FOURIER ANALYSIS TECHNIQUES

LEARNING GOALS

FOURIER SERIES

Fourier series permit the extension of steady state analysis to general periodic signal.

FOURIER TRANSFORM

Fourier transform allows us to extend the concepts of frequency domain to arbitrary non-periodic inputs

GEAUX

FOURIER SERIES

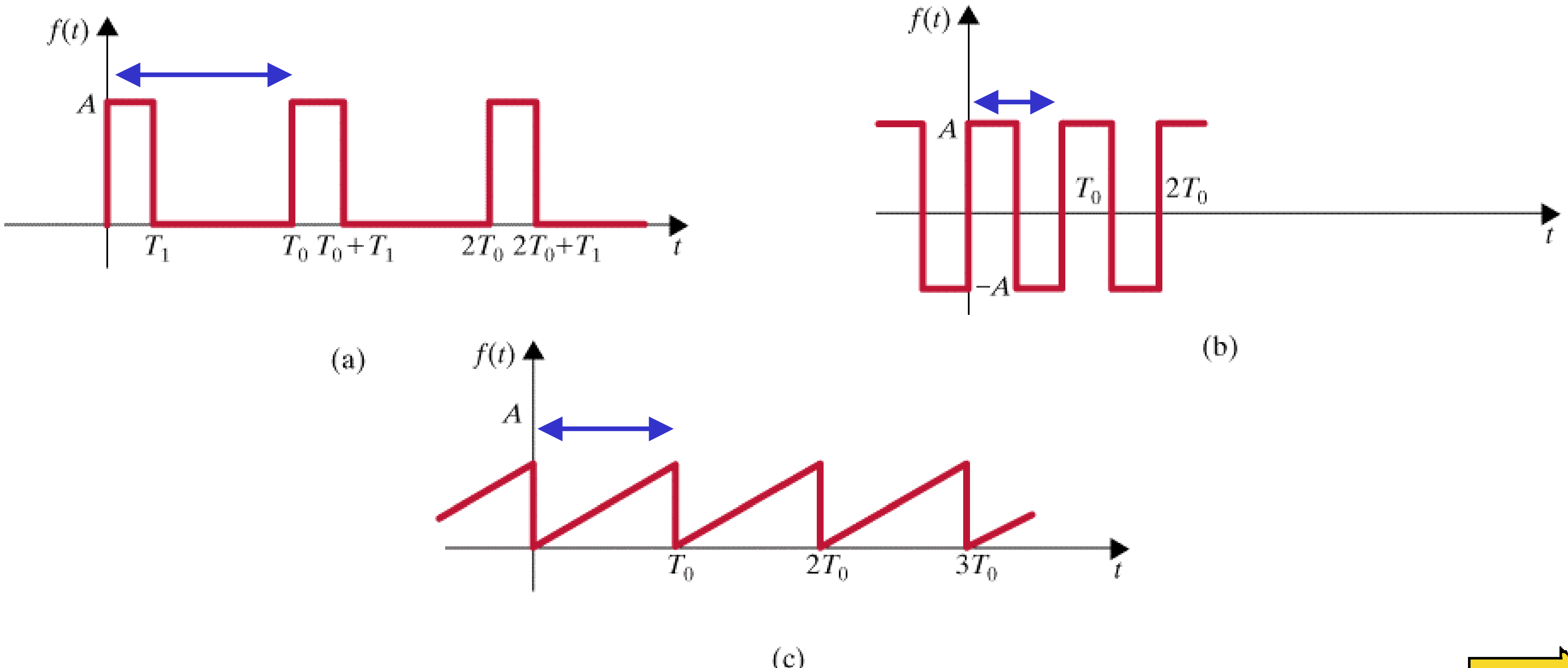
The Fourier series permits the representation of an arbitrary periodic signal as a sum of sinusoids or complex exponentials

Periodic signal

The signal $f(t)$ is periodic iff there exists $T > 0$ such that

$$f(t) = f(t+T), \forall t$$

The *smallest T* that satisfies the previous condition is called the (fundamental) period of the signal



GEAUX

FOURIER SERIES RESULTS

If $f(t)$ is periodic, with period T_0, then $f(t)$ can be expressed in one of the following equivalent forms

$$\omega_0 = \frac{2\pi}{T_0}$$

Cosine expansion

Phasor for n-th harmonic

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_o t + \theta_n) = a_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[D_n \angle \theta_n e^{jn\omega_o t}\right]$$

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Complex exponential expansion

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$e^{j\alpha} + e^{-j\alpha} = 2\cos\alpha$$

$$e^{j\alpha} - e^{-j\alpha} = 2j\sin\alpha$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t$$

