APPLICATION OF THE LAPLACE TRANSFORM TO CIRCUIT ANALYSIS

LEARNING GOALS

Laplace circuit solutions
Showing the usefulness of the Laplace transform

Circuit Element Models
Transforming circuits into the Laplace domain

Analysis Techniques
All standard analysis techniques, KVL, KCL, node, loop analysis, Thevenin's theorem are applicable

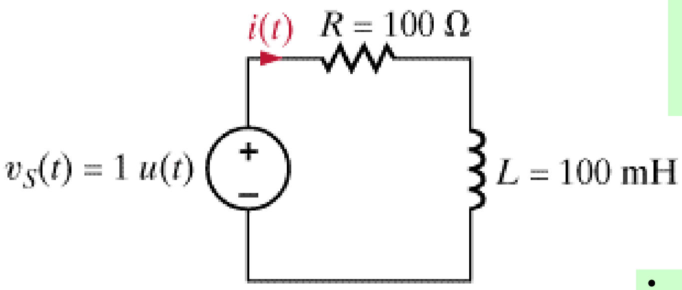
Transfer Function
The concept is revisited and given a formal meaning

Pole-Zero Plots/Bode Plots
Establishing the connection between them

Steady State Response
AC analysis revisited

LAPLACE CIRCUIT SOLUTIONS

We compare a conventional approach to solve differential equations with a technique using the Laplace transform



KVL: $v_S(t) = Ri(t) + L\frac{di}{dt}(t)$

Comple mentary

$i = i_C + i_p$

Particular

Complementary equation

$$Ri_C(t) + L\frac{di_C}{dt}(t) = 0 \Rightarrow i_C(t) = K_C e^{-\alpha t}$$

$$RK_C e^{-\alpha t} + LK_C(-\alpha e^{-\alpha t}) = 0 \Rightarrow \alpha = \frac{R}{L}$$

Particular solution for this case

$$i_p(t) = K_p \quad \Rightarrow v_S = 1 = RK_p$$

$$i(t) = \frac{1}{R} + K_C e^{-\frac{R}{L}t}$$

Use boundary conditions

$$v_S(t) = 0 \; for \; t < 0 \Rightarrow \; i(0) = 0$$

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right); \; t > 0$$

"Take Laplace transform" of the equation

$$v_S(t) = Ri(t) + L\frac{di}{dt}(t)$$

L

$$V_S(s) = RI(s) + L\mathcal{L}\left[\frac{di}{dt}\right]$$

$$\mathcal{L}\left[\frac{di}{dt}\right] = sI(s) - i(0) = sI(s)$$

Initial conditions are automatically included

$$\therefore \frac{1}{s} = RI(s) + LsI(s)$$

$$I(s) = \frac{1}{s(R + Ls)}$$

$$I(s) = \frac{1/L}{s(R/L + s)} = \frac{K_1}{s} + \frac{K_2}{s + R/L}$$

Only algebra is needed

$$K_1 = sI(s)|_{s=0} = \frac{1}{R}$$

$$K_2 = (s + R/L)I(s)|_{s=-R/L} = -\frac{1}{R}$$

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right); \; t > 0$$

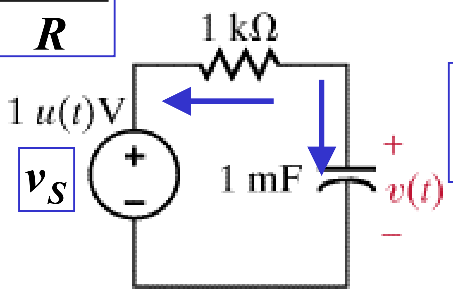
No need to search for particular or comple-mentary solutions

GEAUX

LEARNING BY DOING

Find $v(t),\, t>0$

$$\frac{v-v_S}{R}$$



$$v_S$$

$$C\frac{dv}{dt}$$

Model using KCL

$$C\frac{dv}{dt}+\frac{v-v_S}{R}=0$$

$$RC\frac{dv}{dt}+v=v_S$$

L

$$RC\mathcal{L}\left[\frac{dv}{dt}\right]+V(s)=V_S(s)$$

$$\mathcal{L}\left[\frac{dv}{dt}\right]=sV(s)-v(0)=sV(s)$$

$$v_S(t)=0,\ t<0\Rightarrow v(0)=0$$

Initial condition given in implicit form

$$v_S=u(t)\Rightarrow V_S(s)=\frac{1}{s}$$

In the Laplace domain the differential equation is now an algebraic equation

$$RCsV(s)+V(s)=\frac{1}{s}$$

$$V(s)=\frac{1}{s(RCs+1)}=\frac{1/RC}{s(s+1/RC)}$$

Use partial fractions to determine inverse

$$V(s)=\frac{1/RC}{s(s+1/RC)}=\frac{K_1}{s}+\frac{K_2}{s+1/RC}$$

$$K_1=sV(s)|_{s=0}=1$$

$$K_2=(s+1/RC)V(s)|_{s=-1/RC}=-1$$

$$v(t)=1-e^{-\frac{t}{RC}},\ t\geq 0$$

GEAUX

CIRCUIT ELEMENT MODELS

The method used so far follows the steps:

1. Write the differential equation model
2. Use Laplace transform to convert the model to an algebraic form

For a more efficient approach:

1. Develop s-domain models for circuit elements
2. Draw the “Laplace equivalent circuit” keeping the interconnections and replacing the elements by their s-domain models
3. Analyze the Laplace equivalent circuit. All usual circuit tools are applicable and all equations are algebraic.

Independent sources

$$v_S(t) \to V_S(s)$$

$$i_S(t) \to I_S(s)$$

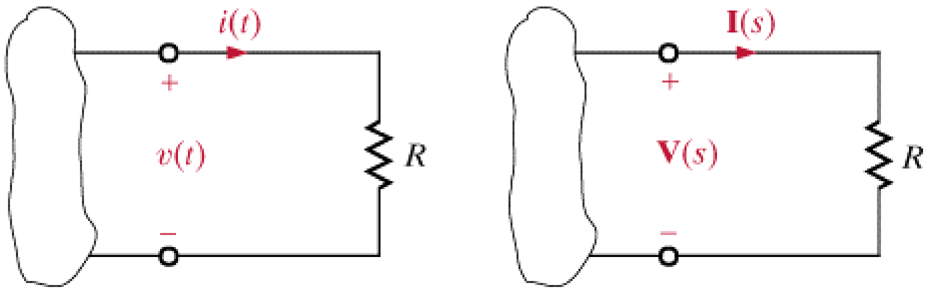
Dependent sources

$$v_D(t) = Ai_C(t) \to V_D(s) = AI_C(s)$$

$$i_D(t) = Bv_C(t) \to I_D(s) = BV_C(s)$$

...

Resistor

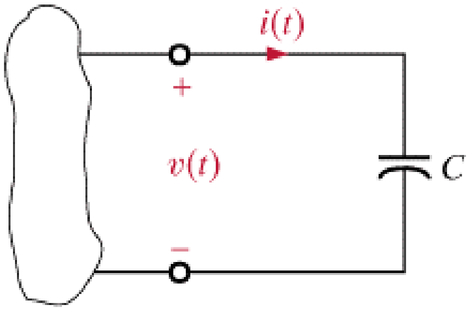


(a)

$$v(t) = Ri(t) \Rightarrow V(s) = RI(s)$$

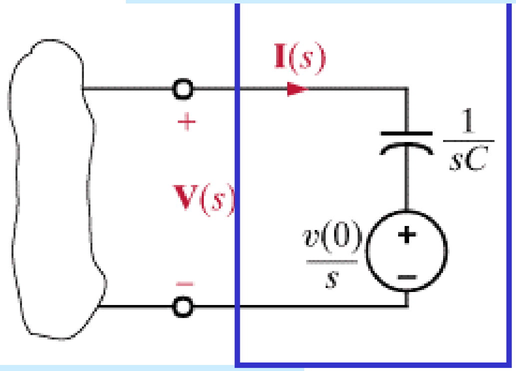
GEAUX

Capacitor: Model 1



$$v(t) = \frac{1}{C}\int_0^t i(x)dx + v(0)$$

Source transformation



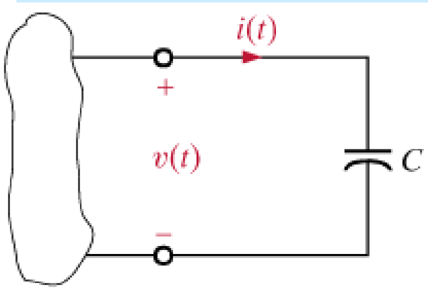
$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s} \qquad \times Cs$$

Impedance in series with voltage source

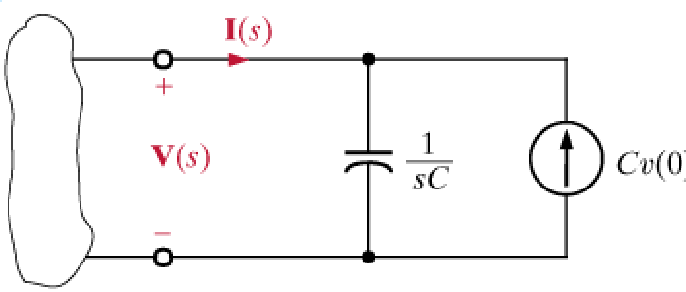
$$\mathcal{L}\left[\int_0^t i(x)dx\right] = \frac{I(s)}{s}$$

$$I_{eq} = \frac{\frac{v(0)}{s}}{\frac{1}{Cs}} = Cv(0)$$

Capacitor: Model 2



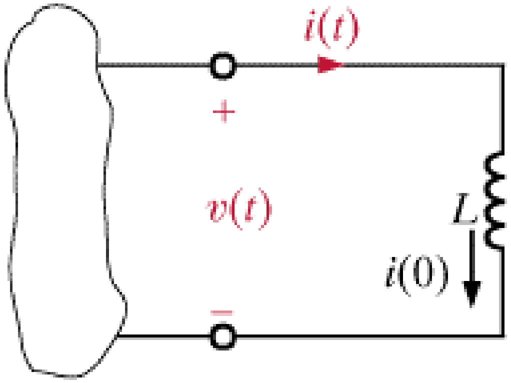
$$v(t) = \frac{1}{C}\int_0^t i(x)dx + v(0)$$



$$I(s) = CsV(s) - Cv(0)$$

Impedance in parallel with current source

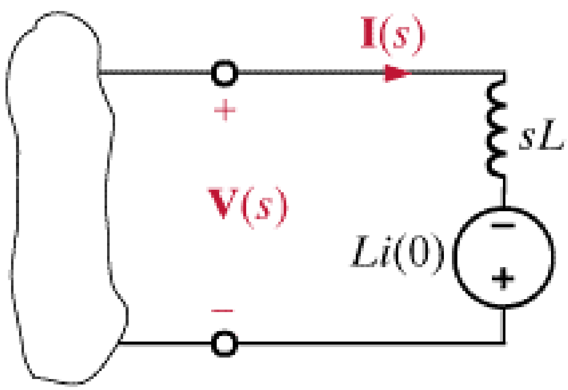
Inductor Models



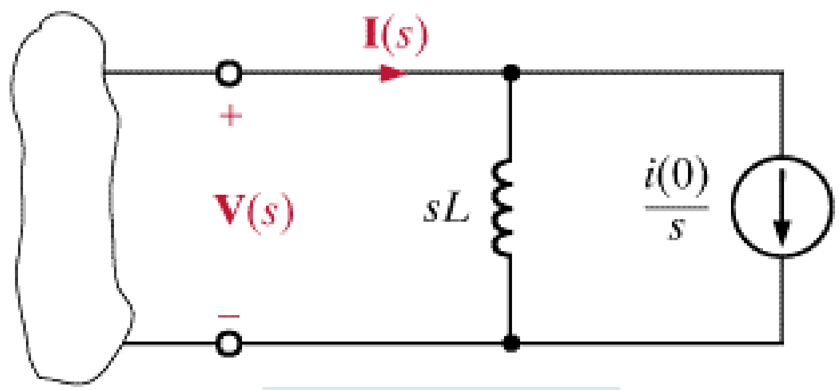
$$\mathcal{L}\left[\frac{di}{dt}\right] = sI(s) - i(0)$$

$$v(t) = L\frac{di}{dt}(t) \Rightarrow V(s) = L(sI(s) - i(0))$$

$$I(s) = \frac{V(s)}{Ls} + \frac{i(0)}{s}$$



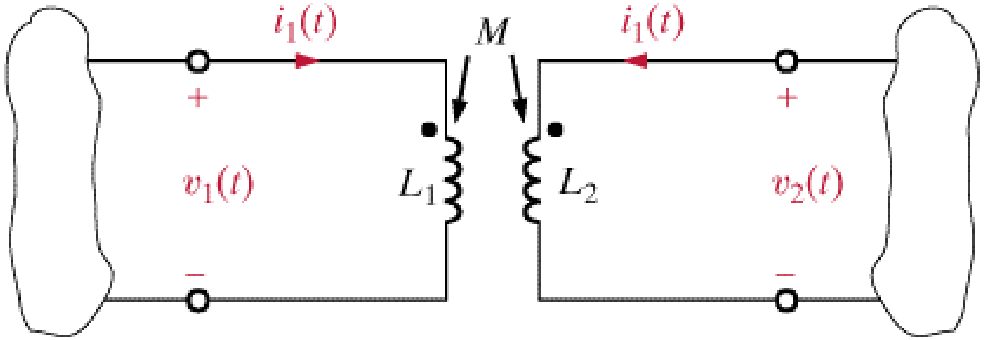
$$V(s) = LsI(s) - Li(0)$$



$$I(s) = \frac{V(s)}{Ls} + \frac{i(0)}{s}$$

GEAUX

Mutual Inductance



$$v_1(t) = L_1 \frac{di_1}{dt}(t) + M \frac{di_2}{dt}(t)$$

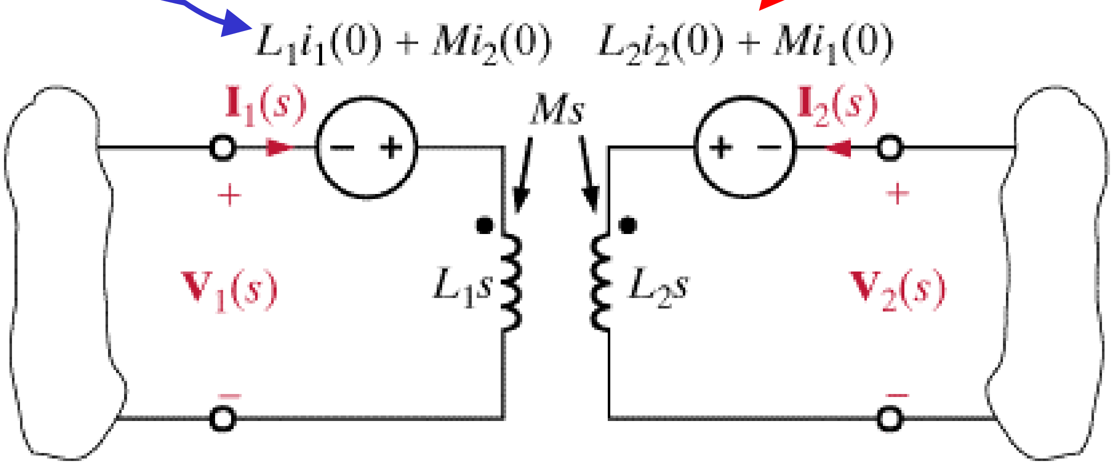
$$v_2(t) = M \frac{di_1}{dt}(t) + L_2 \frac{di_2}{dt}(t)$$

$$V_1(s) = L_1 s I_1(s) - L_1 i_1(0) + M s I_2(s) - M i_2(0)$$

$$V_2(s) = M s I_1(s) - M i_1(0) + L s I_2(s) - L i_2(0)$$

Combine into a single source in the primary

Single source in the secondary



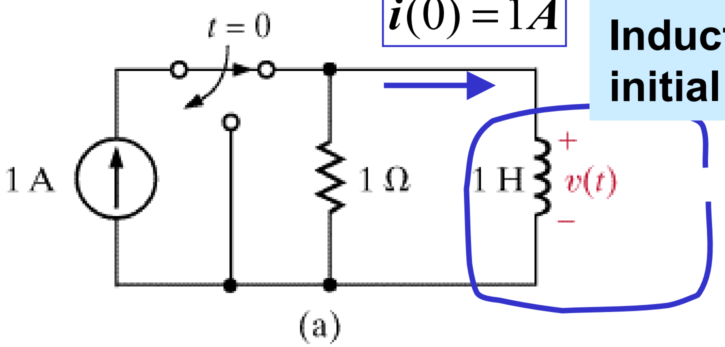
GEAUX

LEARNING BY DOING

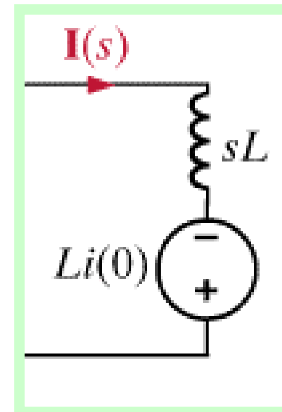
Determine the model in the s-domain and the expression for the voltage across the inductor

