EECE 360
Lecture 28

State Feedback

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Chapter 11.1-11.6, 11.9, 11.10

Today's lecture

- Review
  - Controllability test
  - Ackermann's formula for controller design
- Today
  - Observability
    - Observerability test
    - Observer design through Ackermann's
  - Separation principle
  - Combining observers and controllers

Review: Controllability

- The eigenvalues of (A-BK) can be arbitrarily assigned when the system [A,B,C,D] is **controllable**.
- A system is **controllable** if there exists a control *u(t)* that can transfer any initial state *x(0)* to any desired state *x(t)* in a finite time *T*.
- The controllability matrix

$$S_C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

must have rank *n* for the system [A,B,C,D] to be controllable. (S_C is "full-rank".)

- When S_C is full-rank, $\det(S_C) \neq 0$

Review: Ackermann's Formula

- The state feedback gain matrix

$$K = [k_1 \quad k_2 \quad \cdots \quad k_n] \text{ where } u(t) = r(t) - Kx(t)$$

that produces the desired characteristic equation

is given by $q(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$

where

$$\boxed{K = \begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix} S_C^{-1} q(A)}$$

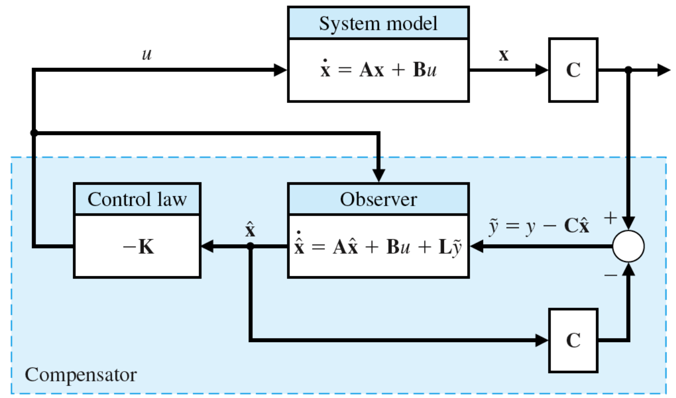
$$S_C = [B \quad AB \quad \cdots \quad A^{n-1}B] \text{ and } q(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

Output feedback

- Often, it is not feasible or even possible to measure all components of the state directly
- The output encapsulates a subset of the states which can be measured.
- For example, in the spring-mass-damper system, only the position of the mass is measured
- In this case, the remaining states must be accurately **estimated** using an **observer**

Output feedback

- Output-based regulation



Output feedback

- Since the control law acts upon the estimated value of the state

$$u = -K\hat{x}$$

- The observer must be designed such that the estimate of the state is guaranteed to converge to the actual value of the state

$$e = x - \hat{x}$$

- The estimate is a dynamic process which evolves over time according to

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$

Output feedback

- We know that $\dot{x} = Ax + Bu$

$$y = Cx$$

- And so we create an estimated system

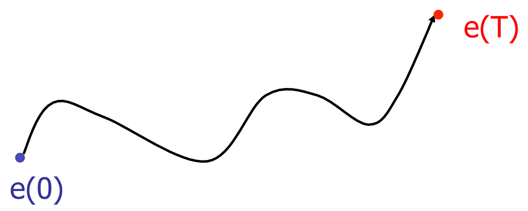
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

- Which is dependent on the difference between the actual output and the output value expected based on the current estimate of the state
- Therefore the error e evolves according to

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax - A\hat{x} + L(Cx - C\hat{x})$$
$$= (A - LC)e$$

Observability

- The eigenvalues of (A-LC) can be arbitrarily assigned when the system is **observable**.
- A system is **observable** if there exists a finite time T such that, given the input u(t), the initial state x(0) can be determined from the observation history y(t).



Observability

- The eigenvalues of **(A-LC)** can be arbitrarily assigned when the system [A,B,C,D] is **observable**.
- A system is **observable** if there exists a finite time T such that, given the input u(t), the initial state x(0) can be determined from the observation history y(t).
- The observability matrix

$$S_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

must have rank *n* for the system [A,B,C,D] to be observable. (S_O is "full-rank".)

Observability vs. Controllability

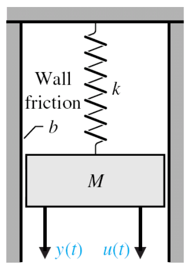
- Note that the observer gain *L* is a matrix of dimension *n* x *p*, where the output matrix *C* is *p* x *n*
- For a SISO system, *L* is *n* x 1
- Therefore *LC* will be an *n* x *n* matrix that can be subtracted, element-wise, from *A*.

- By contrast, recall that the controller gain *K* is a matrix of dimension *m* x *n*, where the input matrix *B* is *m* x *n*
- For a SISO system, *K* is 1 x *n*
- Therefore *BK* will be an *n* x *n* matrix that can be subtracted, element-wise, from *A*.