Trigonometric series

$c_0 = a_0$

For $n > 0$

Relationship between exponential and trigonometric expansions

$$c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} = \underbrace{(c_n + c_{-n})}_{a_n} \cos n\omega_0 t + \underbrace{j(c_n - c_{-n})}_{b_n} \sin n\omega_0 t$$

$$\Rightarrow 2c_n = a_n - jb_n$$
$$2c_{-n} = a_n + jb_n$$

If $f(t)$ is real-valued then $c_{-n} = (c_n)^*$

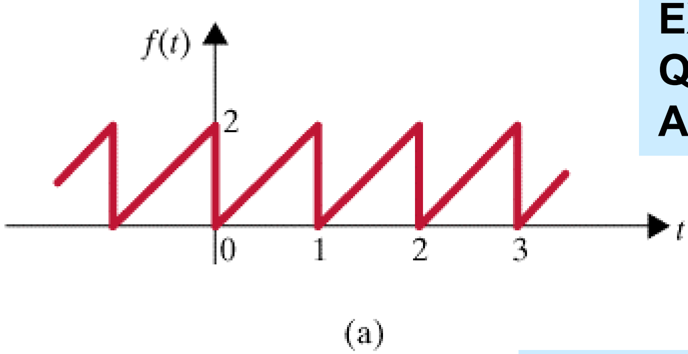
GENERAL STRATEGY:

. Approximate a periodic signal using a Fourier series

. Analyze the network for each harmonic using phasors or complex exponentials

. Use the superposition principle to determine the response to the periodic signal

Original Periodic Signal

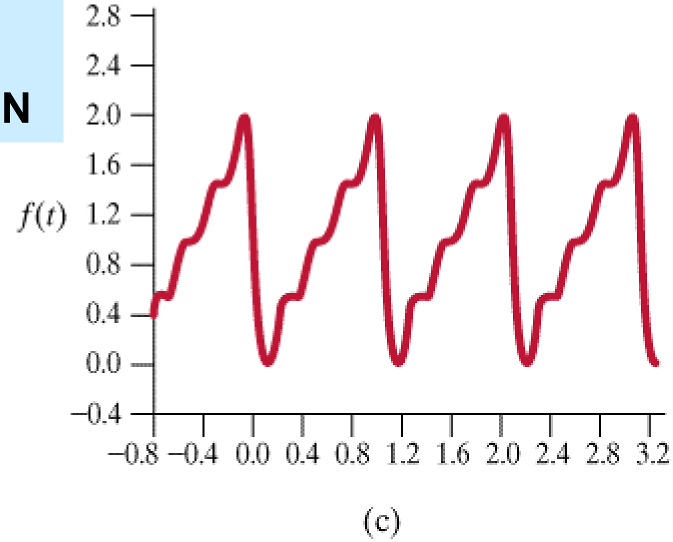


(a)

EXAMPLE OF QUALITY OF APPROXIMATION

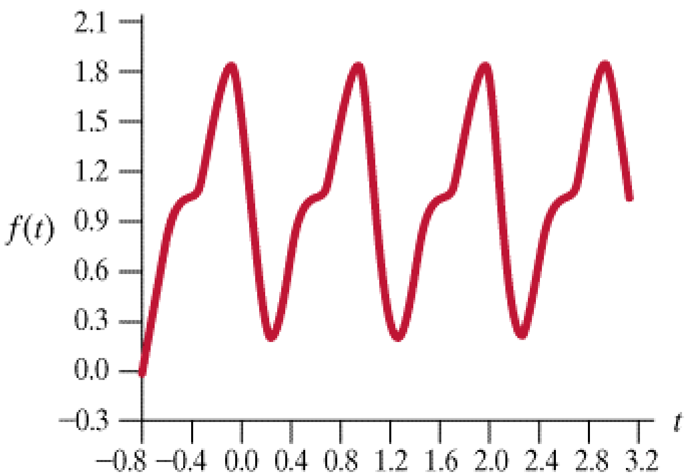
$$f_N(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t}$$

Approximation with 4 terms



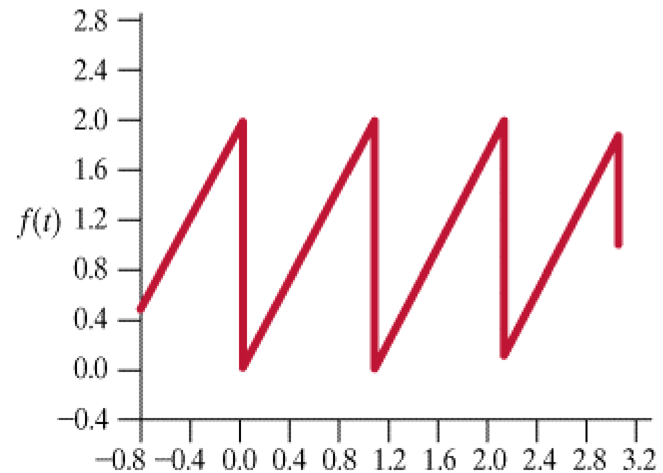
(c)