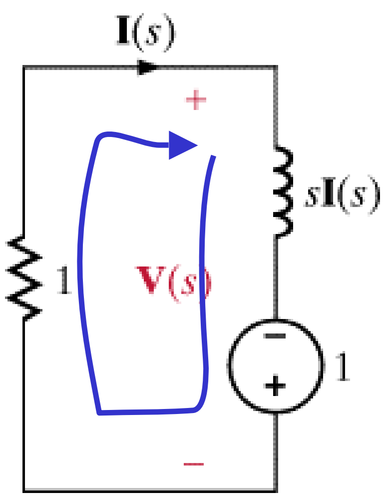
Steady state for t<0

$i(0) = 1A$



Inductor with initial current





KVL : $1 = (1+s)I(s)$

Ohm's Law

$$V(s) = -1 \times I(s) \Rightarrow V(s) = -\frac{1}{s+1}$$

Equivalent circuit in s-domain

GEAUX

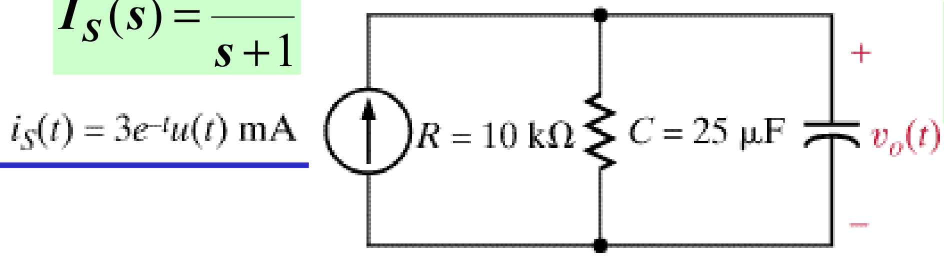
ANALYSIS TECHNIQUES

All the analysis techniques are applicable in the s-domain

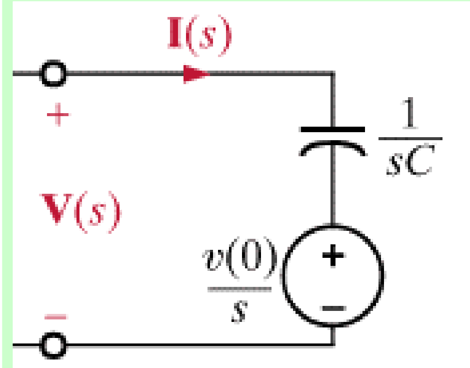
LEARNING EXAMPLE

Draw the s-domain equivalent and find the voltage in both s-domain and time domain

$$I_S(s) = \frac{3}{s+1}$$

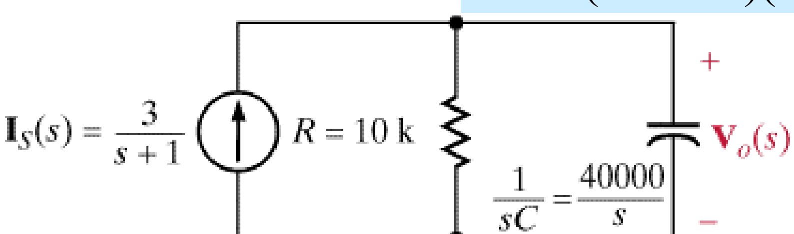


One needs to determine the initial voltage across the capacitor



$$i_S(t) = 0,\ t < 0 \Rightarrow v_o(0) = 0$$

$$RC = (10\times10^3)(25\times10^{-6}) = 0.25$$



$$V_o(s) = \left(R \,\|\, \frac{1}{Cs} \right) I_S(s)$$

$$V_o(s) = \frac{\frac{R}{Cs}}{R + \frac{1}{Cs}} I_S(s) = \frac{1/C}{s + 1/RC} \times \frac{3\times10^{-3}}{s+1}$$

$$V_o(s) = \frac{120}{(s+4)(s+1)} = \frac{K_1}{s+4} + \frac{K_2}{s+1}$$

$$K_1 = (s+4)V_o(s)\,|_{s=-4} = -40$$

$$K_2 = (s+1)V_o(s)\,|_{s=-1} = 40$$

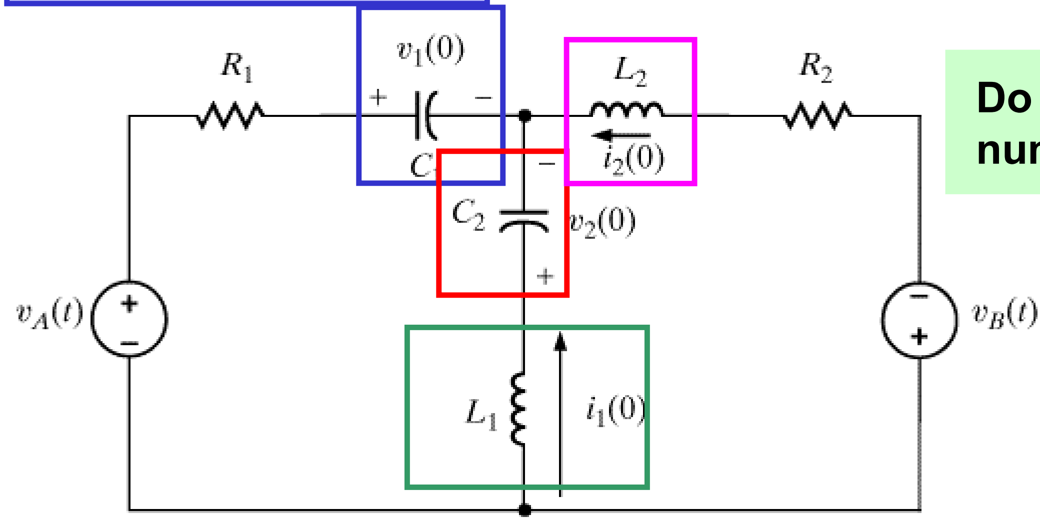
$$v_o(t) = 40\left[e^{-t} - e^{-4t}\right]u(t)$$

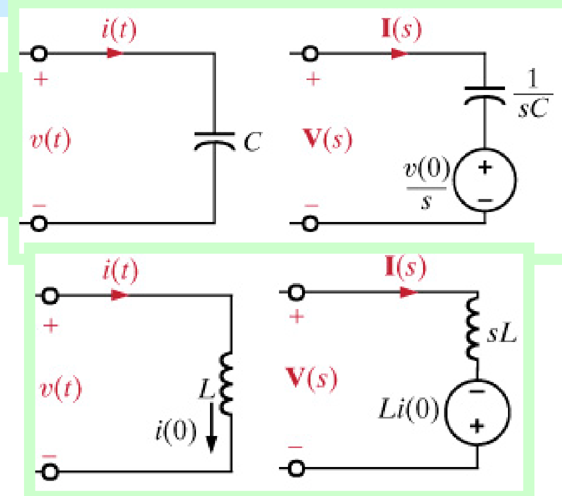
GEAUX

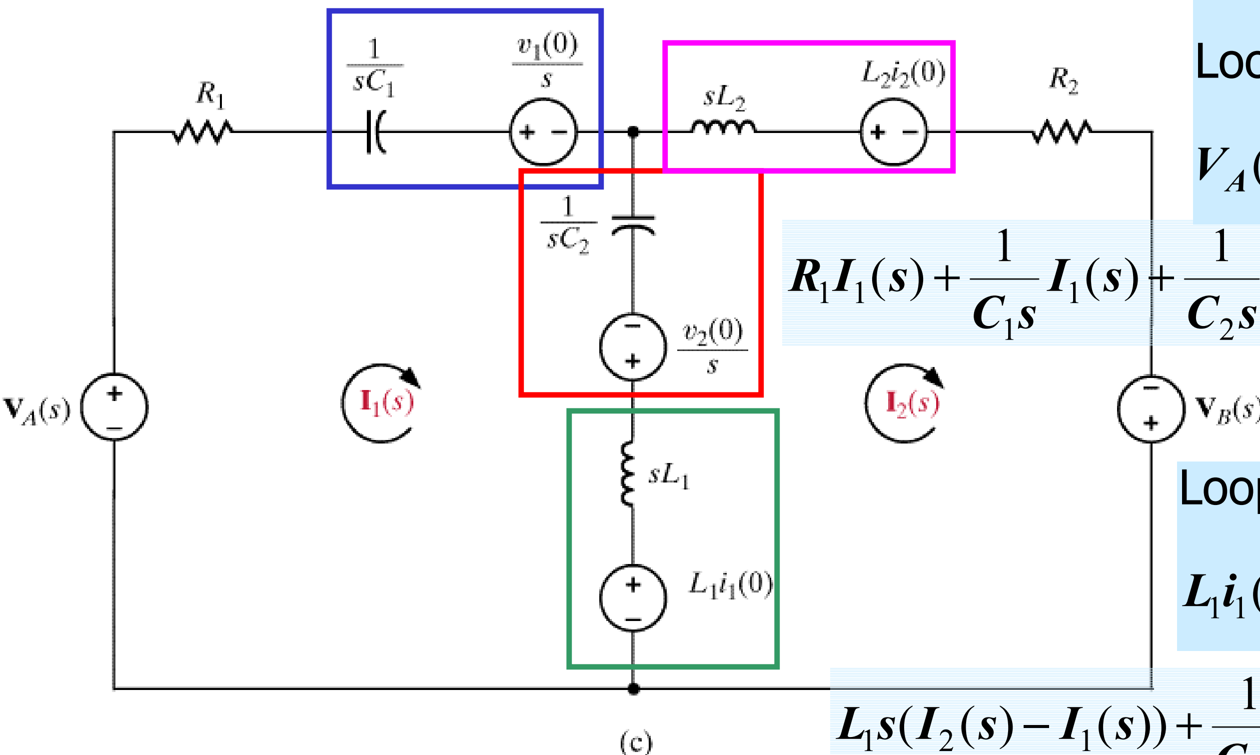
LEARNING EXAMPLE

Write the loop equations in the s-domain

Do not increase number of loops







Loop 1

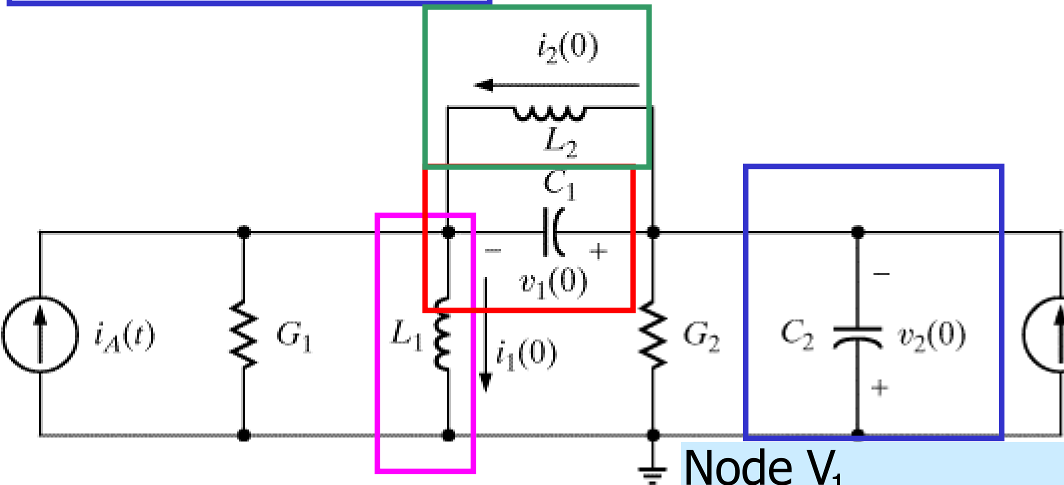
$$V_A(s) - \frac{v_1(0)}{s} + \frac{v_2(0)}{s} - L_1 i_1(0) = R_1 I_1(s) + \frac{1}{C_1 s} I_1(s) + \frac{1}{C_2 s}(I_1(s) - I_2(s)) + L_1 s(I_1(s) - I_2(s))$$

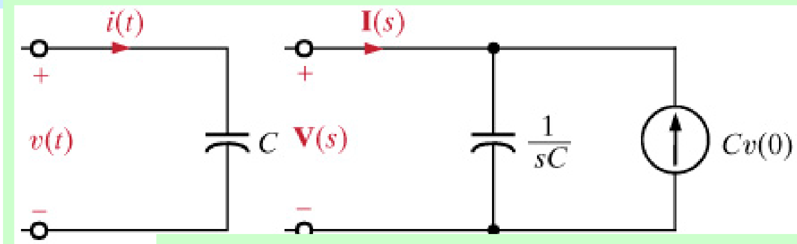
Loop 2

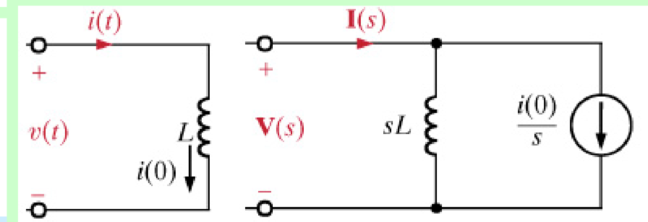
$$L_1 i_1(0) - \frac{v_2(0)}{s} - L_2 i_2(0) + V_B(s) = L_1 s(I_2(s) - I_1(s)) + \frac{1}{C_2 s}(I_2(s) - I_1(s)) + (L_2 s + R_2) I_2(s)$$

LEARNING EXAMPLE

Write the node equations in the s-domain





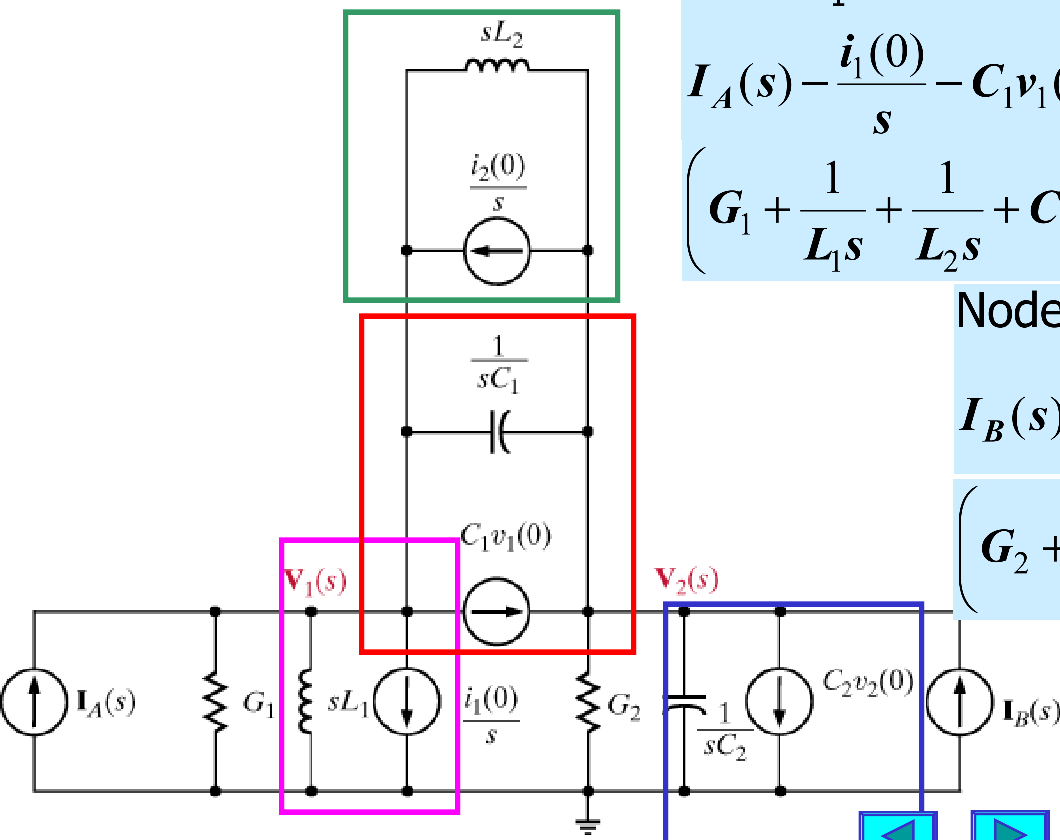


Do not increase number of nodes

Node V_1

$$I_A(s) - \frac{i_1(0)}{s} - C_1 v_1(0) + \frac{i_2(0)}{s} =$$

$$\left(G_1 + \frac{1}{L_1 s} + \frac{1}{L_2 s} + C_1 s\right)V_1(s) - \left(\frac{1}{L_2 s} + C_1 s\right)V_2(s)$$



Node V_2

$$I_B(s) - C_2 v_2(0) + C_1 v_1(0) - \frac{i_2(0)}{s} =$$

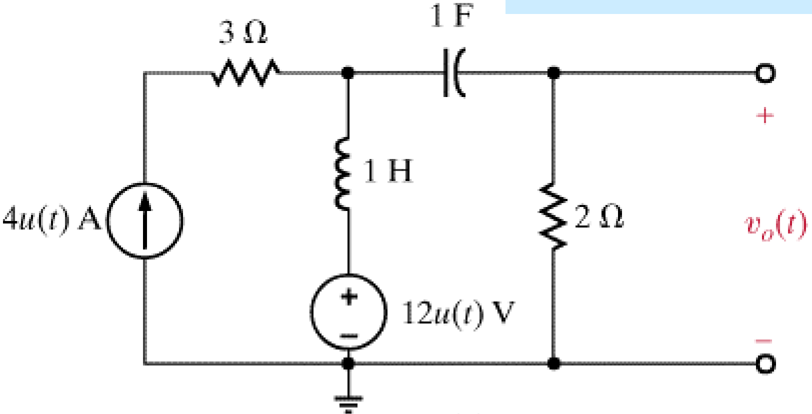
$$\left(G_2 + C_2 s + C_1 s + \frac{1}{L_2 s}\right)V_2(s) - \left(C_1 s + \frac{1}{L_2 s}\right)V_1(s)$$

GEAUX

LEARNING EXAMPLE

Find $v_o(t)$ using node analysis, loop analysis, superposition, source transformation, Thevenin's and Norton's theorem.

Assume all initial conditions are zero



(a)

KCL @ V_1

$$-\frac{4}{s}+\frac{V_1(s)-\frac{12}{s}}{s}+\frac{V_1(s)-V_o(s)}{\frac{1}{s}}=0 \quad \times s$$

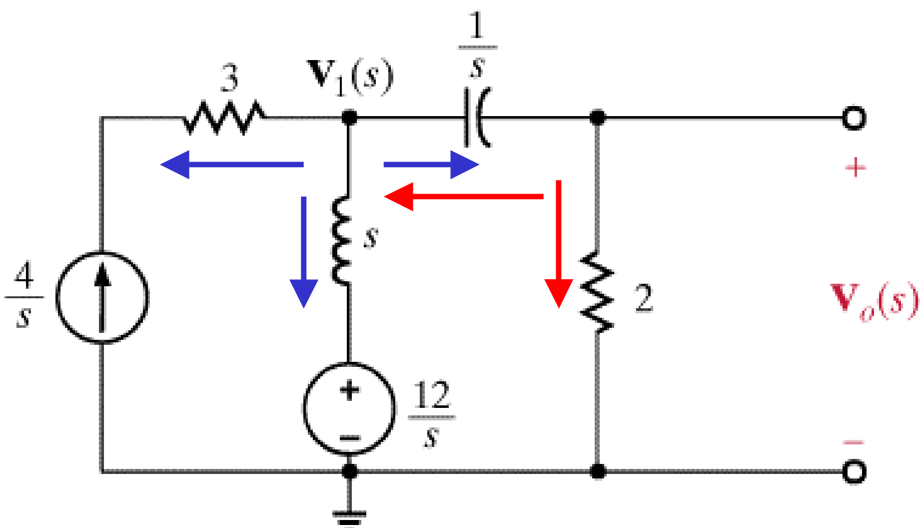
$$\frac{V_1(s)}{2+\frac{1}{s}}$$

KCL@V_o

$$\frac{V_o(s)}{2}+\frac{V_o(s)-V_1(s)}{\frac{1}{s}}=0 \quad \times 2$$

Could have used voltage divider here

Node Analysis



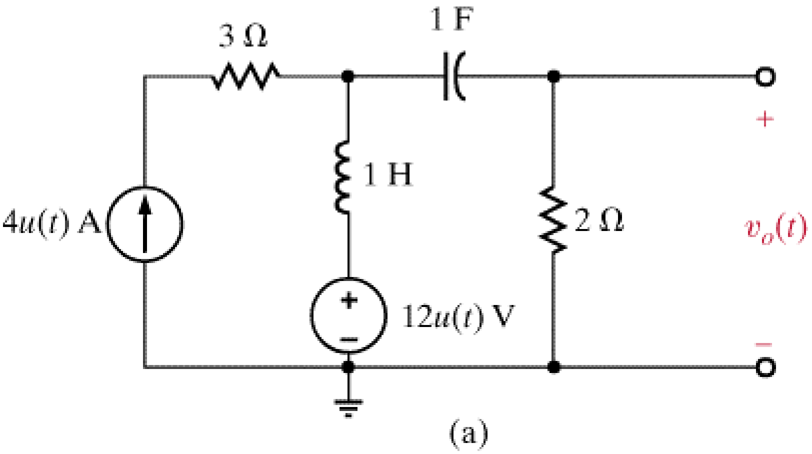
$$(1+s^2)V_1(s)-s^2V_o(s)=\frac{4s+12}{s} \quad \times 2s$$