Example: Spring-Mass-Damper

- System and input matrices

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



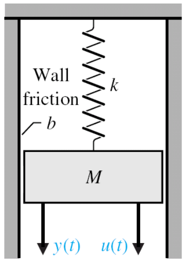
- Observability matrix

$$S_O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- To test for controllability, $|S_O|=1-0=1$
- Therefore the system is **observable.**

Example: Spring-Mass-Damper

- The open-loop observer poles are located where
$$0 = s^2 + \frac{b}{M}s + \frac{k}{M}$$
- With the observer gain L, the closed-loop poles are located where
$$0 = s^2 + \left(\tfrac{b}{M} + l_1\right)s + \left(\tfrac{b}{M}l_1 + \tfrac{k}{M} + l_2\right)$$
- Because the system is observable, the poles of the closed-loop error dynamics can be placed anywhere in the complex plane.



Observability vs. Controllability

"duality"

- Controller:
  - Regulate x -> 0 by choosing K such that
$$\dot{x} = (A - BK)x$$
is stable.
  - Controllability matrix S_C=[B AB A^2B … A^{n-1}B]
- Observer:
  - Regulate e -> 0 by choosing L such that
$$\dot{e} = (A - LC)e$$
is stable.
  - Controllability matrix S_O=[C; CA; CA^2; …; CA^{n-1}]

Observability vs. Controllability

- Controller:
  - Design a control gain K =[k_1 k_2 k_3 … k_n] through Ackermann's formula
$$K = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix} S_C^{-1} q(A)$$
- Observer:
  - Design an observer gain L = [l_1 l_2 l_3 … l_n]^T through Ackermann's formula
$$L = q(A) S_O^{-1} \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}^T$$
  - This takes advantage of the **duality** between the observer and controller

Observer design: Ackermann's

Example: Consider the spring-mass-damper system

- Choose the closed-loop poles of the observer to be 4-10 times faster than the controller poles
- For now, assume that these poles occur at a desired damping ξ and desired natural frequency ω_n, the characteristic equation is
$$q(s) = s^2 + 2\bar{\xi}\bar{\omega}_n s + \bar{\omega}_n^{\,2}$$
- Compute the observability matrix and its inverse
$$S_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$S_O^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ackermann's Formula

- The characteristic equation in terms of A is
  $q(A) = A^2 + 2\bar{\xi}\bar{\omega}_n A + \bar{\omega}_n^{\,2}$, therefore the control gain is

$$L = \left(A^2 + 2\bar{\xi}\bar{\omega}_n A + \bar{\omega}_n^{\,2} I\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}^T$$

$$= \left(\begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix}^2 + 2\bar{\xi}\bar{\omega}_n\begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix} + \bar{\omega}_n^{\,2} I\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \left(\begin{bmatrix}-\frac{k}{M} & -\frac{b}{M}\\ \frac{kb}{M^2} & -\frac{k}{M} + \frac{b^2}{M^2}\end{bmatrix} + 2\bar{\xi}\bar{\omega}_n\begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix} + \bar{\omega}_n^{\,2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{b}{M}\\ -\frac{k}{M} + \frac{b^2}{M^2}\end{bmatrix} + 2\bar{\xi}\bar{\omega}_n\begin{bmatrix}1\\ -\frac{b}{M}\end{bmatrix} + \bar{\omega}_n^{\,2}\begin{bmatrix}0\\ 1\end{bmatrix}$$

Ackermann's Formula

- The observer gain to achieved the desired closed-loop poles for the error dynamics is

$$L = \begin{bmatrix}-\frac{b}{M}\\ \frac{k}{M} + \frac{b^2}{M^2}\end{bmatrix} + 2\bar{\xi}\bar{\omega}_n\begin{bmatrix}1\\ -\frac{b}{M}\end{bmatrix} + \bar{\omega}_n^{\,2}\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$L = \begin{bmatrix}2\bar{\xi}\bar{\omega}_n - \frac{b}{M}\\ \bar{\omega}_n^{\,2} - 2\bar{\xi}\bar{\omega}_n\frac{b}{M} - \frac{k}{M} + \frac{b^2}{M^2}\end{bmatrix}$$

- Note that the observer gain will drive the error dynamics to the desired closed-loop error dynamics poles.

Ackermann's Formula

- The closed-loop system is

$$\dot{e} = (A - LC)e$$

$$= \left(\begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix} - \begin{bmatrix}2\bar{\xi}\bar{\omega}_n - \frac{b}{M}\\ \bar{\omega}_n^{\,2} - 2\bar{\xi}\bar{\omega}_n\frac{b}{M} - \frac{k}{M} + \frac{b^2}{M^2}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)x$$

$$= \left(\begin{bmatrix}0 & 1\\ -\frac{k}{M} & -\frac{b}{M}\end{bmatrix} - \begin{bmatrix}2\bar{\xi}\bar{\omega}_n - \frac{b}{M} & 0\\ \bar{\omega}_n^{\,2} - 2\bar{\xi}\bar{\omega}_n\frac{b}{M} - \frac{k}{M} + \frac{b^2}{M^2} & 0\end{bmatrix}\right)x$$

$$= \begin{bmatrix}-2\bar{\xi}\bar{\omega}_n + \frac{b}{M} & 1\\ \bar{\omega}_n^{\,2} + \left(2\bar{\xi}\bar{\omega}_n - \frac{b}{M}\right)\frac{b}{M} & -\frac{b}{M}\end{bmatrix}x$$