Approximation with 2 terms



$$f_2(t) = a_0 + \sum_{n=1}^{2} a_n \cos n\omega_o t + \sum_{n=1}^{2} b_n \sin n\omega_o t$$

Approximation with 100 terms



$$a_0 + \sum_{n=1}^{100} a_n \cos n\omega_o t + \sum_{n=1}^{100} b_n \sin n\omega_o t$$

EXPONENTIAL FOURIER SERIES

Any "physicallv realizable" periodic signal, with period To, can be represented over the interval $t_1 < t < t_1 + T_0$, the expression

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

The sum of exponential functions is always a continuous function. Hence, the right hand side is a continuous function.
Technically, one requires the signal, *f(t)*, to be at least piecewise continuous. In that case, the equality does not hold at the points where the signal is discontinuous

Computation of the exponential Fourier series coefficients

$$\int_{t_1}^{t_1+T_0} \qquad f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \qquad \times e^{-jk\omega_0 t}$$

$$\int_{t_1}^{t_1+T_0} e^{j(n-k)\omega_0 t} dt = \begin{cases} 0 & \text{for } n \neq k \\ T_0 & \text{for } n = k \end{cases}$$

$$\int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt = \int_{t_1}^{t_1+T_0} \left(\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

$$c_k = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt$$

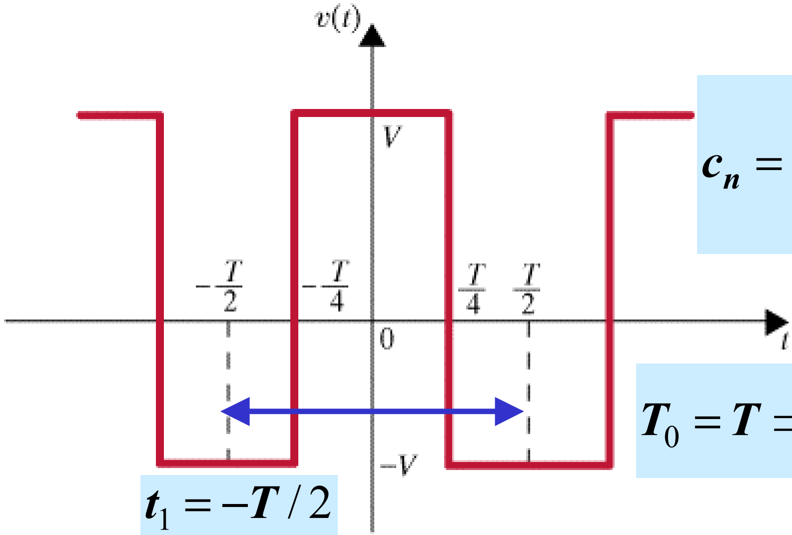
t_1 is arbitray and can be chosen to make computations simpler

LEARNING EXAMPLE

Determine the exponential Fourier series

$$f(t)=\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t};\quad \omega_0=\frac{2\pi}{T_0}$$

$$c_n=\frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)e^{-jn\omega_0 t}dt$$



$$c_n=\begin{cases}0 & n \quad even\\ \dfrac{2V}{\pi n}\sin\dfrac{n\pi}{2} & n \quad odd\end{cases}$$

$$t_1=-T/2$$

$$T_0=T\Rightarrow \omega_0=\frac{2\pi}{T}$$

A strategy :

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)e^{-jn\omega_0 t}dt=\frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{4}}(-V)e^{-jn\omega_0 t}dt+\frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}}Ve^{-jn\omega_0 t}dt-\frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}}Ve^{-jn\omega_0 t}dt$$

$$\frac{e^{j\alpha}-e^{-j\alpha}}{j}=2\sin\alpha$$

$$T\omega_0=2\pi$$

$$c_n=\frac{V}{T(jn\omega_0)}\left[e^{-jn\omega_0 t}\Big|_{-T/2}^{-T/4}-e^{-jn\omega_0 t}\Big|_{-T/4}^{T/4}+e^{-jn\omega_0 t}\Big|_{T/4}^{T/2}\right]$$

This is for $n\neq 0$!