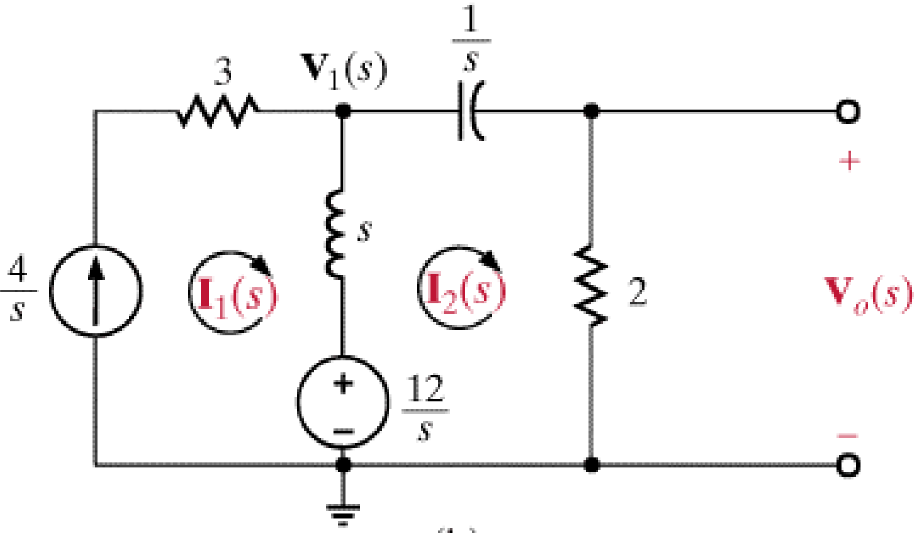
$$-2sV_1(s)+(1+2s)V_o(s)=0 \quad \times(1+s^2)$$

$$V_o(s)=\frac{8(s+3)}{(1+s)^2}$$

GEAUX

Loop Analysis





Loop 1

$$I_1(s) = \frac{4}{s}$$

Loop 2

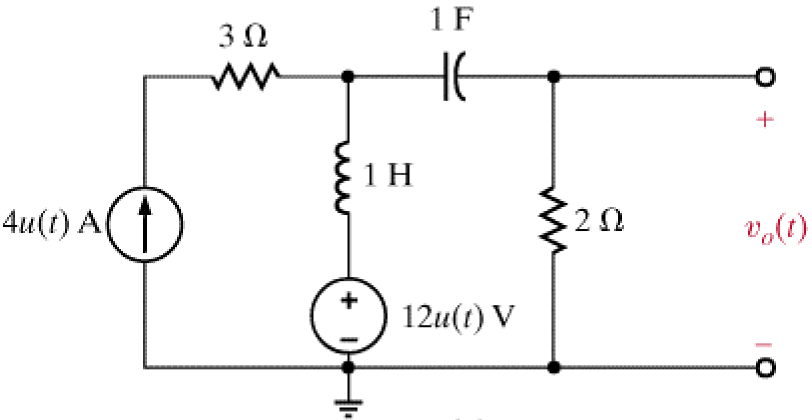
$$s(I_2(s) - I_1(s)) + \frac{1}{s} I_2(s) + 2I_2(s) = \frac{12}{s}$$

$$I_2(s) = \frac{4(s+3)}{(s+1)^2}$$

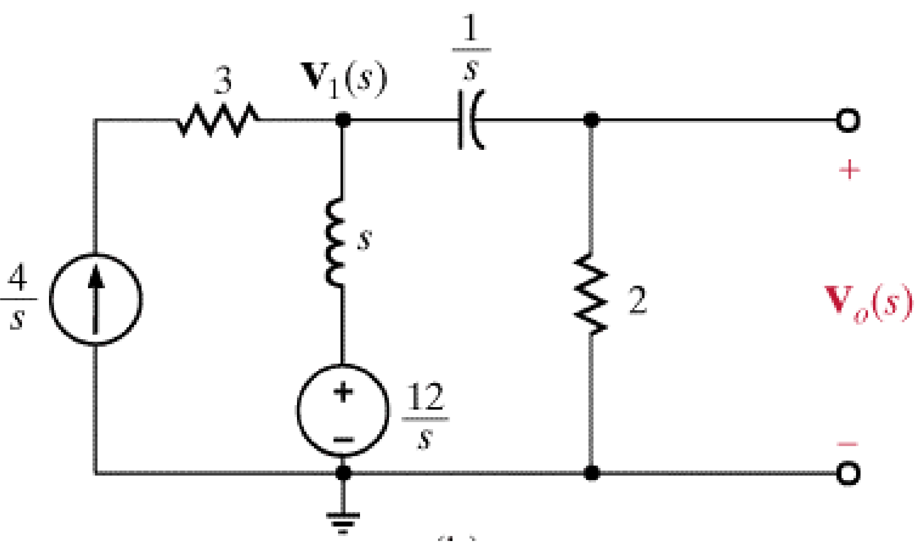
$$V_o(s) = 2I_2(s) = \frac{8(s+3)}{(s+1)^2}$$

GEAUX

Source Superposition

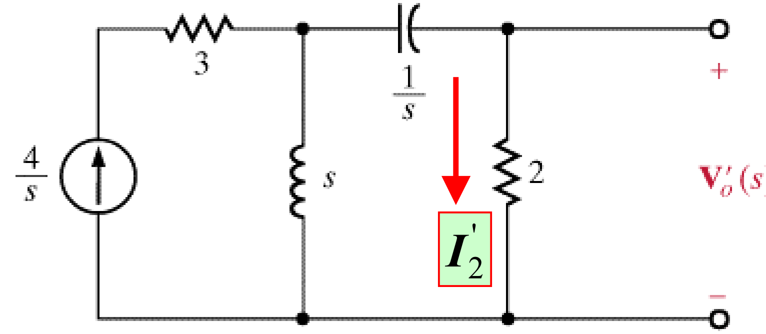


(a)



$$V_o(s) = V_o^{'}(s) + V_o^{"}(s) = \frac{8(s+3)}{(s+1)^2}$$

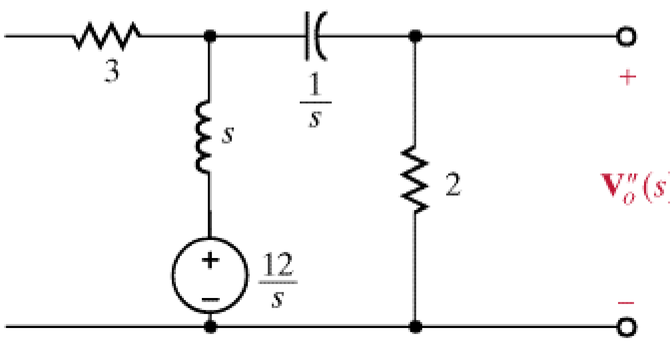
Applying current source



Current divider

$$V_o^{'}(s) = 2 \times \frac{s}{2+\frac{1}{s}+2} \times \frac{4}{s}$$

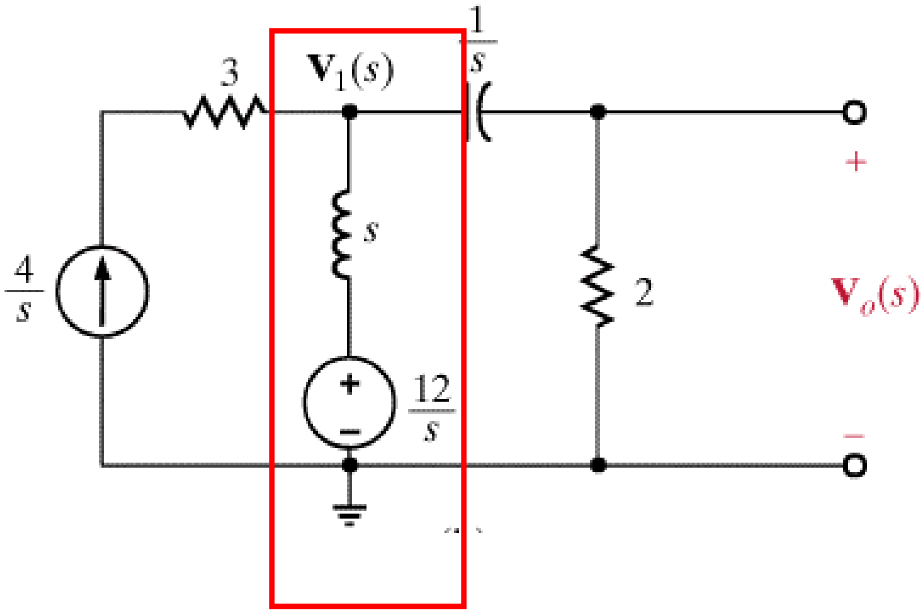
Applying voltage source

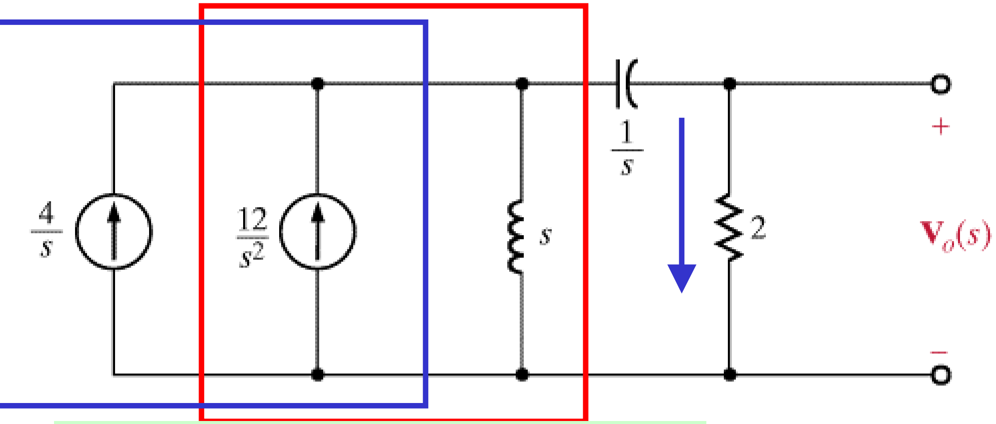


Voltage divider

$$V_o^{"}(s) = \frac{2}{2+\frac{1}{s}+s} \times \frac{12}{s}$$

Source Transformation





The resistance is redundant

Combine the sources and use current divider

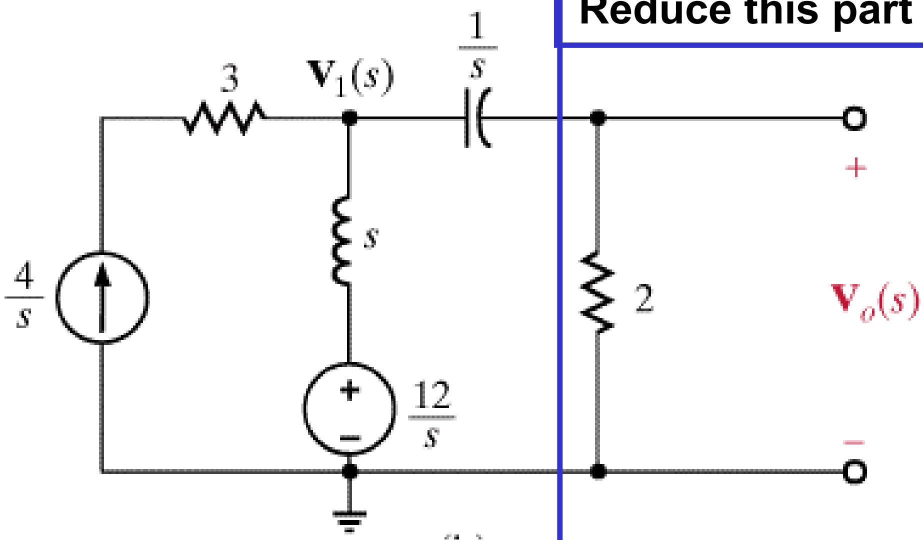
$$V_o(s) = 2 \times \frac{s}{s + \frac{1}{s} + 2}\left(\frac{4}{s} + \frac{12}{s^2}\right)$$

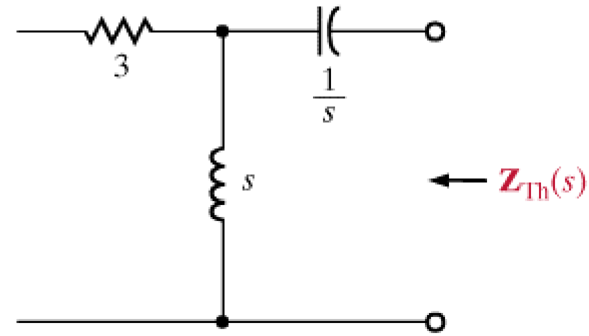
$$V_o(s) = \frac{8(s+3)}{(s+1)^2}$$

GEAUX

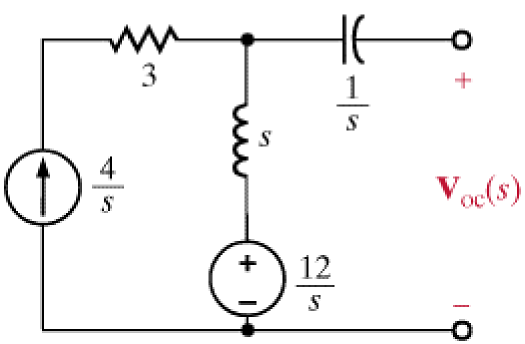
Using Thevenin's Theorem

Reduce this part



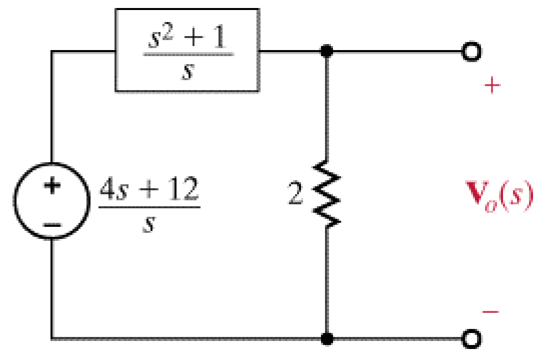


$$Z_{Th} = \frac{1}{s} + s = \frac{s^2+1}{s}$$



$$V_{OC}(s) = \frac{12}{s} + s\frac{4}{s} = \frac{4s+12}{s}$$

Only independent sources



Voltage divider

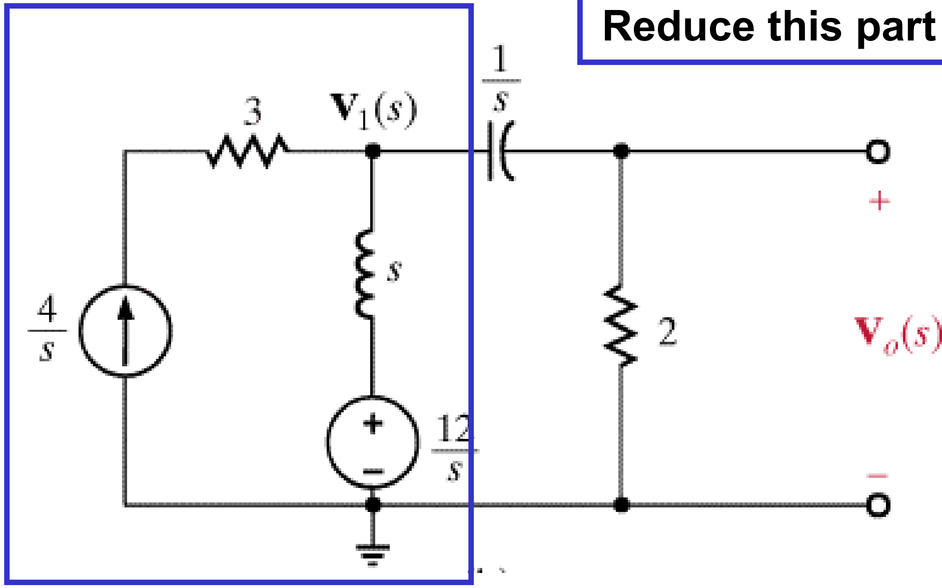
$$V_o(s) = \frac{2}{2+\frac{s^2+1}{s}}\,\frac{4s+12}{s}$$

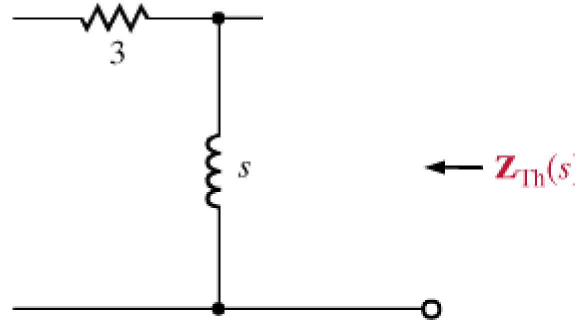
$$V_o(s) = \frac{8(s+3)}{(s+1)^2}$$

GEAUX

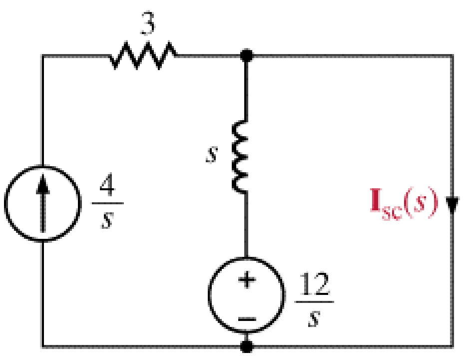
Using Norton's Theorem

Reduce this part

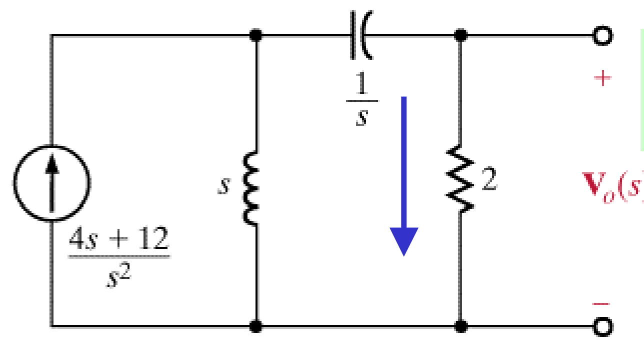




$$Z_{Th} = s$$



$$I_{SC}(s) = \frac{4}{s} + \frac{12/s}{s} = \frac{4s+12}{s^2}$$



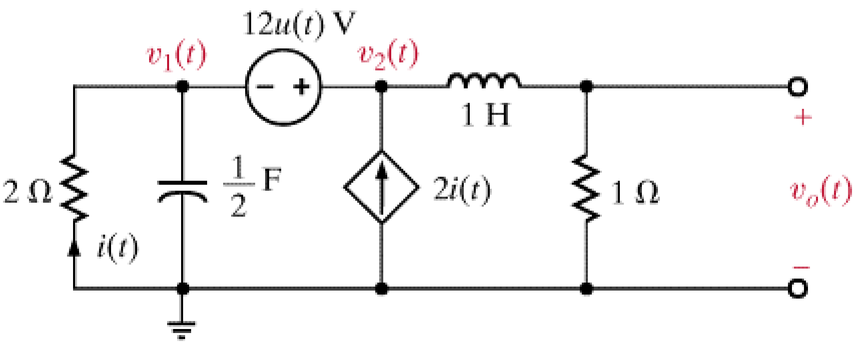
Current division

$$V_o(s) = 2 \times \frac{s}{s + \frac{1}{s} + 2} \frac{4s+12}{s^2}$$

$$V_o(s) = \frac{8(s+3)}{(s+1)^2}$$

GEAUX

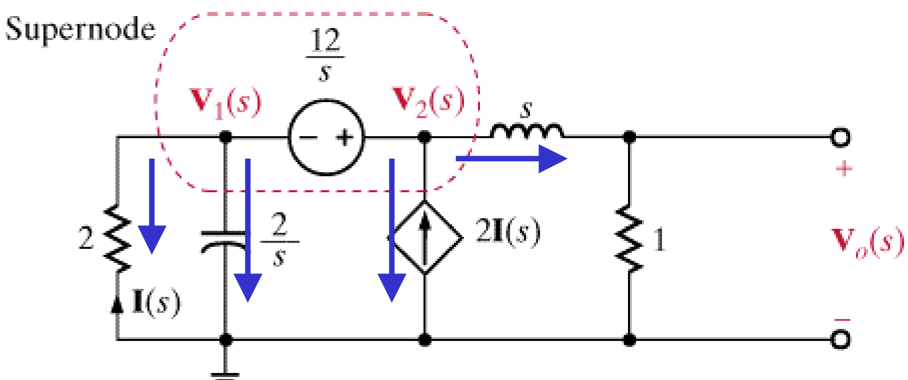
LEARNING EXAMPLE Determine the voltage $v_o(t)$. Assume all initial conditions to be zero



Selecting the analysis technique:

- . Three loops, three non-reference nodes
- . One voltage source between non-reference nodes - supernode
- . One current source. One loop current known or supermesh
- . If v_2 is known, v_o can be obtained with a voltage divider

Transforming the circuit to s-domain



Supernode constraint : $V_2(s) - V_1(s) = \dfrac{12}{s}$

KCL@ supernode : $\dfrac{V_1(s)}{2} + \dfrac{V_1(s)}{2/s} - 2I(s) + \dfrac{V_2(s)}{s+1} = 0$

Controlling variable : $I(s) = -\dfrac{V_1(s)}{2}$

Voltage divider : $V_0(s) = \dfrac{1}{s+1} V_2(s)$

Doing the algebra : $V_1(s) = V_2(s) - 12/s$

$$I(s) = -V_2(s)/2 + 6/s$$

$$(1/2)(s+1)\left(V_2(s) - 12/s\right) - 2(-V_2(s)/2 + 6/s) + V_2(s)/(s+1) = 0$$

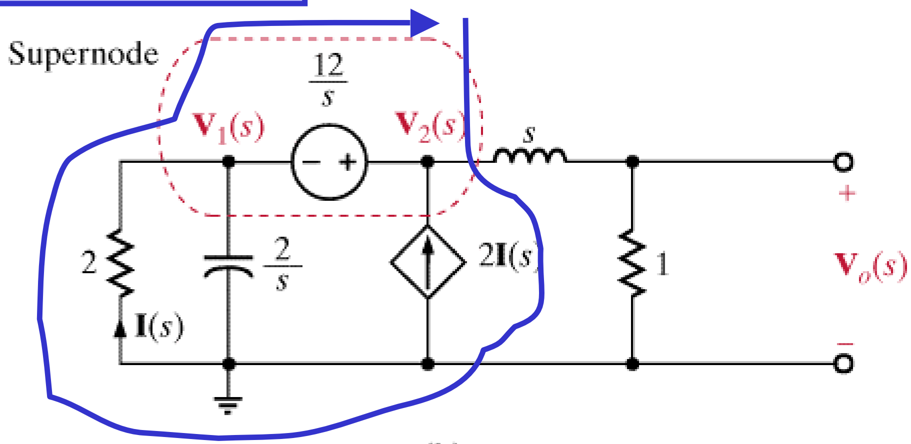
$$V_2(s) = \frac{12(s+1)(s+3)}{s(s^2+4s+5)}$$

$$V_o(s) = \frac{12(s+3)}{s(s^2+4s+5)}$$

Continued ...

