EECE 360
Lecture 27

State Feedback

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Chapters 11.1-11.6, 11.9, 11.10

Review: The Nyquist Criterion**

- Plot Nyquist diagram
  - Evaluate open-loop transfer function G(s) along Nyquist contour
- Evaluate Nyquist criterion
  - Identify P = # of open-loop poles in RHP
  - Identify N = # of clockwise encirclements of -1
- Determine stability
  - If Z = N+P = 0, then the closed-loop system with unity feedback is stable.
  - If not, the closed-loop system with unity feedback is NOT stable.
- Variations:
  - Find the value of K for which the system will be stable or unstable

Context

- Modeling
  - State-space
  - s-domain
- Classical control (s-domain, freq. response)
  - Root Locus
  - Bode
  - Nyquist
- **Modern control (state-space)**
  - **Full-state feedback**
  - **Output feedback**
  - **Controllers and observers**

Today's lecture

- Full-state feedback regulation
  - Controllability
  - Ackerman's formula for controller synthesis
- Next lectures:
  - Output-based regulation
    - Observability
    - Ackermann's for observer synthesis
  - Separation Principle

State Feedback: Regulation

- Consider the state feedback controller where K is constant feedback gain matrix

$$\dot{x} = Ax + Bu, \quad u = -Kx$$

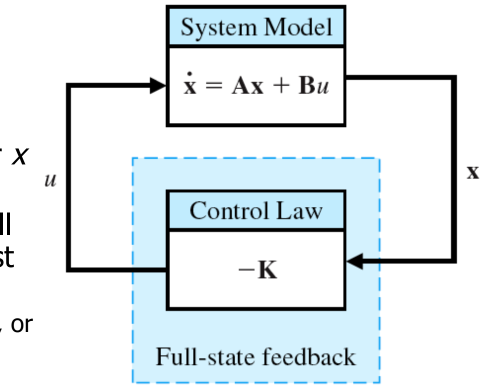
- Then one can write

$$\dot{x} = Ax + B(-Kx) \\ = (A - BK)x$$

- Whereas the poles of the open-loop system are given by the eigenvalues of A, the poles of the closed-loop system are given by the **eigenvalues of *(A-BK)***

State Feedback: Regulation

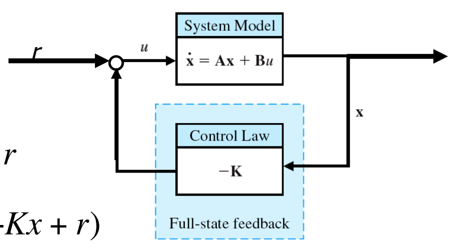
- **Full-state** feedback
- The control law
  u=-Kx is computed by assuming that the entire state vector *x* is available
- This means that all elements of *x* must be either
  - directly measured, or
  - estimated from measurements of other combinations of states



State Feedback: Tracking

- **Full-state** feedback

$$\dot{x} = Ax + Bu,$$
$$u = -Kx + r$$

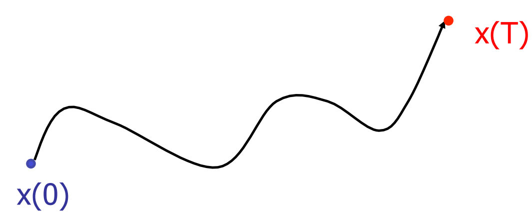


$$\dot{x} = Ax + B(-Kx + r) \\ = (A - BK)x + Br$$

- **Eigenvalues of (A-BK)** determine state-transition matrix

Controllability

- The eigenvalues of (A-BK) can be arbitrarily assigned when the system [A,B,C,D] is **controllable**.
- A system is **controllable** if there exists a control *u(t)* that can transfer any initial state *x(0)* to any desired state *x(t)* in a finite time *T*.



Controllability

- The eigenvalues of (A-BK) can be arbitrarily assigned when the system [A,B,C,D] is **controllable**.
- A system is **controllable** if there exists a control *u(t)* that can transfer any initial state *x(0)* to any desired state *x(t)* in a finite time *T*.
- The controllability matrix

$$S_C = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$$

must have rank *n* for the system [A,B,C,D] to be controllable. (S_C is "full-rank".)

