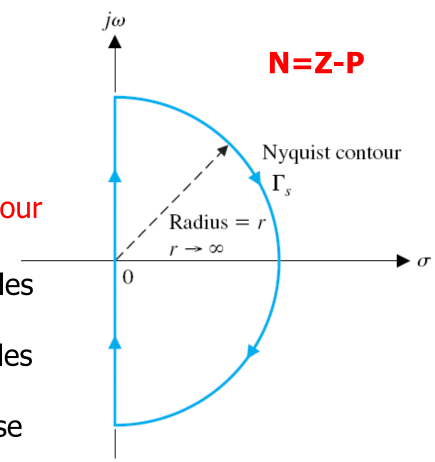
N=Z-P

- Evaluate the function

$$F(s) = 1 + KG_c(s)G(s)$$
$$= \frac{D(s) + KN(s)}{D(s)}$$

  along the Nyquist contour

- Z = zeros of F(s) = poles of closed-loop system
- P = poles of F(s) = poles of open-loop system
- N =number of clockwise encirclements of 0



The Nyquist Criterion

- This is equivalent to analyzing the function

$$F'(s) = F(s) - 1$$
$$= 1 + KG_c(s)G(s) - 1$$
$$= KG_c(s)G(s)$$

for **encirclements about -1.****

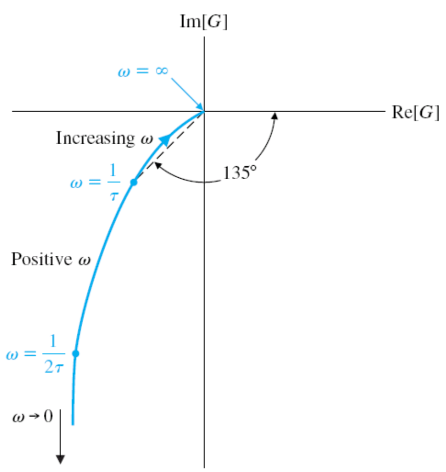
- Thus it is usually more convenient to consider this function than $1+KG_c(s)G(s)$
- The F'(s)-plane plot (aka **the Nyquist plot**) can be easily obtained from Bode diagrams of $G_c(s)G(s)$
- The Nyquist plot is the polar plot of the magnitude and phase of the open-loop system

The Nyquist Criterion**

- The closed-loop system with is **stable** if and only if the number of counter-clockwise encirclements of -1 is equal to the number of open-loop poles in the right-half plane.
- The closed-loop system which is open-loop stable (no open-loop poles in RHP) is stable if and only if there are **no** encirclements of -1.

(Recall that Z = number of roots of characteristic equation of closed-loop system in the RHP, so for stability we want to have Z=0.)

Relationship to Bode diagram

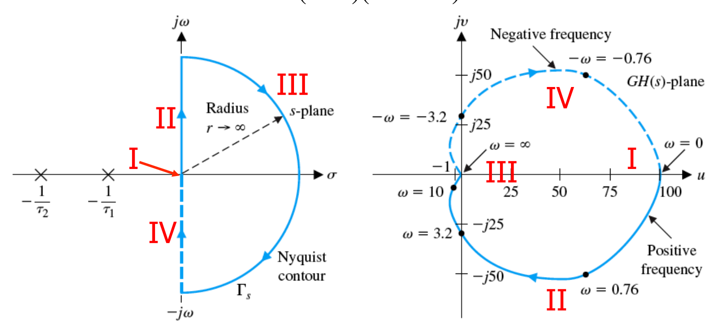


$$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau}$$
$$= \frac{-K\omega^2\tau}{\omega^2 + \omega^4\tau^2} - \frac{jK\omega}{\omega^2 + \omega^4\tau^2}$$

$$|G(\omega)| = \frac{K}{(\omega^2 + \omega^4\tau^2)^{\frac{1}{2}}}$$
$$\phi(\omega) = -\tan^{-1}\left(\frac{1}{-\omega\tau}\right)$$

Example 1: Two real poles

- Consider $G(s)H(s) = \dfrac{100}{(s+1)(s/10+1)}$



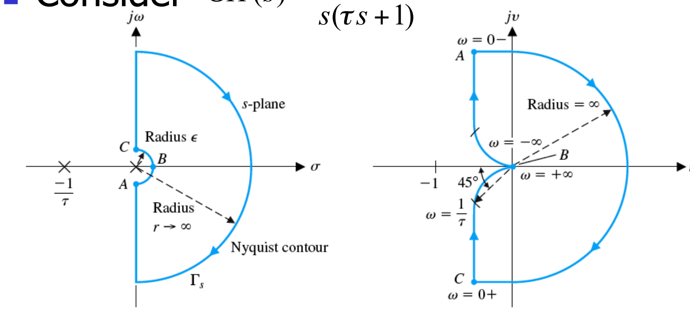
$P = 0$, hence for stability we require $Z = N = 0$,
i.e. the contour must not encircle the -1 point in the $GH(s)$-plane.