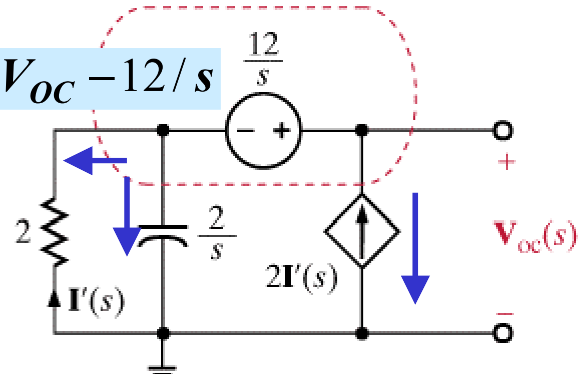
Compute $V_o(s)$ using Thevenin's theorem

- keep dependent source and controlling variable in the same sub-circuit
- Make sub-circuit to be reduced as simple as possible
- Try to leave a simple voltage divider after reduction to Thevenin equivalent



(b)

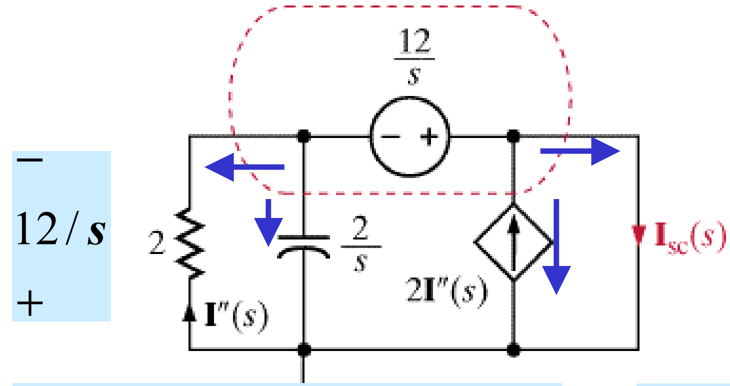
$V_{OC} - 12/s$



$$\frac{V_{OC} - 12/s}{2} + \frac{V_{OC} - 12/s}{2/s} - 2I' = 0$$

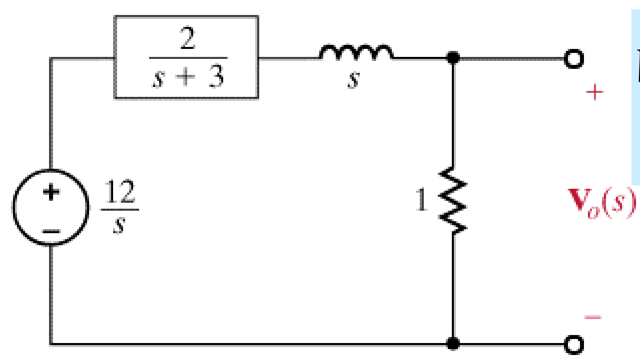
$$I' = -\frac{V_{OC} - 12/s}{2} \qquad V_{OC}(s) = \frac{12}{s}$$

$$-I' + (-2I')/(2/s) - 2I' = 0 \Rightarrow I' = 0$$



$$I_{SC} - 2I'' - I'' - 2I''/(2/s) = 0 \qquad I'' = 6/s$$

$$I_{SC} = \frac{6(s+3)}{s} \qquad Z_{TH} = \frac{V_{OC}(s)}{I_{SC}(s)} = \frac{2}{s+3}$$



(e)

$$V_o(s) = \frac{1}{1 + s + \frac{2}{s+3}} \times \frac{12}{s}$$

GEAUX

Continued …

Computing the inverse Laplace transform

Analysis in the s-domain has established that the Laplace transform of the output voltage is

$$V_o(s)=\frac{12(s+3)}{s(s^2+4s+5)}$$

$$s^2+4s+5=(s+2-j1)(s+2+j1)=(s+2)^2+1$$

$$V_o(s)=\frac{12(s+3)}{s(s+2-j1)(s+2+j1)}=\frac{K_o}{s}+\frac{K_1}{(s+2-j1)}+\frac{K_1^*}{(s+2+j1)}$$

$$K_o=sV_o(s)|_{s=0}=36/5$$

$$\frac{K_1}{(s+\alpha-j\beta)}+\frac{K_1^*}{(s+\alpha+j\beta)}\leftrightarrow 2|K_1|e^{-\alpha t}\cos(\beta t+\angle K_1)u(t)$$

$$K_1=(s+2-j1)V_o(s)|_{s=-2+j1}=\frac{12(1+j1)}{(-2+j1)(j2)}=\frac{12\sqrt{2}\angle 45^\circ}{\sqrt{5}\angle 153.43^\circ(2\angle 90^\circ)}$$

$$=3.79\angle -198.43^\circ=3.79\angle 161.57^\circ$$

One can also use quadratic factors...

$$v_o(t)=\left(\frac{36}{5}+7.59e^{-2t}\cos(t+161.57^\circ\right)u(t)$$

$$V_o(s)=\frac{12(s+3)}{s\left[(s+2)^2+1\right]}=\frac{C_o}{s}+\frac{C_1(s+2)}{(s+2)^2+1}+\frac{C_2}{(s+2)^2+1}$$

$$C_o=sV_o(s)|_{s=0}=36/5$$

$$\frac{C_1(s+\alpha)}{(s+\alpha)^2+\beta^2}+\frac{C_2\beta}{(s+\alpha)^2+\beta^2}\leftrightarrow e^{-\alpha t}[C_1\cos\beta t+C_2\sin\beta t]u(t)$$

$$12(s+3)=C_o((s+2)^2+1)+s[C_1(s+2)+C_2]$$

$$s=-2\Rightarrow 12=C_o-2C_2\Rightarrow C_2=36/10-6=-12/5$$

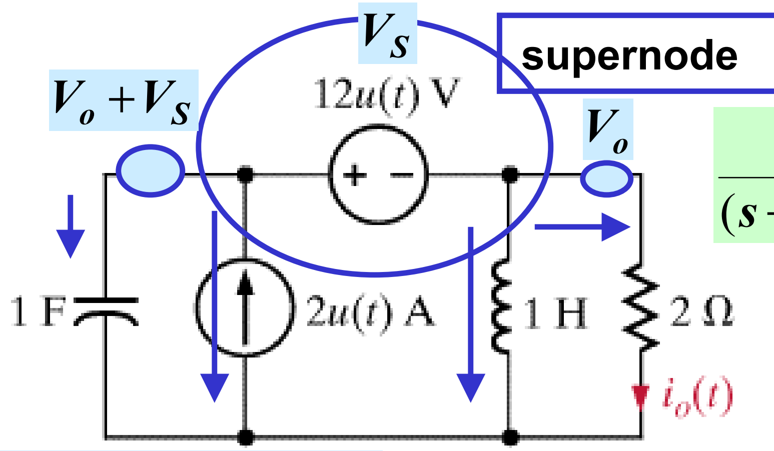
Equating coefficien ts of $s^2: 0=C_o+C_1\Rightarrow C_1=-36/5$

$$v_o(t)=\left[\frac{36}{5}(1-e^{-2t}\cos t)-\frac{12}{5}e^{-2t}\sin t\right]u(t)$$

GEAUX

LEARNING EXTENSION

Find $i_o(t)$ using node equations



Assume zero initial conditions
Implicit circuit transformation to s-domain

$$\frac{K_1}{(s+\alpha-j\beta)}+\frac{K_1^*}{(s+\alpha+j\beta)}\leftrightarrow 2|K_1|e^{-\alpha t}\cos(\beta t+\angle K_1)u(t)$$

KCL at supernode

$$Cs(V_o(s)+V_S(s))-\frac{2}{s}+\frac{V_o(s)}{Ls}+\frac{V_o(s)}{2}=0$$

$$V_S(s)=\frac{12}{s},\quad I_o(s)=\frac{V_o(s)}{2}$$

Doing the algebra

$$I_o(s)=\frac{1-6s}{s^2+0.5s+1}=\frac{1-6s}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

$$I_o(s)=\frac{1-6s}{\left(s+\frac{1}{4}-j\frac{\sqrt{15}}{4}\right)\left(s+\frac{1}{4}+j\frac{\sqrt{15}}{4}\right)}=\frac{K_1}{\left(s+\frac{1}{4}-j\frac{\sqrt{15}}{4}\right)}+\frac{K_1^*}{\left(s+\frac{1}{4}+j\frac{\sqrt{15}}{4}\right)}$$

$$K_1=\left(s+\frac{1}{4}-j\frac{\sqrt{15}}{4}\right)I_o(s)\Big|_{s=-\frac{1}{4}+j\frac{\sqrt{15}}{4}}=\frac{1-6\left(-\frac{1}{4}+j\frac{\sqrt{15}}{4}\right)}{2j\frac{\sqrt{15}}{4}}$$

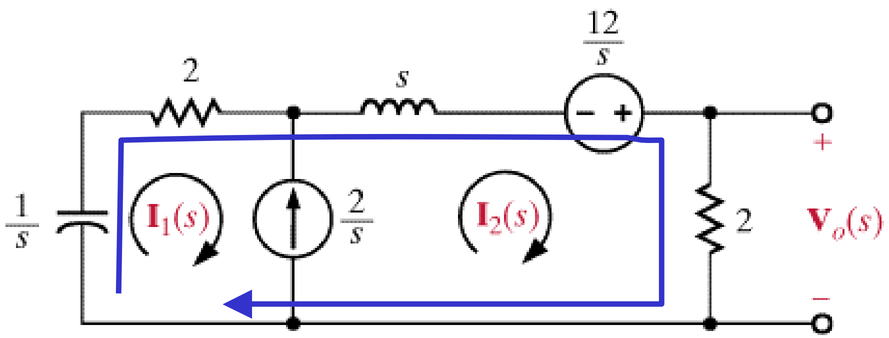
$$K_1=\frac{6.33\angle-66.72^\circ}{0.97\angle 90^\circ}=6.53\angle-156.72^\circ$$

$$i_o(t)=13.06e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}}{4}t-156.72^\circ\right)$$

GEAUX

LEARNING EXTENSION

Find $v_o(t)$ using loop equations



supermesh

constraint due to source

$$\frac{2}{s} = I_2 - I_1$$

KVL on supermesh

$$\frac{1}{s}I_1 + 2I_1 + sI_2 - \frac{12}{s} + 2I_2 = 0$$

Solve for I2

$$I_2(s) = \frac{16s+2}{s(s^2+4s+1)} = \frac{16s+2}{s(s+0.27)(s+3.73)}$$

Determine inverse transform

$$I_2(s) = \frac{K_0}{s} + \frac{K_1}{s+0.27} + \frac{K_2}{s+3.73}$$

$$K_0 = sI_2(s)|_{s=0} = 2$$

$$K_1 = (s+0.27)I_2(s)|_{s=-0.27} = \frac{16(-0.27)+2}{(-0.27)(-0.27+3.73)} = 2.48$$

$$K_2 = (s+3.73)I_2(s)|_{s=-3.73} = \frac{16(-3.73)+2}{(-3.73)(-3.73+0.27)} = -4.47$$

$$i_2(t) = \left(2 + 2.48e^{-0.27t} - 4.47e^{-3.73t}\right)u(t)$$

$$v_o(t) = 2i_2(t)$$

GEAUX

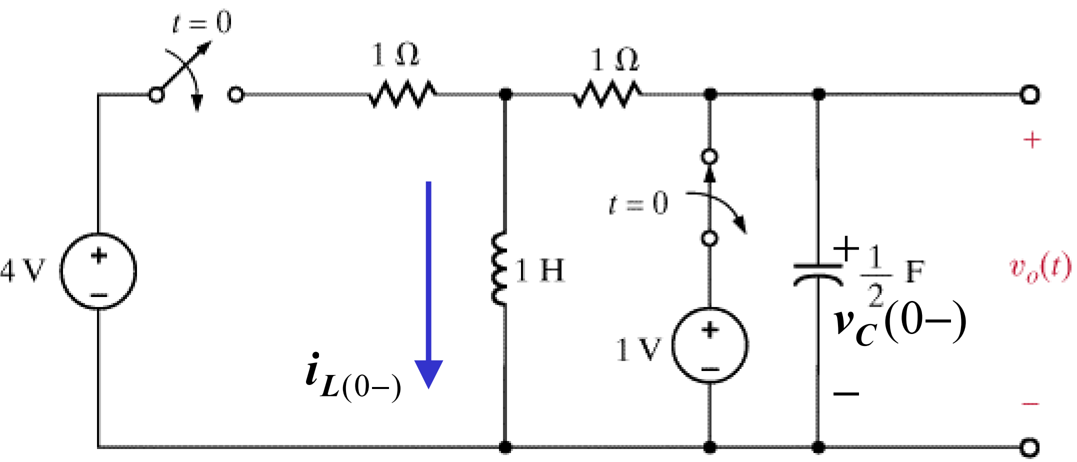
TRANSIENT CIRCUIT ANALYSIS USING LAPLACE TRANSFORM

For the study of transients, especially transients due to switching, it is important to determine initial conditions. For this determination, one relies on the properties:

1. **Voltage across capacitors cannot change discontinuously**
2. **Current through inductors cannot change discontinuously**

LEARNING EXAMPLE

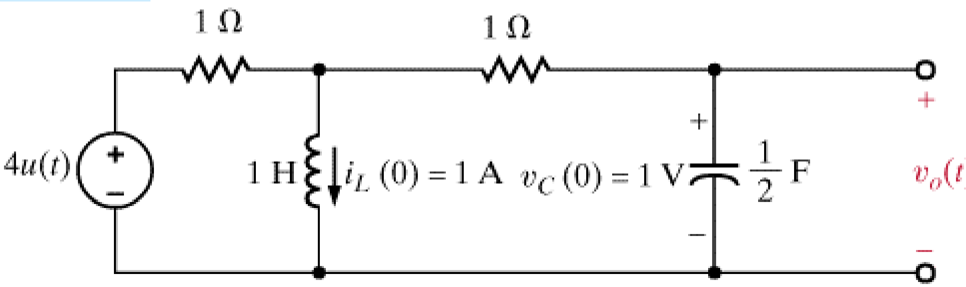
Determine $v_o(t), t > 0$



Assume steady state for t<0 and determine voltage across capacitors and currents through inductors

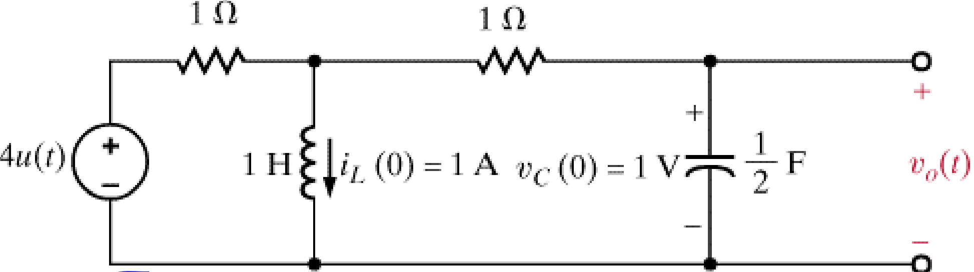
For DC case capacitors are open circuit
inductors are shortcircuit

$v_C(0-) = 1V,\ i_L(0-) = 1A$



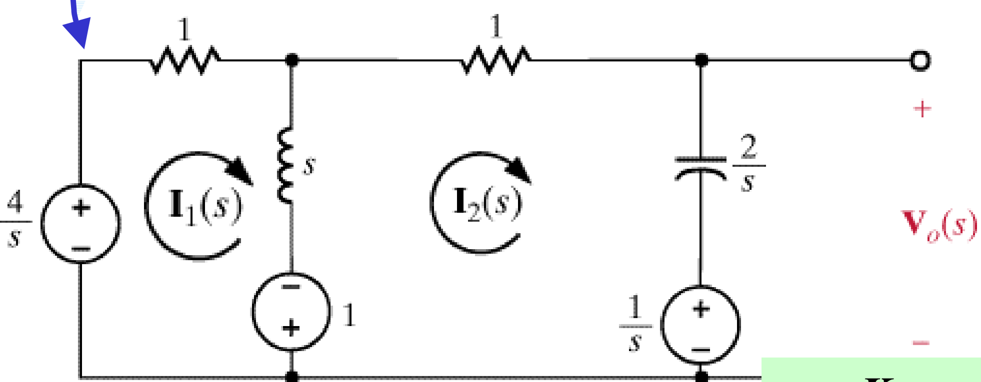
Circuit for t>0

GEAUX



Laplace

Circuit for t>0



Use mesh analysis

$$(s+1)I_1 - sI_2 = \frac{4}{s} + 1 \quad \times s$$

Solve for I2

$$-sI_1 + (s+1+\frac{2}{s})I_2 = -\frac{1}{s} - 1 \quad \times (s+1)$$

$$I_2(s) = \frac{2s-1}{2s^2+3s+2}$$

$$V_o(s) = \frac{2}{s}I_2(s) + \frac{1}{s}$$

$$V_o(s) = \frac{2s+7}{2s^2+3s+2}$$

Now determine the inverse transform

$b^2 - 4ac < 0 \Rightarrow$ complex conjugate roots

$$V_o(s) = \frac{K_1}{s + \frac{3}{4} - j\frac{\sqrt{7}}{4}} + \frac{K_1^*}{s + \frac{3}{4} + j\frac{\sqrt{7}}{4}}$$