- When S_C is full-rank, $\det(S_C) \neq 0$

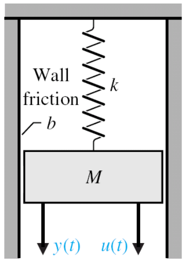
Example: Spring-Mass-Damper

- System and input matrices

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Controllability matrix

$$S_C = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{b}{M} \end{bmatrix}$$

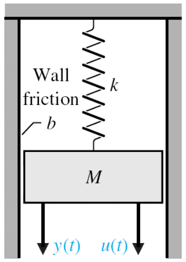


- To test for controllability, $|S_C| = 0 - 1 = -1$
- Therefore the system is **controllable.**

Example: Spring-Mass-Damper

- The open-loop poles are located where

$$0 = s^2 + \frac{b}{M}s + \frac{k}{M}$$



- With the control u = -Kx, the closed-loop poles are located where

$$0 = s^2 + \left(\frac{b}{M} + k_2\right)s + \left(\frac{k}{M} + k_1\right)$$

- Because the system is controllable, the poles of the closed-loop can be placed anywhere in the complex plane.

Example 2

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y = u$$

With $x_1 = y,\; x_2 = dy/dt,\; x_3 = d^2y/dt^2$

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Control canonical form

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} x$$ (state feedback, regulator)

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -3-k_2 & -5-k_3 \end{bmatrix}$$

$$\det(sI - A + BK) = s^3 + (5+k_3)s^2 + (3+k_2)s + (2+k_1)$$

Example 2

- Controllability matrix:

$$AB = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & -5 \\ 10 & 13 & 22 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 22 \end{bmatrix}$$

$$|S_C| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -5 & 22 \end{vmatrix} = 0 - 0 + 1(0-1) = -1$$

Example 2

Choosing characteristic equation as

$(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \zeta\omega_n)$ ← Want rapid response and a low overshoot

Specifying $\zeta = 0.8$ and $T_s < 1$, then

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.8\omega_n} < 1 \text{ is satisfied with } \omega_n = 6$$

$$(s^2 + 9.6s + 36)(s + 4.8) = s^3 + 14.4s^2 + 82.1s + 172.8$$

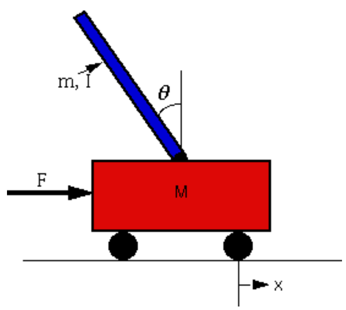
Comparing with $s^3 + (5 + k_3)s^2 + (3 + k_2)s + (2 + k_1)$

gives $k_1 = 170.8,\ k_2 = 79.1,\ k_3 = 9.4$

$$K = [170.8 \quad 79.1 \quad 9.4]$$

Example: Pendulum on a Cart

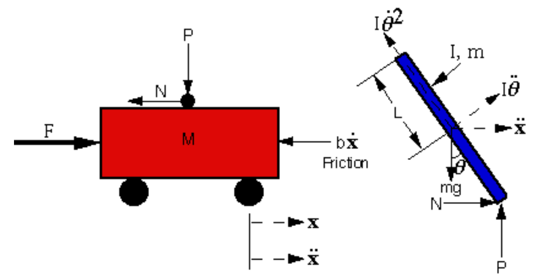
- Control input: force acting on the cart
- State: position and velocity of the cart and rotational position and rotational velocity of the mass.



http://www.engin.umich.edu/group/ctm/examples/pend/invpen.html

Example: Pendulum on a Cart

- Obtain equations of motion by summing forces acting on the two bodies
- Linearize around θ = π



Example: Pendulum on a Cart

- State-space equations

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\Phi}\\ \ddot{\Phi}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \Phi\\ \dot{\Phi}\end{bmatrix} + \begin{bmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \Phi\\ \dot{\Phi}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u$$

- (Note this is a single input, multi-output system.)

Example: Pendulum on a Cart

- **Problem statement:** Design a controller to stabilize the system.

- First: Is the system controllable?
  Check by finding the rank of the controllability matrix

$$S_C = [B \;\; AB \;\; A^2B \;\; A^3B]$$

- Second: Then design a controller, or if the system is not controllable, determine if it is stabilizable.

Controllability

- In Matlab,
  - use 'ctrb' to find the controllability matrix numerically
  - use 'rank' to find the rank of the controllability matrix
  - *Note:* Using 'det' to find the determinant of the controllability matrix is not numerically robust and generally not a good idea. (e.g., How do you distinguish between low-magnitude eigenvalues of an ill-conditioned matrix and 0 eigenvalues of a matrix that is genuinely singular?)

Ackermann's Formula

- Method for pole placement for SISO systems
- Presented without derivation here
- Uses the Cayley-Hamilton theorem, which states that a matrix must satisfy its own characteristic equation
- Related: Bass-Gura formula

- For MIMO systems, K is not unique. Other methods must be used.

