MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
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PROBLEM SET 1
Solutions

Problem 1

For example, the cruise control system in a car: the input is the desired speed and the output is the actual speed of the car. The controller is the cruise control computer, which compares the actual speed with the desired speed, and sets the throttle accordingly. The plant is the car driving on the road. The sensor is the tachometer. A source of disturbance could be a gear change, and a source of noise could be voltage fluctuations on the power bus. (Of course, real cruise control systems are much more complicated and incorporate more than one sensor, but the basic idea is the same.)

Without feedback, the controller would have to "guess" at a throttle setting, based only on the desired speed. If the model of the plant was absolutely perfect, then it could work, but the instant something about the car changed–e.g. it started going up a hill, or the engine began to wear out and run less efficiently–the car's speed would also change and the controller would have no idea how to correct for it.

Problem 2

1. Using the superposition property of linear systems we can compute the transfer functions from each input the the output individually and then obtain the overall transfer function by adding the individual contributions.

   Let's start with the transfer function from r to c. Disregard the other two inputs, i.e. set $D(s) = 0$ and $N(s) = 0$.

   $$C(s) = K(s)G(s)E(s) = K(s)G(s)(R(s) - B(s)) = K(s)G(s)\,R(s) - K(s)G(s)H(s)\,C(s) \tag{1}$$

   rearranging:

   $$C(s) = \frac{K(s)G(s)}{1 + K(s)G(s)H(s)}\,R(s) \tag{2}$$

   In a similar fashion, considering one input at the time, for disturbance we get:

   $$C(s) = \frac{G(s)}{1 + K(s)G(s)H(s)}\,D(s) \tag{3}$$

   and for noise:

   $$C(s) = \frac{-K(s)G(s)H(s)}{1 + K(s)G(s)H(s)}\,N(s) \tag{4}$$

   As a final result, the overall transfer function is:

   $$C(s) = \frac{1}{1 + K(s)G(s)H(s)}\,[G(s)D(s) + K(s)G(s)R(s) - K(s)G(s)H(s)N(s)] \tag{5}$$

2. If there is no noise and no disturbance, and $H(s) = 1$, the error signal is given by:

   $$E(s) = R(s) - B(s) = R(s) - C(s) = R(s) - \frac{1}{1 + KG(s)}\,[KG(s)R(s)] = \frac{1}{1 + KG(s)}R(s) \tag{6}$$

3. If our input is the unit step, then the steady state error is the following:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \, \frac{1}{1 + KG(s)} \, \frac{1}{s} = \lim_{s \to 0} \frac{1}{1 + KG(s)} \tag{7}$$

then:

$$e_{ss_1} = \lim_{s \to 0} \frac{1}{1 + \frac{K}{s^2+2}} = \lim_{s \to 0} \frac{s+2}{s+2+K} = \frac{2}{2+K} \tag{8}$$

$$e_{ss_2} = \lim_{s \to 0} \frac{1}{1 + \frac{K}{s(s+2)}} = \lim_{s \to 0} \frac{s(s+2)}{s+2+K} = 0 \tag{9}$$

4. For the first plant G_1, the steady-state error decreases as the gain K increases. For the second plant G_2, the steady-state error is zero regardless of the size of the gain. Later in this course, we will learn how the steady-state error is affected by the presence of an integrator ($1/s$ term) in the system.

Problem 3

3.1

Refer to Euler's formula:

$$e^{j\phi} = \cos\phi + j \sin\phi \tag{10}$$

Using eq. (10) we can derive the *polar representation* of a complex number. Consider the complex number $z = x + jy$; and define its modulus $\rho = |z|$ and argument $\phi = \arg(z)$ as follows:

$$\begin{aligned} \rho &= \sqrt{(x^2 + y^2)} \\ \phi &= \tan^{-1}(y/x) \end{aligned}$$

Note that the inverse tangent function used in the above definition has to take into account the quadrant in which z lies.