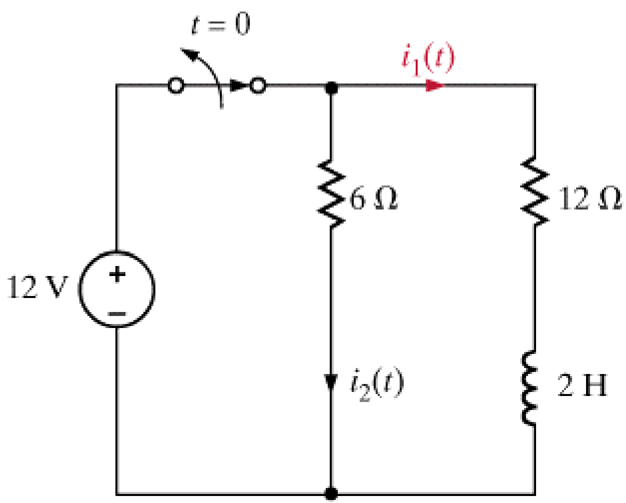
$$K_1 = \left(s + \frac{3}{4} - j\frac{\sqrt{7}}{4}\right)V_o(s)\Big|_{s=-\frac{3}{4}+j\frac{\sqrt{7}}{4}} = 2.14\angle -76.5°$$

$$\frac{K_1}{(s+\alpha-j\beta)} + \frac{K_1^*}{(s+\alpha+j\beta)} \leftrightarrow 2|K_1|e^{-\alpha t}\cos(\beta t + \angle K_1)u(t)$$

$$v_o(t) = 4.28\cos(\frac{\sqrt{7}}{4}t - 76.5°)$$

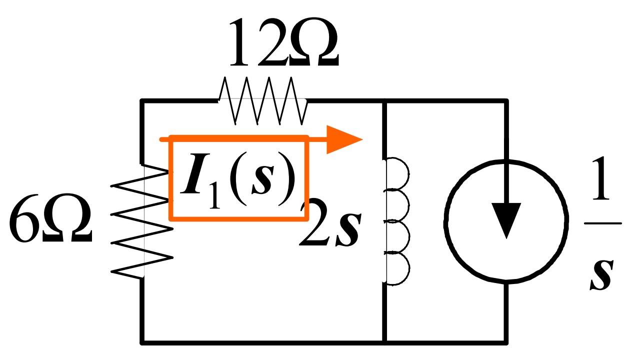
GEAUX

LEARNING EXTENSION Determine $i_1(t),\ t>0$



Initial current through inductor

$$i_L(0-) = i_L(0+) = 1A$$



$$I_1(s) = \frac{2s}{2s+18} \times \frac{1}{s}$$

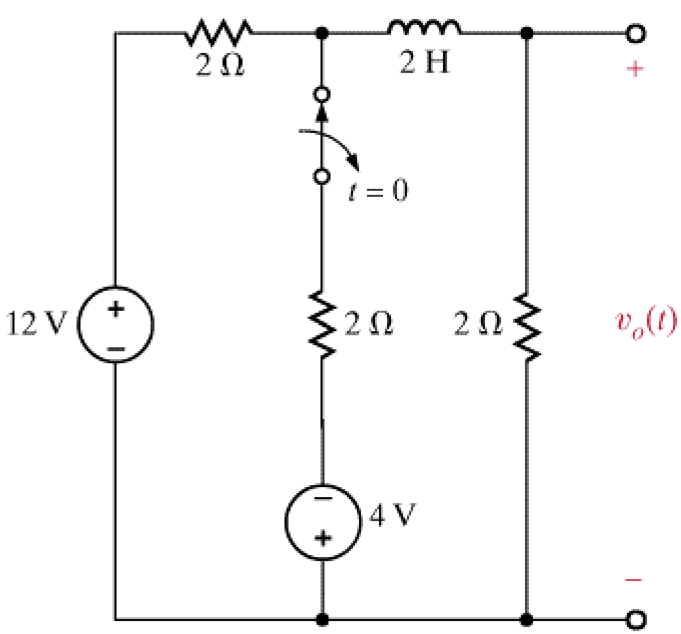
Current divider

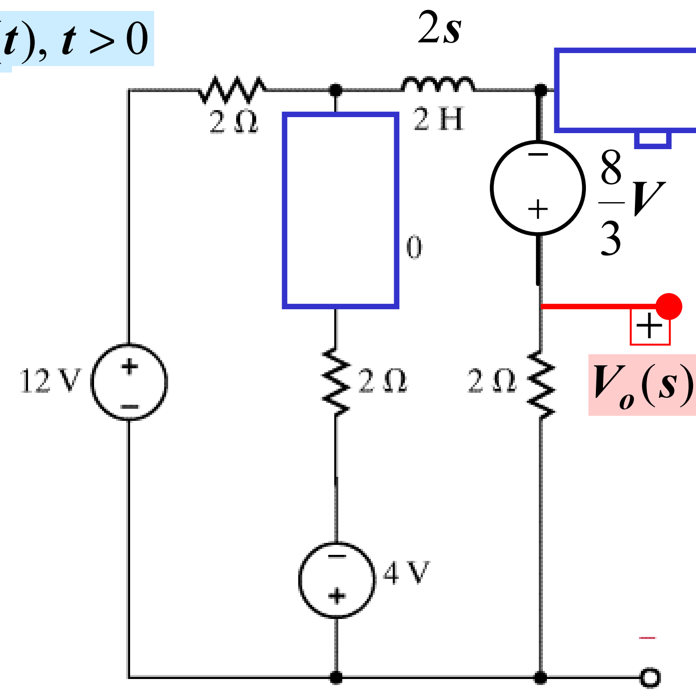
$$I_1(s) = \frac{s}{s+9} \to i_1(t) = e^{-9t}u(t)$$

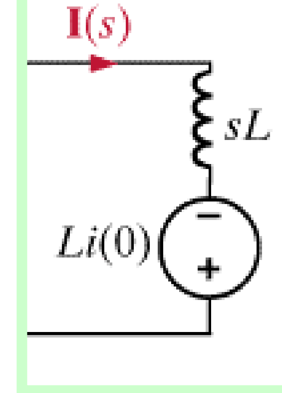
GEAUX

LEARNING EXTENSION

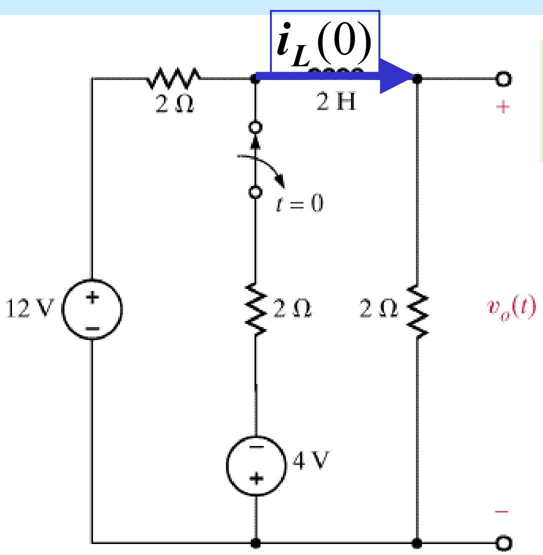
Determine $v_o(t),\ t > 0$







Determine initial current through inductor



Use source superposition

$$i_{12V} = 2A$$

$$i_{4V} = -\frac{2}{3}A$$

$$i_L(0) = \frac{4}{3}A$$

$$V_o(s) = \frac{2}{4+2s} \times \left(\frac{12}{s} + \frac{8}{3}\right) \quad \text{(voltage divider)}$$

$$V_o(s) = \frac{(8s+36)}{3s(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+2}$$

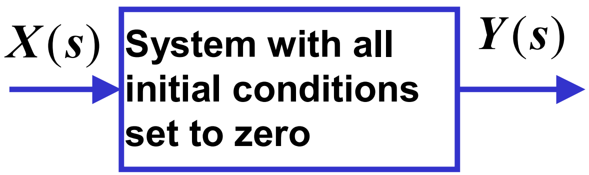
$$K_1 = sV_o(s)|_{s=0} = 6$$

$$K_2 = (s+2)V_o(s)|_{s=-2} = -\frac{10}{3}$$

$$v_o(t) = \left(6 - \frac{8}{3}e^{-2t}\right)u(t)$$

GEAUX

TRANSFER FUNCTION



$$H(s) = \frac{Y(s)}{X(s)}$$

If the model for the system is a differential equation

$$b_n \frac{d^n y}{dt^n} + b_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + b_1 \frac{dy}{dt} + b_o y$$

$$= a_m \frac{d^m x}{dt^m} + a_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x$$

If all initial conditions are zero

$$\mathrm{L}\left[\frac{d^k y}{dt^k}\right] = s^k Y(s)$$

$$b_n s^n Y(s) + \ldots + b_1 s Y(s) + b_0 Y(s)$$

$$= a_m s^m X(s) + \ldots + a_1 s X(s) + a_0 X(s)$$

$$Y(s) = \frac{b_n s^n + \ldots + b_1 s + b_0}{a_m s^m + \ldots + a_1 s + a_0} X(s)$$

$$H(s) = \frac{b_n s^n + \ldots + b_1 s + b_0}{a_m s^m + \ldots + a_1 s + a_0}$$

For the impulse function

$$x(t) = \delta(t) \Rightarrow X(s) = 1$$

H(s) can also be interpreted as the Laplace transform of the output when the input is an impulse and all initial conditions are zero

The inverse transform of H(s) is also called the impulse response of the system

If the impulse response is known then one can determine the response of the system to ANY other input

LEARNING EXAMPLE A network has impulse response $h(t) = e^{-t}u(t)$

Determine the response, $v_o(t)$, for the input $v_i(t) = 10e^{-2t}u(t)$

In the Laplace domain, *Y(s)=H(s)X(s)*

$$\therefore V_o(s) = H(s)V_i(s)$$

$$h(t) = e^{-t}u(t) \Rightarrow H(s) = \frac{1}{s+1}$$

$$v_i(t) = 10e^{-2t}u(t) \Rightarrow V_i(s) = \frac{10}{s+2}$$

$$V_o(s) = \frac{10}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$K_1 = (s+1)V_o(s)|_{s=-1} = 10$$

$$K_2 = (s+2)V_o(s)|_{s=-2} = -10$$

$$v_o(t) = 10\left(e^{-t} - e^{-2t}\right)u(t)$$

GEAUX

Impulse response of first and second order systems

First order system $H(s)=\dfrac{K\tau}{\tau s+1}\Rightarrow h(t)=Ke^{-\frac{t}{\tau}}$

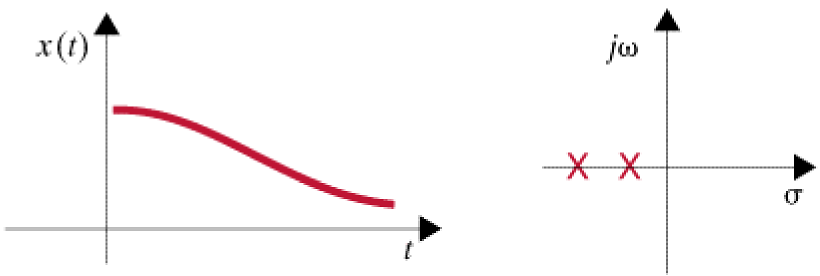
Normalized second order system

$$H(s)=\frac{\omega_0^2}{s^2+2\varsigma\omega_0 s+\omega_0^2}$$

poles: $s_{1,2}=-\varsigma\omega_0\pm\omega_0\sqrt{\varsigma^2-1}$

Case 1: $\varsigma>1$: Overdamped network

$$h(t)=K_1e^{-(\varsigma\omega_0+\omega_0\sqrt{\varsigma^2-1})t}+K_2e^{-(\varsigma\omega_0-\omega_0\sqrt{\varsigma^2-1})t}$$

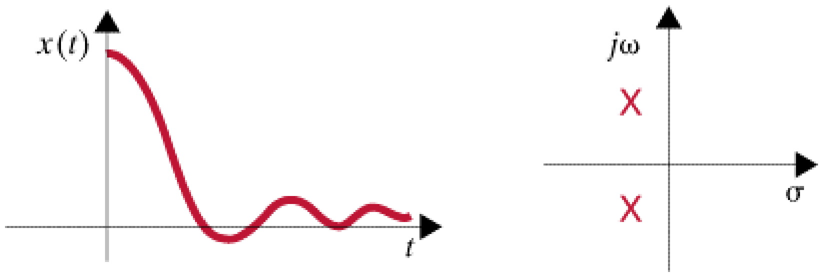


(a)

Case 2: $\varsigma<1$: Underdamped network

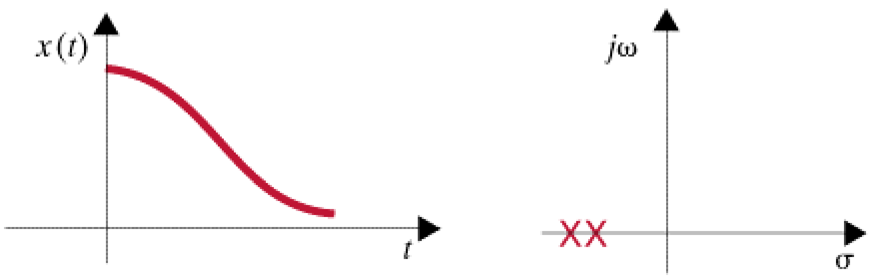
poles: $s_{1,2}=-\varsigma\omega_0\pm j\omega_0\sqrt{1-\varsigma^2}$

$$h(t)=Ke^{-\varsigma\omega_o t}\cos(\omega_o\sqrt{1-\varsigma^2}t+\phi)$$



Case 3: $\varsigma=1$: Critically damped network

$$h(t)=K_1te^{-\omega_o t}+K_2e^{-\omega_o t}$$

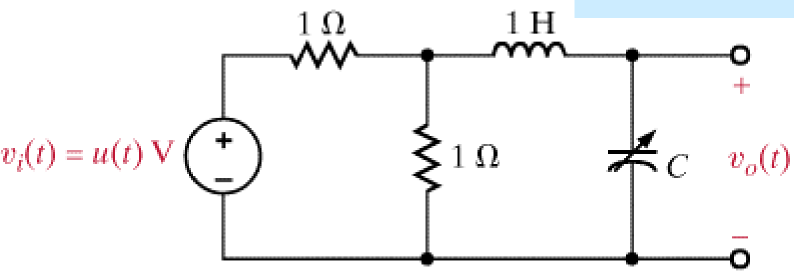


(c)

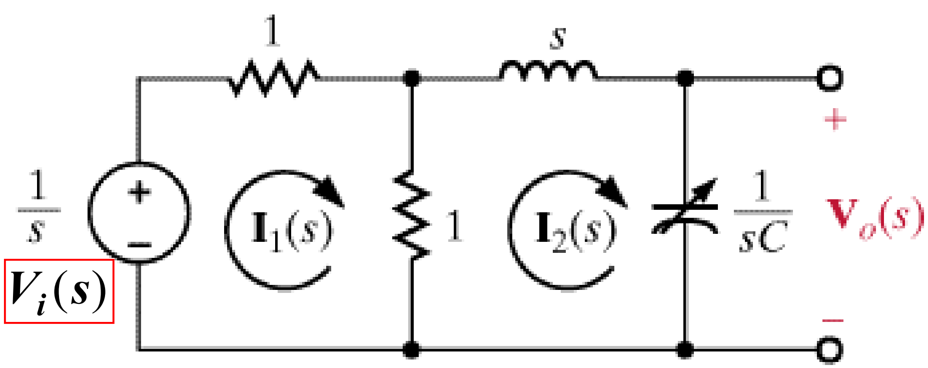
GEAUX

LEARNING EXAMPLE

Determine the transfer function $H(s) = \dfrac{V_o(s)}{V_i(s)}$



Transform the circuit to the Laplace domain. All initial conditions set to zero



Mesh analysis

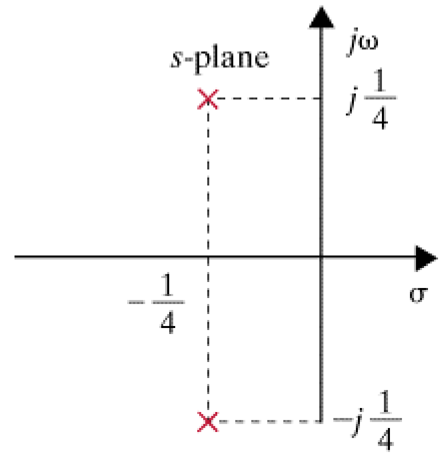
$$V_i(s) = 2I_1 - I_2$$

$$0 = -I_1 + \left(1 + s + \frac{1}{sC}\right) I_2$$

$$V_o(s) = \frac{1}{sC} I_2(s)$$

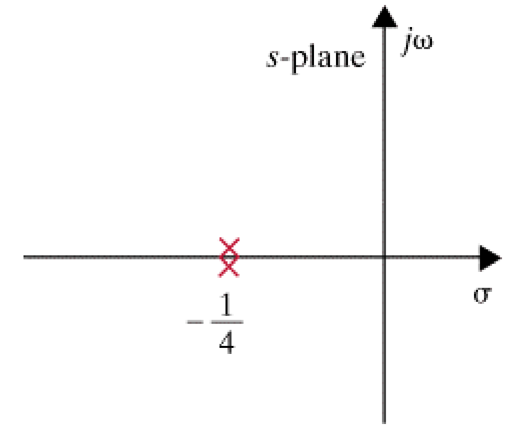
$$V_o(s) = \frac{(1/2C)}{s^2 + (1/2)s + 1/C}$$

a) $C = 8\text{F} \Rightarrow \text{poles}: s_{1,2} = -0.25 \pm j0.25$

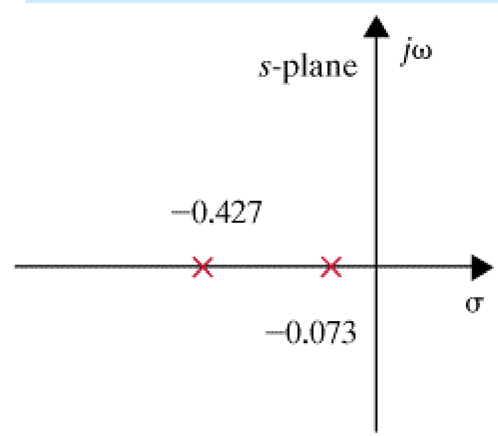


(c)

b) $C = 16\text{F} \Rightarrow \text{poles}: s_{1,2} = -0.25$



c) $C = 32\text{F} \Rightarrow \text{poles}: s_{1,2} = -0.427, -0.073$

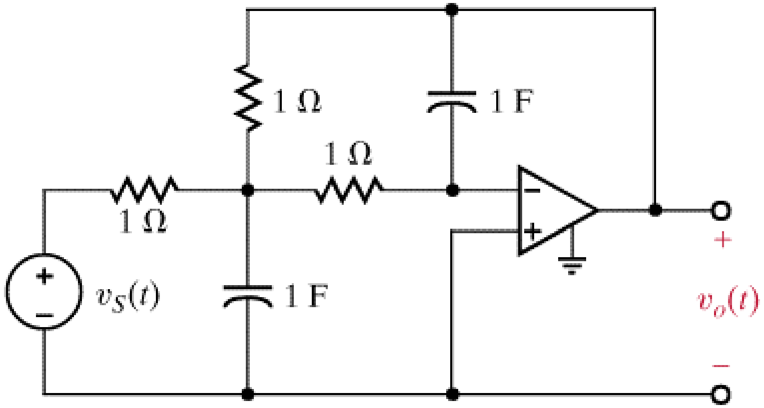


(e)