Ackermann's Formula

- The state feedback gain matrix

$$K = [k_1 \quad k_2 \quad \cdots \quad k_n] \text{ where } u(t) = r(t) - Kx(t)$$

that produces the desired characteristic equation

is given by $$q(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

where $$K = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} S_C^{-1} q(A)$$

$$S_C = [B \quad AB \quad \cdots \quad A^{n-1}B] \text{ and } q(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

Example: Spring-Mass-Damper

- Consider a spring-mass damper system with control law $u = -Kx$. Find K such that the closed-loop system has damping ratio ζ and natural frequency ω_n.
- The desired closed-loop characteristic equation is

$$q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

- Compute the controllability matrix and its inverse

$$S_C = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{b}{M} \end{bmatrix}$$

$$S_C^{-1} = \begin{bmatrix} \frac{b}{M} & 1 \\ 1 & 0 \end{bmatrix}$$

Example: Ackermann's Form.

- The characteristic equation in terms of A is

$q(A) = A^2 + 2\zeta\omega_n A + \omega_n^2$, therefore the control gain is

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{b}{M} & 1 \\ 1 & 0 \end{bmatrix} \left(A^2 + 2\zeta\omega_n A + \omega_n^2 I \right)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}^2 + 2\zeta\omega_n \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + \omega_n^2 I \right)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} -\frac{k}{M} & -\frac{b}{M} \\ \frac{kb}{M^2} & -\frac{k}{M} + \frac{kb}{M^2} \end{bmatrix} + 2\zeta\omega_n \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + 2\zeta\omega_n \begin{bmatrix} 0 & 1 \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example: Ackermann's Form.

- The control gain to achieved the desired closed-loop poles is

$$K = \begin{bmatrix} -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} + 2\zeta\omega_n \begin{bmatrix} 0 & 1 \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \omega_n^2 - \frac{k}{M} & 2\zeta\omega_n - \frac{b}{M} \end{bmatrix}$$

- Note that the control gain is the difference between the desired closed-loop and actual open-loop coefficients of the characteristic equation.

Example: Ackermann's Form.

- Check: The closed-loop system is

$$\dot{x} = (A - BK)x$$
$$= \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \omega_n^2 - \frac{k}{M} & 2\zeta\omega_n - \frac{b}{M} \end{bmatrix} \right) x$$
$$= \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \omega_n^2 - \frac{k}{M} & 2\zeta\omega_n - \frac{b}{M} \end{bmatrix} \right) x$$
$$= \begin{bmatrix} 0 & 1 \\ \omega_n^2 & 2\zeta\omega_n \end{bmatrix} x$$

- which has poles at $0=|s-(A-BK)|=s(s+2\zeta\omega_n)+\omega_n^2$

Using Matlab

ACKER Pole placement gain selection using Ackermann's formula.

K = ACKER(A,B,P) calculates the feedback gain matrix K such that the single input system

$$\dot{x} = Ax + Bu$$

with a feedback law of u = -Kx has closed loop poles at the values specified in vector P, i.e., P = eig(A-B*K).

Note: This algorithm uses Ackermann's formula. This method is NOT numerically reliable and starts to break down rapidly for problems of order greater than 10, or for weakly controllable systems. A warning message is printed if the nonzero closed-loop poles are greater than 10% from the desired locations specified in P.

See also PLACE.

Using Matlab

PLACE Pole placement technique

K = PLACE(A,B,P) computes a state-feedback matrix K such that the eigenvalues of A-B*K are those specified in vector P. No eigenvalue should have a multiplicity greater than the number of inputs.

[K,PREC,MESSAGE] = PLACE(A,B,P) returns PREC, an estimate of how closely the eigenvalues of A-B*K match the specified locations P (PREC measures the number of accurate decimal digits in the actual closed-loop poles). If some nonzero closed-loop pole is more than 10% off from the desired location, MESSAGE contains a warning message.

See also ACKER.

Controllability Summary

- A system (A,B,C,D) is controllable if its controllability matrix S_C is full rank.
- The closed-loop poles of a controllable system can be placed anywhere in the complex plane.
- Choose the desired pole location, then compute the gain K required to achieve those locations
- Ackermann's formula for SISO systems (Matlab's 'acker')
- Matlab's 'place' for MIMO systems

Summary

- Full-state feedback for regulation or tracking
- For a controllable system, we can arbitrarily assign the closed-loop poles through full-state feedback
- To test for controllability: the controllability matrix S_C should be full-rank.

- Next time:
  - Observability
  - Designing observers