Summary so far…

- Map the Nyquist contour Γ_s to Γ_L using the loop gain $L(s)=G_c(s)G(s)$
- Count the net number of encirclements of the point (-1,0) by drawing a line from -1 to infinity in any direction. This is N.
- For a closed-loop system to be stable, N=-P, where P is the number of open-loop poles in the RHP.
- If N≠-P, the closed-loop system is not stable.

Example 2: Pole + integrator

- Consider $GH(s) = \dfrac{K}{s(\tau s + 1)}$



Cauchy's Theorem requires that the contour cannot pass through the pole → Modified Nyquist contour which skirts the origin

Nyquist plot for pole + int.

- Calculate the Nyquist plot by examining parts of the s-plane contour individually
  - 1. The portion at the origin
  - 2. Along the positive imaginary axis: from $\omega = 0^+$ to $\omega=+\infty$
  - 3. From $\omega=+\infty$ to $\omega = -\infty$
  - 4. Along the negative imaginary axis: from $\omega = -\infty$ to 0^-

Nyquist plot for $G(s) = \dfrac{1}{s(\tau s + 1)}$

- 1.The origin
  - Cannot have poles occurring on the contour, so adjust the contour slightly
  - Add small circular detour in the s-plane

    $s = \varepsilon e^{j\phi}, \quad \phi = -90^\circ \text{ at } \omega = 0^- \text{ to } \phi = +90^\circ \text{ at } \omega = 0^+$
  - Therefore the Nyquist plot will be

$$\lim_{\varepsilon \to 0} KG(s) = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon e^{j\phi}} = \left(\lim_{\varepsilon \to 0} \frac{K}{\varepsilon} \right) e^{-j\phi}$$

a semi-circle of infinite radius from $+90^0$ at $\omega=0^-$ to -90^0 at $\omega=0^+$.

Nyquist plot for $G(s) = \dfrac{1}{s(\tau s + 1)}$

- 2. Plot the frequency response G(jω) for positive ω.
- 3. Portion from $\omega = +\infty$ to $\omega = -\infty$

$$\lim_{r\to\infty} GH(s) = \lim_{r\to\infty} \left| \frac{K}{(\tau r^2)} \right| e^{-2j\phi}$$

-180° at $\omega = +\infty$ to $+180^\circ$ at $\omega = -\infty$

Magnitude is zero when r is infinite

Example: 2 poles + integrator

$$GH(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$GH(j\omega) = \frac{-K(\tau_1 + \tau_2) - jK(1/\omega)(1 - \omega^2 \tau_1 \tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4 \tau_1^2 \tau_2^2}$$

$$|GH(\omega)| = \frac{K}{\sqrt{\omega^4(\tau_1 + \tau_2)^2 + \omega^2(1 - \omega^2 \tau_1 \tau_2)^2}}$$

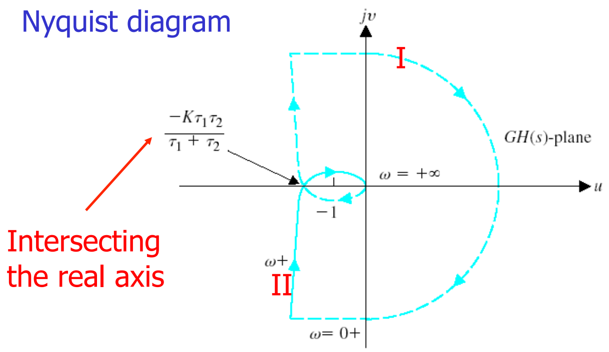
$$\phi(\omega) = -\tan^{-1}\omega\tau_1 - \tan^{-1}\omega\tau_2 - \pi/2$$

$$\lim_{\omega\to\infty} |GH(\omega)| = 0$$

$$\lim_{\omega\to\infty} \phi(\omega) = -3\pi/2 = -270^\circ$$

Example: 2 poles + integrator

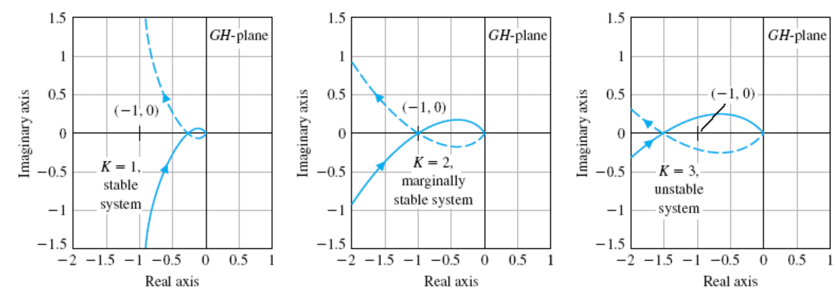
Nyquist diagram



It is possible to encircle the -1 point. → The system is unstable with roots in the right-hand s-plane.

Example: 2 poles + integrator

- The system is stable when $\dfrac{-K\tau_1\tau_2}{\tau_1 + \tau_2} \geq -1$



- Consider the case when $\tau_1 = 1$, $\tau_2 = 1$. For stability, $K \leq 2$ is required.