GEAUX

LEARNING EXAMPLE

Determine the transfer function, the type of damping and the unit step response



Transform the circuit to the Laplace domain. All initial conditions set to zero

$$\frac{V_1(s)-V_S(s)}{1}+\frac{V_1(s)}{1/s}+\frac{V_1(s)}{1}+\frac{V_1(s)-V_0(s)}{1}=0$$

$$\frac{V_o(s)}{V_S(s)}=\frac{\frac{1}{32}}{s^2+\frac{1}{2}s+\frac{1}{16}}$$

$$\omega_o^2 \Rightarrow \omega_0=0.25$$

$$2\varsigma\omega_o \Rightarrow \varsigma=1$$

Unit step response $\Rightarrow V_S(s)=\frac{1}{s}$

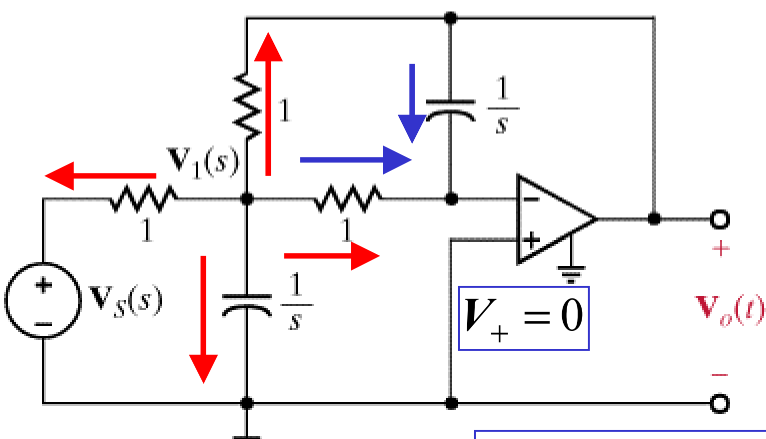
$$V_o(s)=\frac{(1/32)}{s\left(s+\frac{1}{4}\right)^2}=\frac{K_o}{s}+\frac{K_{11}}{s+0.25}+\frac{K_{12}}{(s+0.25)^2}$$

$$K_o=sV_o(s)|_{s=0}=0.5$$

$$K_{12}=(s+0.25)^2V_o(s)|_{s=-0.25}=0.125$$

$$K_{11}=\left.\frac{d\left[s^2V_o(s)\right]}{ds}\right|_{s=-0.25}=-0.5$$

$$v_o(t)=\left(0.5-(0.125t+0.5)e^{-0.25t}\right)u(t)$$



(b)

$$V_1(s)=-sV_o(s) \Leftarrow \frac{V_1(s)}{1}+\frac{V_o(s)}{1/s}=0$$

GEAUX

LEARNING EXTENSION

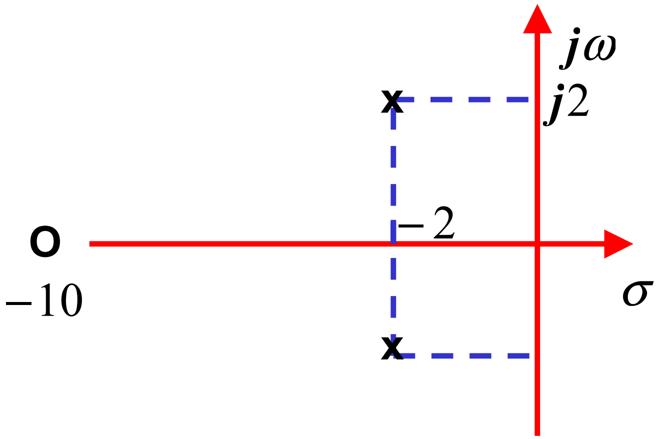
Determine the pole-zero plot, the type of damping and the unit step response

$$H(s)=\frac{s+10}{s^2+4s+8}$$

zero: $z=-10$

poles:

$$s^2+4s+8=0 \Rightarrow s_{1,2}=-2\pm j2$$



$$s^2+\boxed{4}\,s+\boxed{8}$$

$4 = 2\varsigma\omega_o$, $8 = \omega_o^2$

$$\Rightarrow \varsigma=\frac{\sqrt{2}}{2}$$

$$Y(s)=H(s)\frac{1}{s}=\frac{s+10}{s(s^2+4s+8)}$$

$$s^2+4s+8=(s+2-j2)(s+2+j2)$$

$$Y(s)=\frac{K_1}{s}+\frac{K_2}{s+2-j2}+\frac{K_2^*}{s+2+j2}$$

$$\frac{K_1}{(s+\alpha-j\beta)}+\frac{K_1^*}{(s+\alpha+j\beta)}\leftrightarrow 2|K_1|e^{-\alpha t}\cos(\beta t+\angle K_1)u(t)$$

$$K_1=sY(s)|_{s=0}=\frac{10}{8}$$

$$K_2=(s+2-j2)V_o(s)|_{s=-2+j2}=\frac{8+j2}{(-2+j2)(j4)}$$

$$K_2=\frac{8.25\angle 14}{2.83\angle 135^\circ\times 4\angle 90^\circ}=0.73\angle-211^\circ$$

$$v_o(t)=\left(\frac{10}{8}+1.46\cos(2t-211^\circ)\right)$$

GEAUX

Second order networks: variation of poles with damping ratio

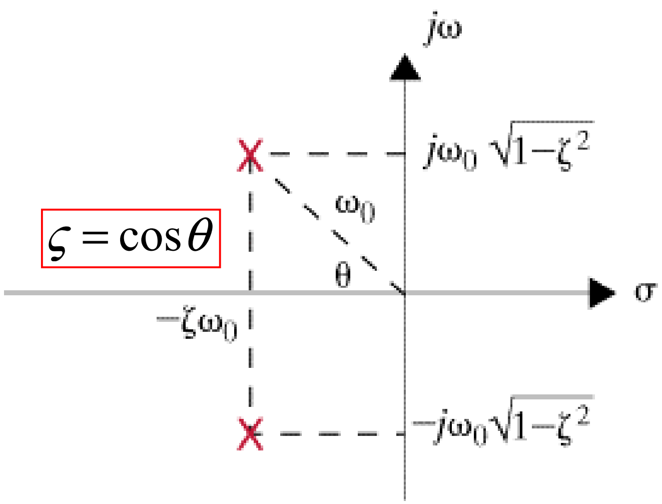
Normalized second order system

$$H(s) = \frac{\omega_0^2}{s^2 + 2\varsigma\omega_0 s + \omega_0^2}$$

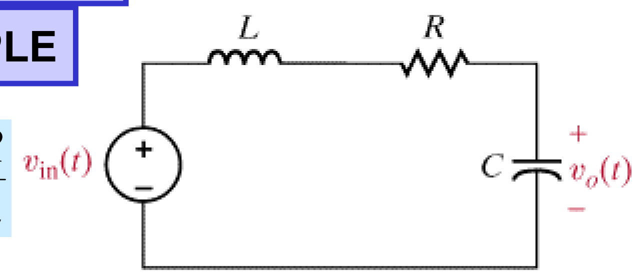
poles: $s_{1,2} = -\varsigma\omega_0 \pm \omega_0\sqrt{\varsigma^2 - 1}$

Case 2: $\varsigma < 1$: Underdamped network

poles: $s_{1,2} = -\varsigma\omega_0 \pm j\omega_0\sqrt{1-\varsigma^2}$

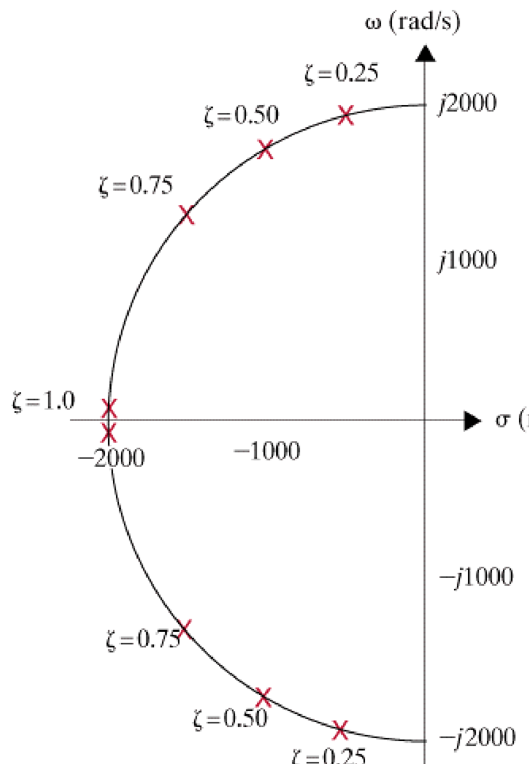


LEARNING EXAMPLE



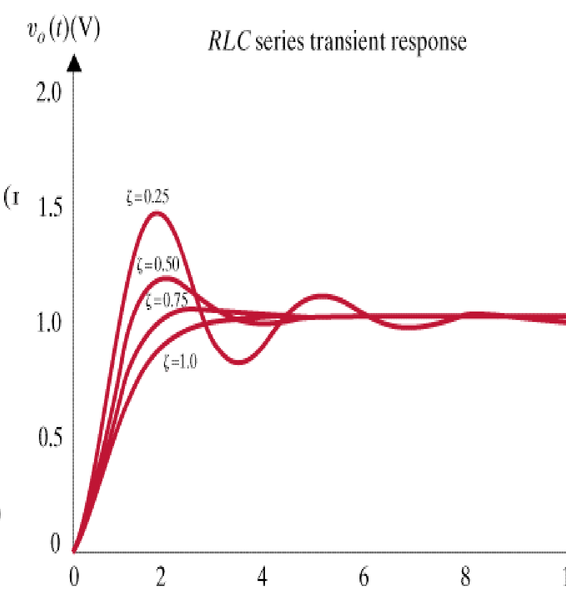
$$\omega_o^2 = \frac{1}{LC}, \qquad 2\varsigma\omega_o = \frac{R}{L}$$

$$G_v(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{\frac{1}{Cs}}{\frac{1}{Cs} + Ls + R} = \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$



Variation of poles.

Use $\omega_o = 2000$



GEAUX

LEARNING EXAMPLE

The Tacoma Narrows Bridge Revisited

Previously the event was modeled as a resonance problem. More detailed studies show that a model with a wind-dependent damping ratio provides a better explanation

$$\frac{d^2\theta}{dt^2} + 2\varsigma\omega_o \frac{d\theta}{dt} + \omega_o^2\theta = 0$$

$$\varsigma = 0.0046 - 0.00013U$$

U = wind speed (mph)

Torsional Resonance Model

Conditions at failure

wind speed = 42mph

twist = 12°

time to collapse = 45min

Problem: Develop a circuit that models this event

GEAUX

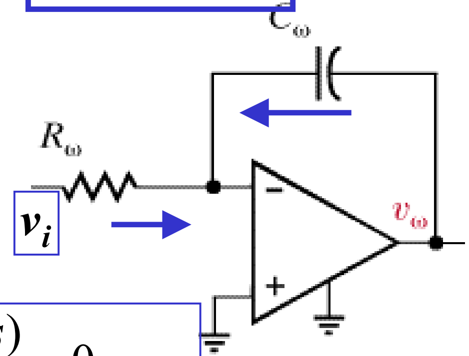
model $\dfrac{d^2\theta}{dt^2} + 2\varsigma\omega_o \dfrac{d\theta}{dt} + \omega_o^2\theta = 0$

$$\ddot{\theta} + 2\varsigma\omega_o\dot{\theta} + \omega_o^2\theta = 0 \Rightarrow \ddot{\theta} = -(2\varsigma\omega_o\dot{\theta} + \omega_o^2\theta)$$

$$\ddot{\theta} = -(0.001156 - 00013U)\dot{\theta} - 1.579\theta$$

Using numerical values

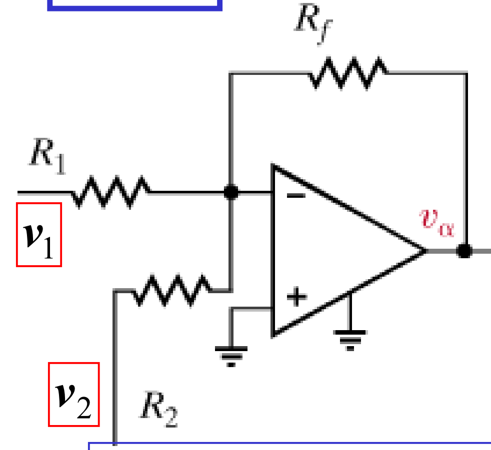
integrator



$$\frac{V_i(s)}{R_\omega} + \frac{V_\omega(s)}{\left(1/Cs\right)} = 0$$

$$V_\omega(s) = -\frac{1}{R_\omega C_\omega s} V_i(s)$$

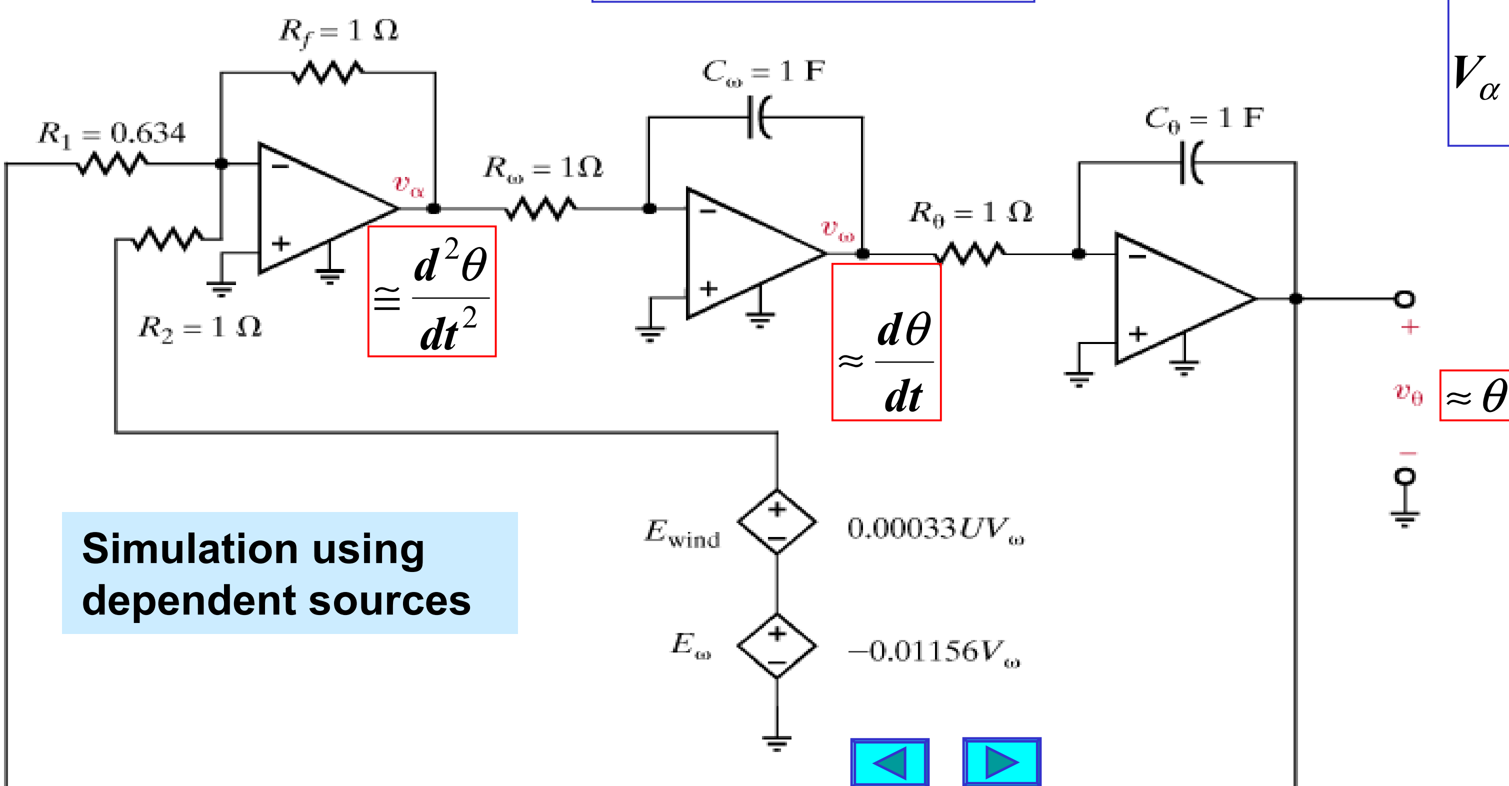
adder



$$\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_\alpha}{R_f} = 0$$

$$V_\alpha = -\left(\frac{R_f}{R_1}V_1 + \frac{R_f}{R_2}V_2\right)$$

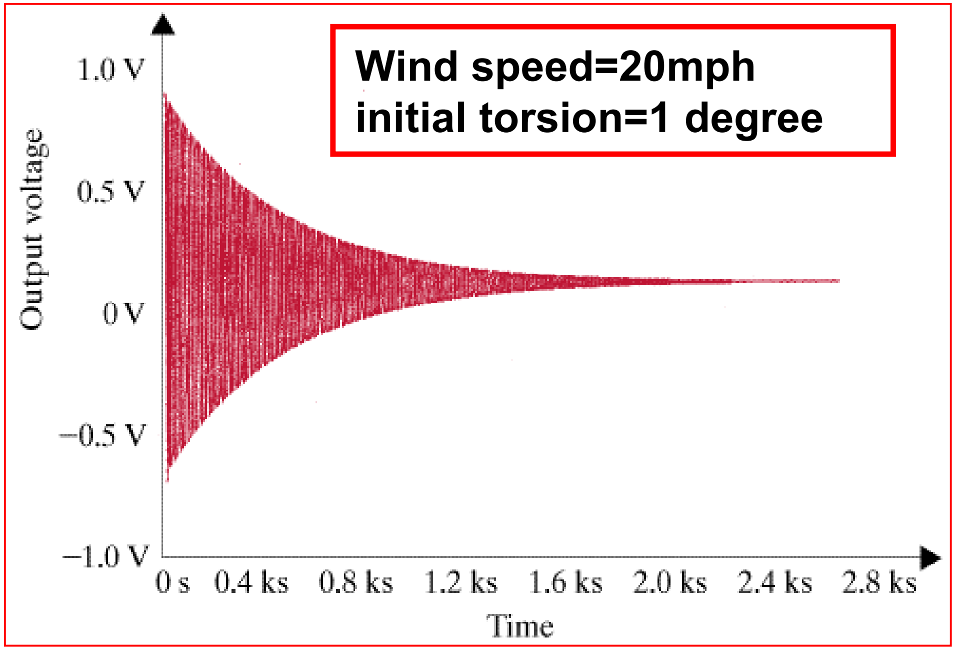
Simulation building blocks

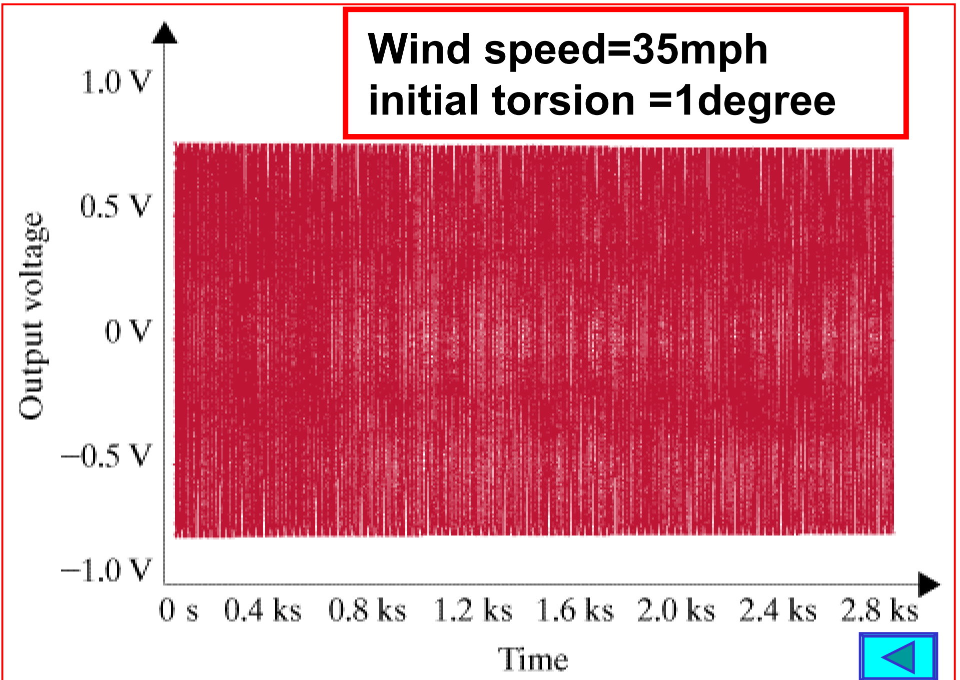


Simulation using dependent sources

GEAUX

Simulation results



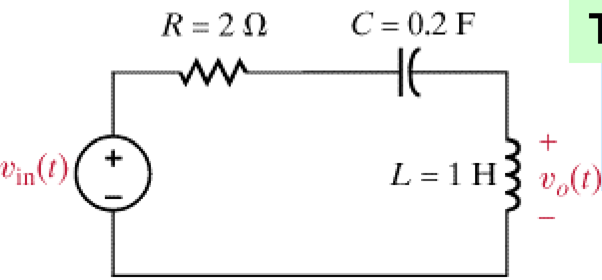


GEAUX

POLE-ZERO PLOT/BODE PLOT CONNECTION

Bode plots display magnitude and phase information of $G(s)|_{s=j\omega}$

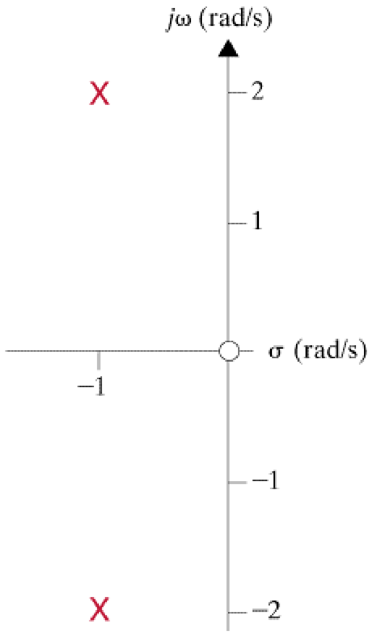
They show a cross section of G(s)