Using Euler's formula:

1. $z_1 = e^{-\pi j} = \cos(-\pi) + j\sin(-\pi) = -1$
   $\mid z_1 \mid = 1, \arg(z_1) = -\pi$

2. $z_2 = 2e^{-\frac{\pi}{3}j} = 2\cos(-\frac{\pi}{3}) + 2j\sin(-\frac{\pi}{3}) = 1 - j\sqrt{3}$
   $\mid z_2 \mid = 2, \arg(z_2) = -\frac{\pi}{3}$

3. $z_3 = \frac{2+j}{1+3j} = \frac{(2+j)(1-3j)}{(1+3j)(1-3j)} = \frac{5-5j}{10} = \frac{1}{2} - \frac{1}{2}j$
   $\mid z_3 \mid = \frac{1}{\sqrt{2}}, \arg(z_3) = -\frac{\pi}{4}$

4. $z_4 = \frac{1}{(1+j)^2} = \frac{(1-j)^2}{(1+j)^2(1-j)^2} = \frac{-2j}{4} = -\frac{1}{2}j$
   $\mid z_4 \mid = \frac{1}{2}, \arg(z_4) = -\frac{\pi}{2}$

Note: to check modulus and argument in MATLAB use **abs** and **angle**.

3.2

$$
\begin{aligned}
F_1(s) &= \mathcal{L}[f_1(t)] = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\frac{1}{s}\lim_{t\to\infty}\left[e^{-st} - e^0\right] = \frac{1}{s} \\
F_2(s) &= \mathcal{L}[f_2(t)] = \int_0^\infty tu(t)e^{-st}dt = -\left[\frac{t}{s}e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}dt = 0 - \frac{1}{s^2}\lim_{t\to\infty}\left[e^{-st} - e^0\right] = \frac{1}{s^2} \\
F_3(s) &= \mathcal{L}[f_3(t)] = \int_0^\infty e^{-t}u(t)e^{-st}dt = \int_0^\infty e^{-(s+1)t}dt = -\frac{1}{s+1}\lim_{t\to\infty}\left[e^{-(s+1)t} - e^0\right] = \frac{1}{s+1}
\end{aligned}
$$

3.3

$$
\begin{aligned}
f_1(t) &= \mathcal{L}^{-1}[F_1(s)] = \mathcal{L}^{-1}\left[\frac{-3/4}{s+3} + \frac{3/4}{s-1}\right] = -\frac{3}{4}e^{-3t}u(t) + \frac{3}{4}e^{t}u(t) \\
f_2(t) &= \mathcal{L}^{-1}[F_2(s)] = \mathcal{L}^{-1}[\frac{1}{(s^2+4)}] = \frac{1}{2}\ sin(2t)u(t) \\
f_3(t) &= \mathcal{L}^{-1}[F_3(s)] = \mathcal{L}^{-1}[\frac{s^2+s+2}{(s+1)^3}] = \mathcal{L}^{-1}[\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}] = \\
&= e^{-t}u(t) - te^{-t}u(t) + 2\frac{t^2}{2!}e^{-t}u(t) = e^{-t}u(t) - te^{-t}u(t) + t^2e^{-t}u(t)
\end{aligned}
$$

Note: to check partial fractions exapansion in MATLAB use **residue**, and **laplace** and **ilaplace** to find Laplace and inverse Laplace transforms. If you have trouble doing partial fractions by hand, ask one of the TAs for a quick tutorial.

3.4

1. Taking the Laplace transform (remember the initial conditions), we get:

$$2sY(s) - 2y(0) + 3Y(s) = \frac{1}{s+1} \tag{11}$$

   Solving for $Y(s)$ we get:

$$Y(s) = \frac{1}{2s+3}\left[\frac{1}{s+1} + 2\right] = \frac{1}{s+1} \tag{12}$$

   Finally, going back to the time domain:

$$y(t) = e^{-t} \tag{13}$$

2. Proceed in the same way for the second problem:

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 5(sY(s) - y(0)) + 6Y(s) = \frac{1}{s} \tag{14}$$

$$(s^2Y(s) + s - \frac{1}{2}) + 5(sY(s) + 1) + 6Y(s) = \frac{1}{s} \tag{15}$$

   Solving for $Y(s)$:

$$Y(s) = \frac{1}{2}\left[\frac{-2s^2 - 9s + 2}{(s+3)(s+2)s}\right] \tag{16}$$

   Expanding into partial fractions:

$$Y(s) = \frac{11/6}{s+3} - \frac{3}{s+2} + \frac{1/6}{s} \tag{17}$$

   Going back to the time domain:

$$y(t) = \frac{11}{6}e^{-3t} - 3e^{-2t} + \frac{1}{6} \tag{18}$$