- which has poles at $0=|s-(A-LC)|=s^2+2\underline{\zeta}\underline{\omega}_n s+\underline{\omega}_n^{\,2}$

Using Matlab

- Designing controller gains
  - $K = \begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix} S_C^{-1} q(A)$
  - K = acker(A,B,Pk)

Use 'place' for MIMO systems

- Designing observer gains
  - $L = q(A) S_O^{-1}\begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix}^T$
  - LT = acker(A′,C′,Pl)
  - L = LT′

** Note that the transpose of both A and C required!

Controllability Summary

- A system (A,B,C,D) is controllable if its controllability matrix S_C is full rank.
- The closed-loop poles of a controllable system can be placed anywhere in the complex plane.
- Choose the desired pole location, then compute the gain K required to achieve those locations
- Ackermann's formula for SISO systems (Matlab's 'acker')
- Matlab's 'place' for MIMO systems

Observability Summary

- A system (A,B,C,D) is observable if its observability matrix S_O is full rank.
- The closed-loop poles of the error dynamics of an observable system can be placed anywhere in the complex plane.
- This allows arbitrarily fast convergence of the state estimate to the actual value of the state.
- Choose the desired error pole location, then compute the gain L required to achieve those locations
- Ackermann's formula for SISO systems (Matlab's 'acker') with **transposed matrices**

Observers/controllers

- The dynamics for *dx/dt* and *de/dt* are **coupled**
  - State dynamics

$$\dot{x} = Ax + Bu, \quad u = -K(x+e)$$
$$= (A - BK)x + BKe$$

  - Error dynamics

$$\dot{e} = \dot{x} - \dot{\hat{x}}, \quad \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$= Ax + Bu - A(x+e) - Bu - LCx + LC(x+e)$$
$$= (A - LC)e$$

Observers/controllers

- In state-space form, with

$$\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix}$$

- The closed-loop system and observer dynamics are

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}}$$

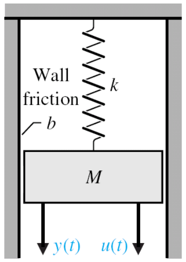
- **The eigenvalues of this system are eig(A-BK) and eig(A-LC)**

Separation Principle **

- Although the state dynamics and observer dynamics are coupled, **the controller and the observer can be designed independently**
- Standard procedure:
  - Design a controller with gain K to place the roots of (A-BK) at desired locations in the LHP.
  - Design an observer with gain L to place the roots of (A-LC) at desired locations in the LHP.
- Generally the observer poles are placed such that the observer dynamics are 4-10 times faster than the state dynamics.

Example: Spring-Mass-Damper

- Using the controller and observer designed previously



$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A-BK & BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & 0 & 1\\ \omega_n^{\,2} & 2\zeta\omega_n & \omega_n^{\,2}-\frac{k}{M} & 2\zeta\omega_n-\frac{b}{M}\\ 0 & 0 & -2\bar{\zeta}\varpi_n+\frac{b}{M} & 1\\ 0 & 0 & \varpi_n^{\,2}+\left(2\bar{\zeta}\varpi_n-\frac{b}{M}\right)\frac{b}{M} & \frac{b}{M}\end{bmatrix}\begin{bmatrix}x_M\\ v_M\\ e_x\\ e_v\end{bmatrix}$$

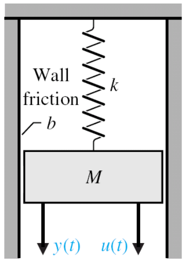
Example: Spring-Mass-Damper

- The open-loop system poles are located where

$$0 = s^2 + \frac{b}{M}s + \frac{k}{M}$$

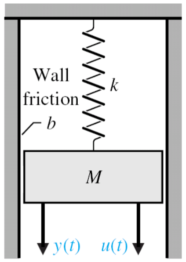
- With controller gain *K* and observer gain *L*, the closed-loop poles of the extended system are located where

$$0 = \left(s^2 + 2\zeta\omega_n s + \omega_n^{\,2}\right)\left(s^2 + 2\bar{\zeta}\varpi_n s + \varpi_n^{\,2}\right)$$



Example: Spring-Mass-Damper

- Because the system is controllable and observable, the closed-loop poles of the error dynamics and the system dynamics can be placed arbitrarily.
- However, the further away the closed-loop poles are placed from the open-loop poles, the higher the control effort.
- Additionally, excessively high observer gains can lead to amplification of noise inherent to the output measurements.



Summary

- Controllability matrix S_C to test whether it is possible to put the poles of the closed-loop state dynamics in any desired location
- Observability matrix S_O to test whether it is possible to put the poles of the closed-loop error dynamics in any desired location
- Duality of controller (with gain K) and observer (with gain L)
- **Separation principle** allows independent design of the controller and observer