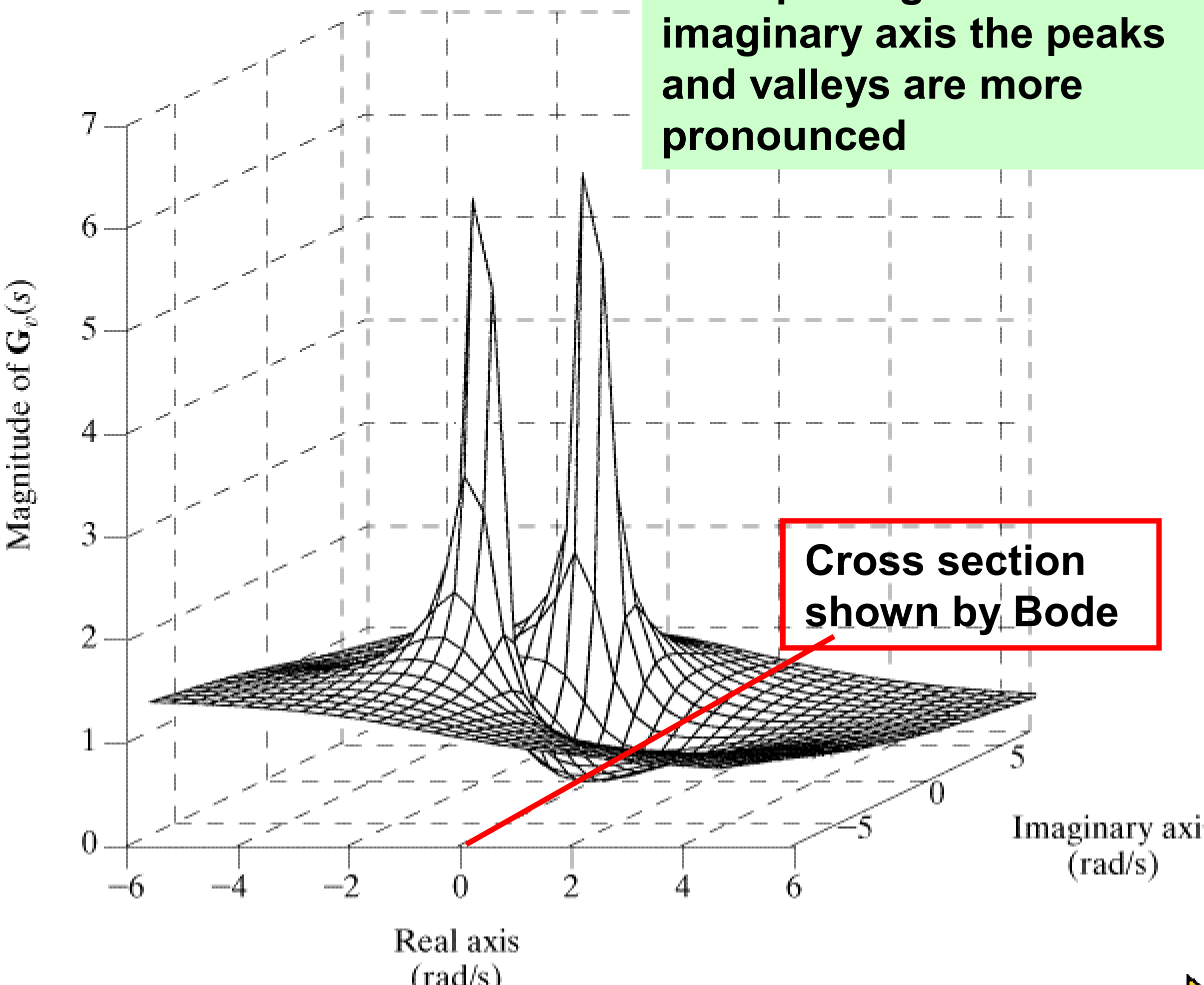
$$G(s)=\frac{s^2}{s^2+2s+5}$$

$$G(s)=\frac{V_o(s)}{V_{in}(s)}=\frac{\frac{1}{LC}}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}}$$

If the poles get closer to imaginary axis the peaks and valleys are more pronounced

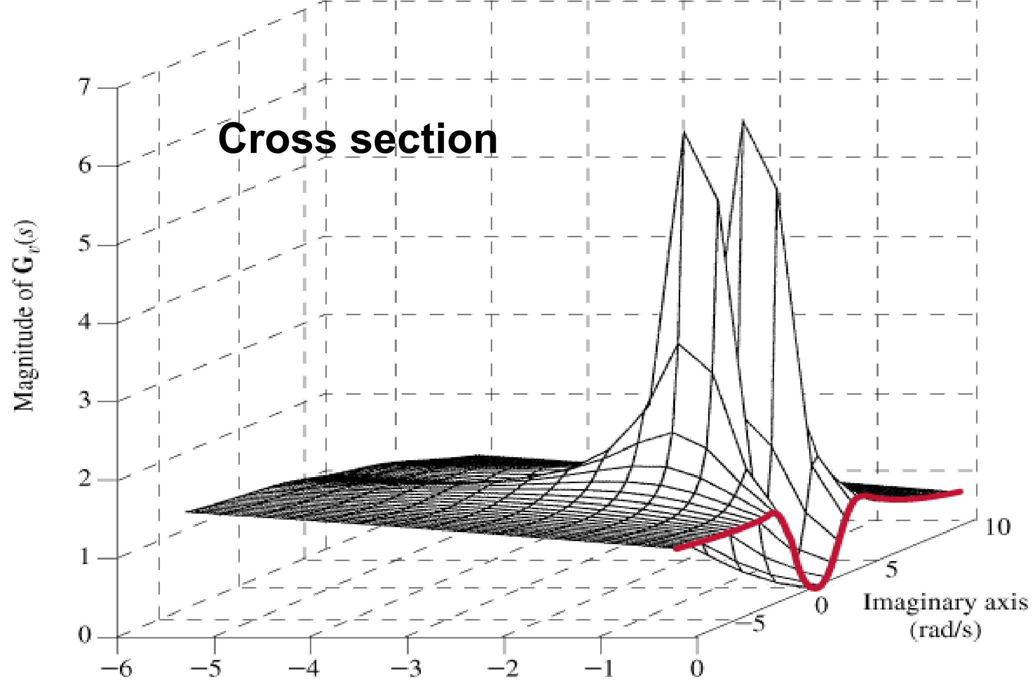


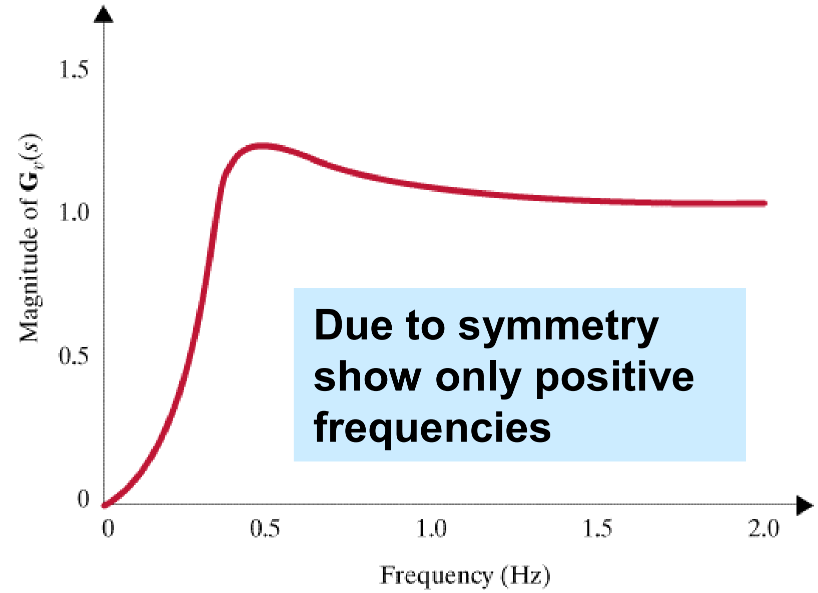
(a)



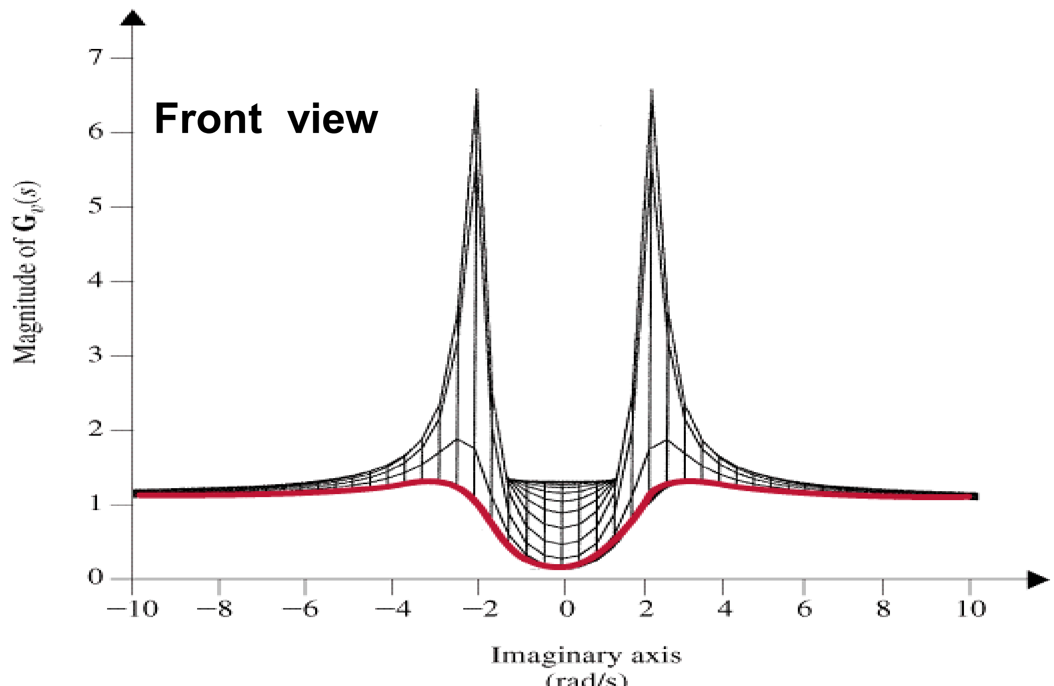
(b)

GEAUX

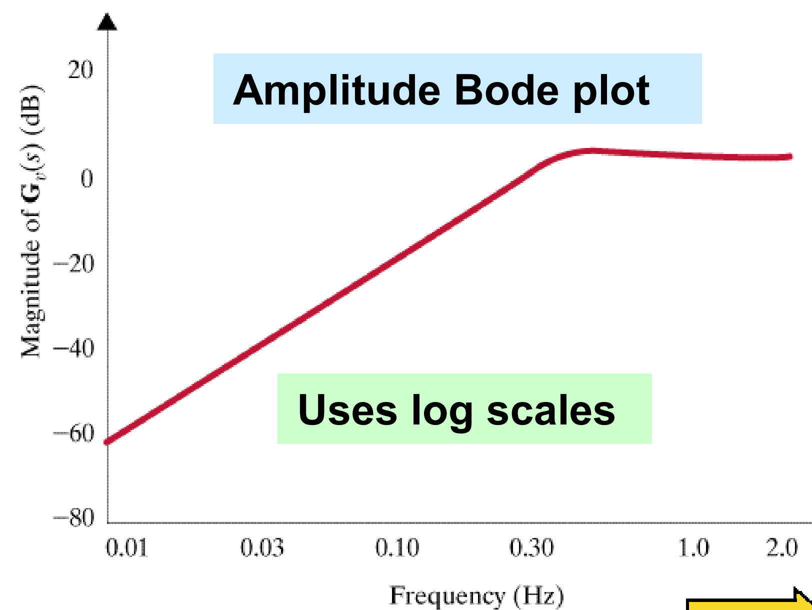

Cross section
Magnitude of $\mathbf{G}_v(s)$
7
6
5
4
3
2
1
0
−6
−5
−4
−3
−2
−1
0
−5
0
5
10
Imaginary axis
(rad/s)


Magnitude of $\mathbf{G}_v(s)$
1.5
1.0
0.5
0
0
0.5
1.0
1.5
2.0
Frequency (Hz)
Due to symmetry show only positive frequencies

(e)


Front view
Magnitude of $\mathbf{G}_v(s)$
7
6
5
4
3
2
1
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
Imaginary axis
(rad/s)

(d)


Amplitude Bode plot
Magnitude of $\mathbf{G}_v(s)$ (dB)
20
0
−20
−40
−60
−80
0.01
0.03
0.10
0.30
1.0
2.0
Frequency (Hz)
Uses log scales

(f)

STEADY STATE RESPONSE

$Y(s) = H(s)X(s)$

Response when all initial conditions are zero

Laplace uses positive time functions. Even for sinusoids the response contains transitory terms

EXAMPLE

$$H(s) = \frac{1}{s+1}, X(s) = \frac{s}{s^2+\omega^2} \; (\Rightarrow x(t) = [\cos \omega t]u(t))$$

$$Y(s) = \frac{s}{(s+1)(s+j\omega)(s-j\omega)} = \frac{K_1}{s+1} + \frac{K_2}{s+j\omega} + \frac{K_2^*}{s-j\omega}$$

If interested in the steady state response only, then don't determine residues associated with transient terms

$$y(t) = \left(Ke^{-t} + 2|K_2|\cos(\omega t + \angle K_2 °)\right)u(t)$$

transient

Steady state response

If $x(t) = X_M \cos(\omega_o t + \theta)u(t)$

$$y_{ss}(t) = |X_M| H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o) + \theta)$$

For the general case

$$X_M \cos \omega t\, u(t) = \frac{X_M}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \Rightarrow X(s) = \frac{1}{2}\left(\frac{X_M}{s-j\omega_o} + \frac{X_M}{s+j\omega_o}\right)$$

$$Y(s) = H(s)\left[\frac{1}{2}\left(\frac{X_M}{s-j\omega_o} + \frac{X_M}{s+j\omega_o}\right)\right] = \frac{K_x}{s-j\omega_o} + \frac{K_x^*}{s+j\omega_o} + \text{transient terms}$$

$$K_x = (s-j\omega_o)Y(s)|_{s=j\omega_o} = \frac{1}{2}X_M H(j\omega_o)$$

$$y(t) = 2|K_x|\cos(\omega_o t + \angle K_2) + \text{transient terms}$$

$$y_{ss}(t) = X_M |H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o))$$

GEAUX

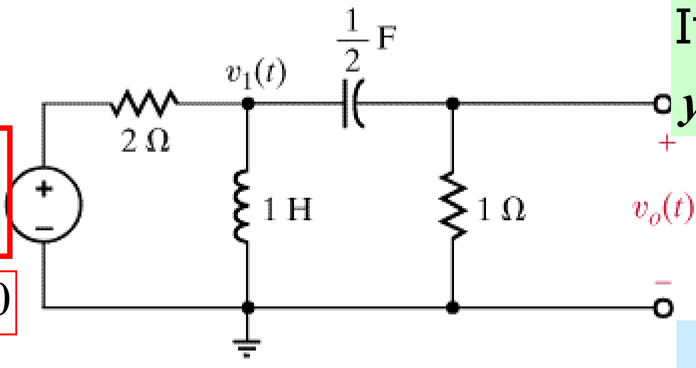
LEARNING EXAMPLE

Determine the steady state response

If $x(t) = X_M \cos(\omega_o t + \theta) u(t)$

$y_{ss}(t) = |X_M| H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o) + \theta)$

$v_i(t) = 10 \cos 2t\, u(t)$ V



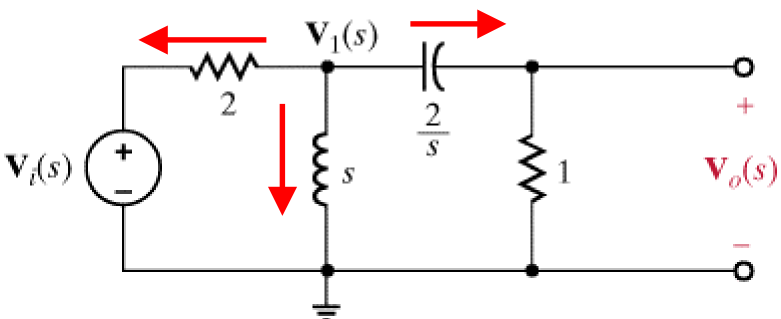
(a)

$\omega_o = 2,\ X_M = 10$

$$V_o(s) = \frac{s^2}{3s^2 + 4s + 4} V_i(s) \Rightarrow H(s) = \frac{s^2}{3s^2 + 4s + 4}$$

Transform the circuit to the Laplace domain. Assume all initial conditions are zero

$$H(j2) = \frac{(j2)^2}{3(j2)^2 + 4(j2) + 4} = 0.354\angle 45°$$



$$\therefore y_s(t) = 3.54\cos(2t + 45°)V$$

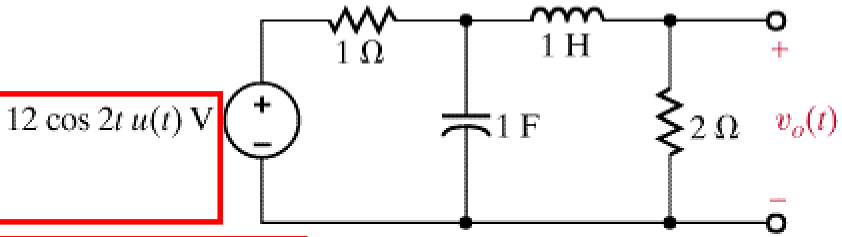
$$\text{KCL@V}_1: \frac{V_1 - V_i}{2} + \frac{V_1}{2} + \frac{V_1}{\frac{2}{s} + 1} = 0$$

$$\text{Voltage divider: } V_o = \frac{1}{\frac{2}{s} + 1} V_1$$

GEAUX

LEARNING EXTENSION

Determine $v_{oss}(t), t > 0$



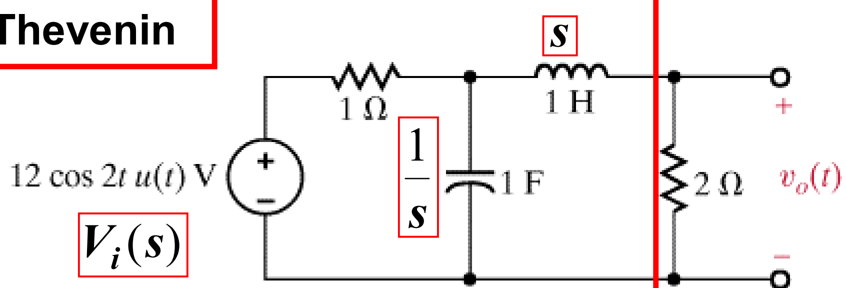
$\omega_o = 2,\ X_M = 12$

If $x(t) = X_M \cos(\omega_o t + \theta)u(t)$

$y_{ss}(t) = |X_M|H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o) + \theta)$

Transform circuit to Laplace domain.
Assume all initial conditions are zero

Thevenin



$$V_{OC}(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}} V_i(s) = \frac{1}{s+1} V_i(s)$$

$$Z_{Th}(s) = s + \| 1, \frac{1}{s} \| = s + \frac{1}{s+1} = \frac{s^2+s+1}{s+1}$$

$$V_o(s) = \frac{2}{2 + Z_{Th}(s)} V_{OC}(s)$$

$$V_o(s) = \frac{2}{2 + \frac{s^2+s+1}{s+1}} \times \frac{1}{s+1} V_i(s)$$

$$V_o(s) = \frac{2}{s^2+3s+3} V_i(s)$$

$H(s)$

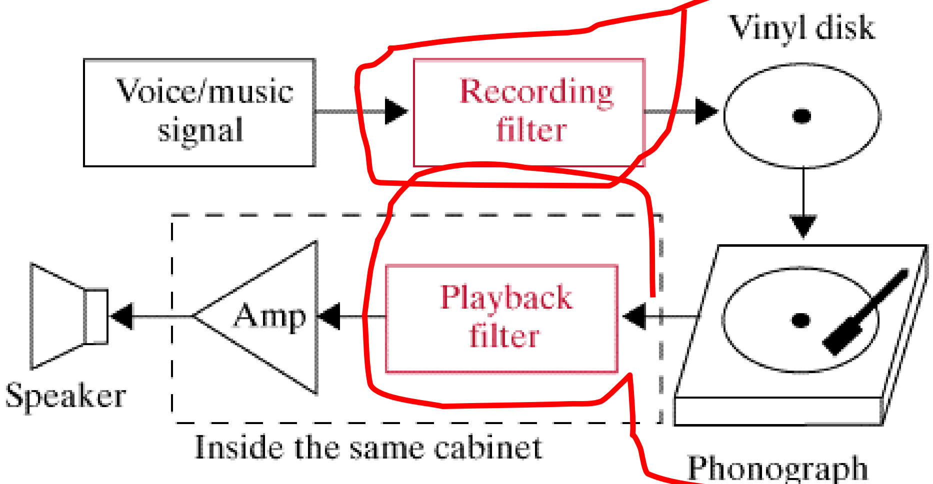
$$H(j2) = \frac{2}{-4+6j+3} = \frac{2}{-1+6j} = \frac{2}{6.08\angle 99.46^\circ}$$

$$v_{oss}(t) = 12 \times \frac{2}{6.08} \cos(2t - 99.46^\circ)$$

GEAUX

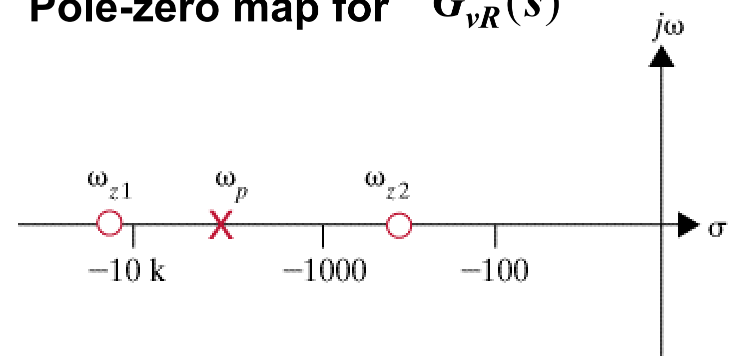
LEARNING BY APPLICATION

De-emphasize bass



Enhances bass to original level

Pole-zero map for $G_{vR}(s)$



(a)

RIAA recording filter

$$G_{vR}(s) = \frac{K(1+s\tau_{z1})(1+s\tau_{z2})}{(1+s\tau_p)}$$

$\tau_{z1} = 75\mu s$

$\tau_{z2} = 3180\mu s$

$\tau_p = 318\mu s$

$$|KG(s)|_{s=j2\pi\times 1000} = 1$$

zeros: $\omega_{z1} = 13.33kr/s[2.12kHz]$,

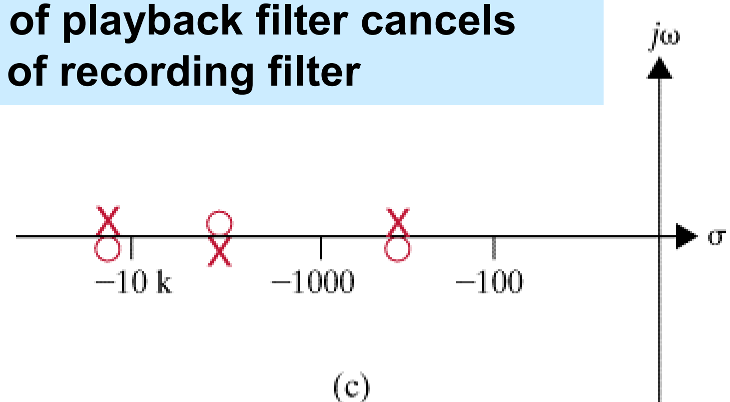
$\omega_{z1} = 313.46r/s[50Hz]$

pole: $\omega_p = 3{,}1346r/s[500Hz]$

The playback filter is the reciprocal

$$G_{vp}(s) = \frac{1}{G_{rp}(s)}$$

Pole/zero of playback filter cancels pole/zero of recording filter



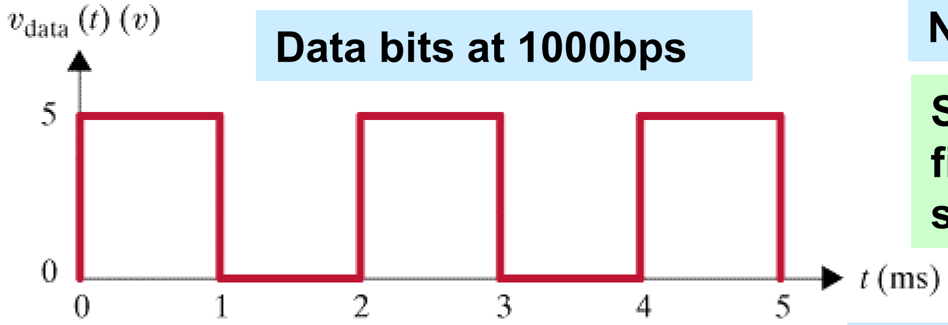
(c)

GEAUX

LEARNING BY DESIGN

Filtering noise in a data transmission line

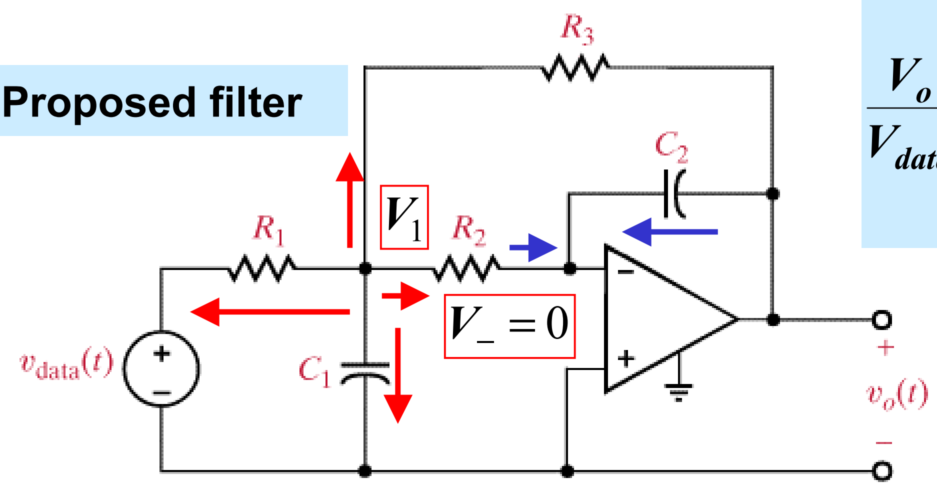
Data bits at 1000bps



Noise source is 100kHz

SOLUTION: Insert a second order low-pass filter in the path. Should not affect data signal and should attenuate noise

Proposed filter



$$\frac{V_1 - V_{data}}{R_1} + \frac{V_1}{(1/C_1 s)} + \frac{V_1}{R_2} + \frac{V_1 - V_o}{R_3} = 0$$

$$\frac{V_1}{R_2} + \frac{V_o}{(1/C_2 s)} = 0$$

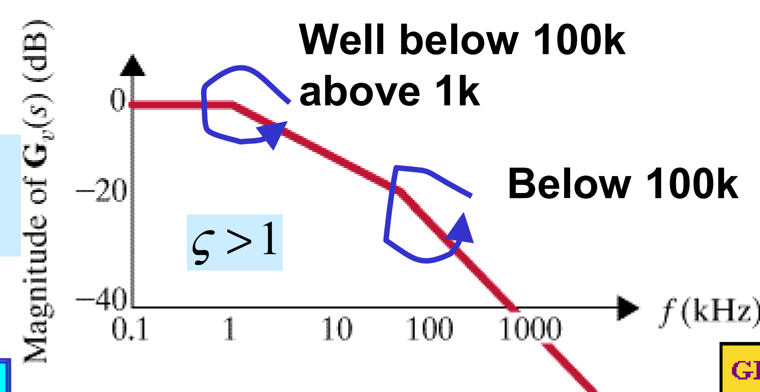
$$\frac{V_o(s)}{V_{data}(s)} = \frac{-\left(\frac{R_3}{R_1}\right)\left(\frac{1}{R_2 R_3 C_1 C_2}\right)}{s^2 + s\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_3 C_1}\right) + \frac{1}{R_2 R_3 C_1 C_2}}$$

$$s^2 + 2\varsigma\omega_o s + \omega_o^2$$

$$R_1 = R_2 = R_3 \Rightarrow \omega_o = \frac{1}{R\sqrt{C_1 C_2}}, \ \varsigma = \frac{3}{2}\sqrt{\frac{C_2}{C_1}}$$

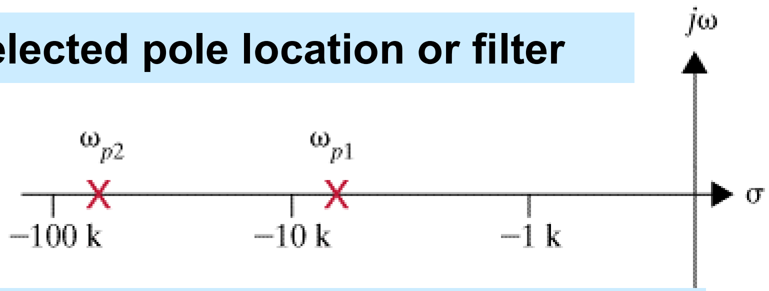
Design equations

Filter design criteria



GEAUX

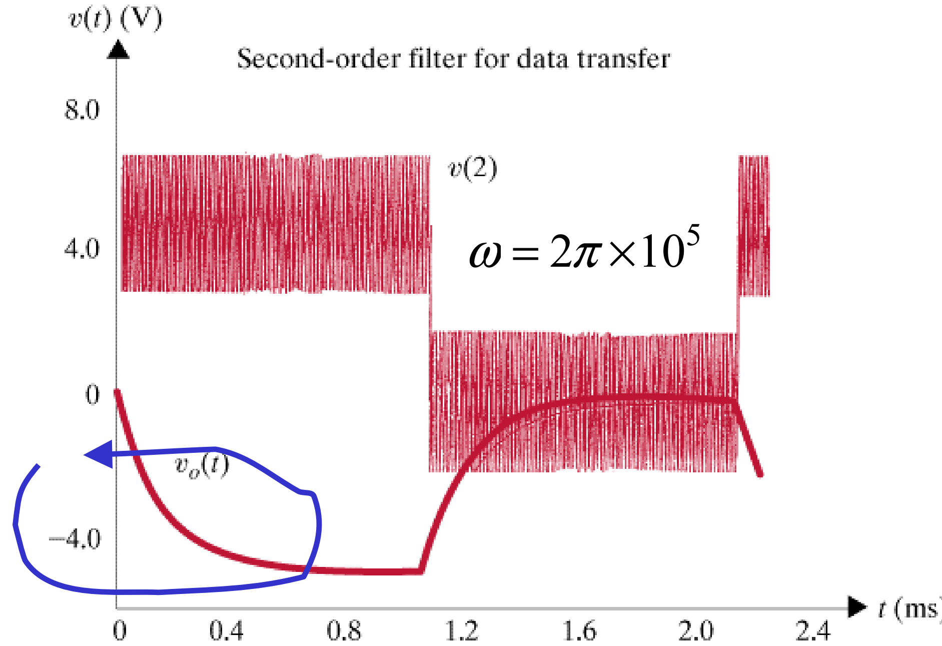
Selected pole location or filter



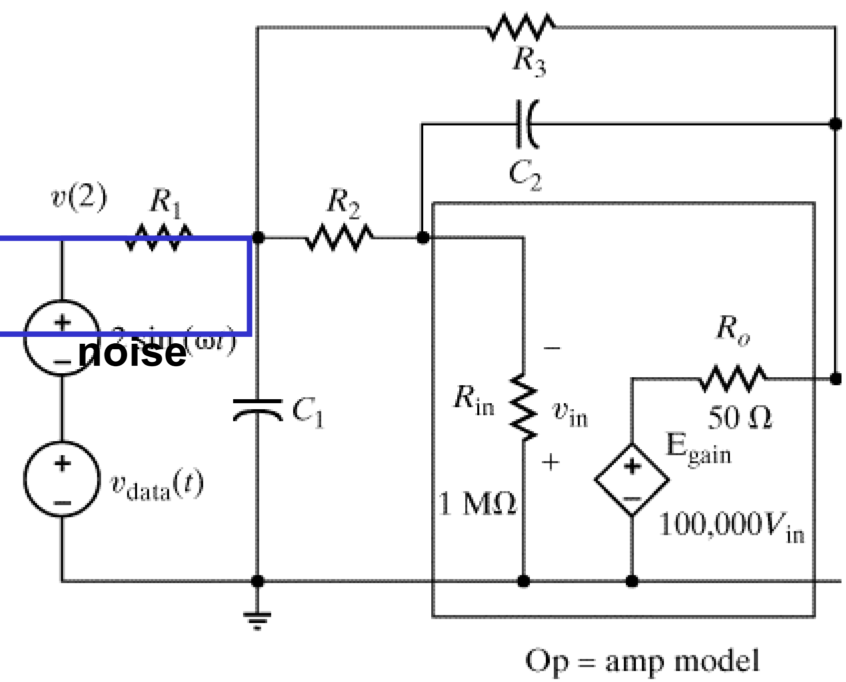
Select $R = 40k\Omega$, $\omega_0 = 25{,}000, \varsigma = 2$.

Use design equations and determine

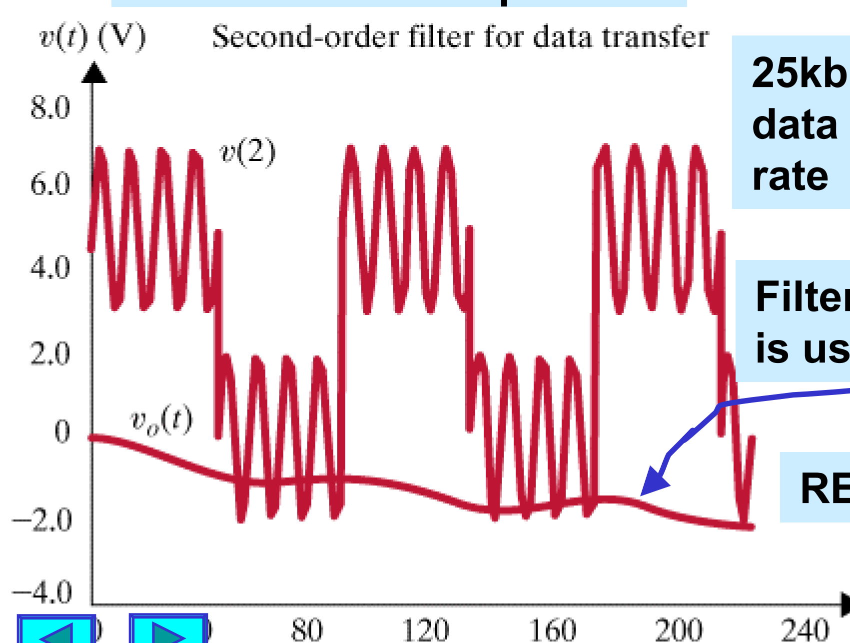
$C_1 = 0.75nF, C_2 = 1.33nF$



The filter eliminates noise but smooths data pulse



Circuit simulating the filter

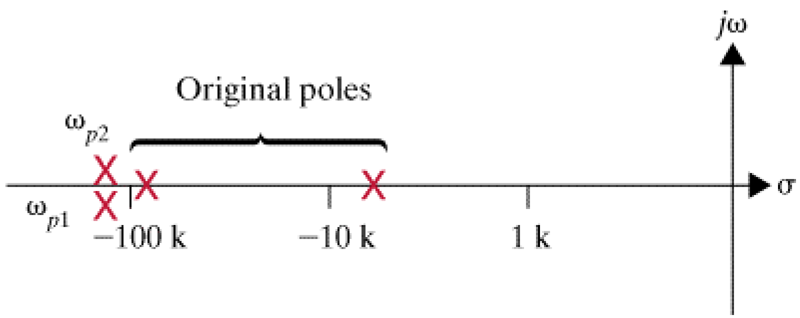


25kbps data transfer rate

Filtered signal is useless

REDESIGN!

New pole-zero selection $\varsigma = 1$

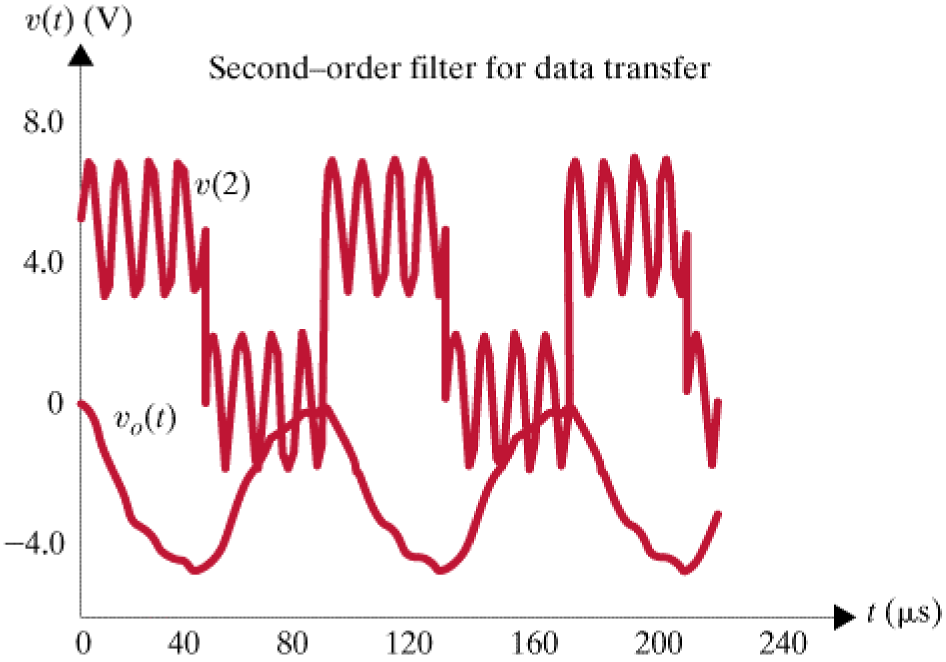


Design equations

$$\omega_0 = 150{,}000 = \frac{1}{40{,}000\sqrt{C_1 C_2}}$$

$$\varsigma = 1 = \frac{3}{2}\sqrt{\frac{C_2}{C_1}}$$

Simulation for 25kbps



APPLICATION LAPLACE

STOP