$c_0=0$ in this case

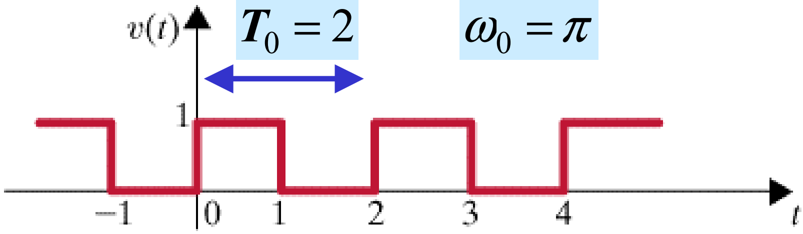
$$c_n=\frac{V}{jTn\omega_0}\left[e^{-jn\omega_0\left(-\frac{T}{4}\right)}-e^{-jn\omega_0\left(-\frac{T}{2}\right)}-e^{-jn\omega_0\left(\frac{T}{4}\right)}+e^{-jn\omega_0\left(-\frac{T}{4}\right)}+e^{-jn\omega_0\left(\frac{T}{2}\right)}-e^{-jn\omega_0\left(\frac{T}{4}\right)}\right]$$

$$c_n=\frac{V}{j2\pi n}\left[2e^{j\frac{n\pi}{2}}-2e^{-j\frac{n\pi}{2}}+e^{-jn\pi}-e^{jn\pi}\right]=\frac{V}{2\pi n}\left[4\sin\left(\frac{n\pi}{2}\right)-2\sin(n\pi)\right]$$

GEAUX

LEARNING EXTENSION

Determine the exponential Fourier series



$T_0 = 2$ $\quad \omega_0 = \pi$

$t_1 = 0$

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

A strategy :

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_n = \frac{1}{2}\int_0^2 v(t)dt = \frac{1}{2}\left[\int_0^1 e^{-j\pi n t} dt\right] = \frac{1}{2j\pi n}\left[-e^{-j\pi n t}\right]_0^1 = \frac{1-e^{-j\pi n}}{2j\pi n};$$

For $n \neq 0$

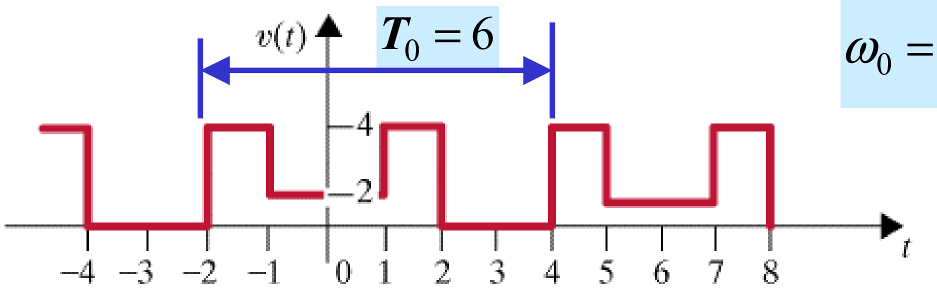
For $n = 0$

$$c_0 = \frac{1}{2}$$

GEAUX

LEARNING EXTENSION

Determine the exponential Fourier series



$t_1 = -2$

$$\omega_0 = \frac{\pi}{3}$$

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

A strategy:

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_0 = \frac{1}{6} \int_{-2}^{2} v(t)dt = 2$$

$$c_n = \frac{1}{6} \int_{-2}^{4} v(t)dt = \frac{1}{6}\left[\int_{-2}^{-1} 4e^{-j\frac{n\pi}{3}t} dt + \int_{-1}^{1} 2e^{-j\frac{n\pi}{3}t} dt + \int_{1}^{2} 4e^{-j\frac{n\pi}{3}t} dt \right]$$

$$\frac{e^{j\alpha} - e^{-j\alpha}}{j} = 2\sin\alpha$$

$$c_n = \frac{n\pi}{2j}\left[4e^{j\frac{2n\pi}{3}} - 4e^{j\frac{n\pi}{3}} + 2e^{j\frac{n\pi}{3}} - 2e^{-j\frac{n\pi}{3}} + 4e^{-j\frac{n\pi}{3}} - 4e^{-j\frac{2n\pi}{3}} \right]$$

$$c_n = n\pi\left[4\sin\frac{2\pi n}{3} - 2\sin\frac{\pi n}{3} \right]$$

GEAUX

TRIGONOMETRIC FOURIER SERIES

$$f_2(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$
$$e^{j\alpha} + e^{-j\alpha} = 2\cos\alpha$$
$$e^{j\alpha} - e^{-j\alpha} = 2j\sin\alpha$$

$c_0 = a_0$

For $n > 0$

Relationship between exponential and trigonometric expansions

$$c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} = \underbrace{(c_n + c_{-n})}_{a_n} \cos n\omega_0 t + \underbrace{j(c_n - c_{-n})}_{b_n} \sin n\omega_0 t$$

$$\Rightarrow 2c_n = a_n - jb_n$$
$$2c_{-n} = a_n + jb_n$$

If $f(t)$ is real-valued then $c_{-n} = (c_n)^*$

$$c_n = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)e^{-jn\omega_0 t}dt$$

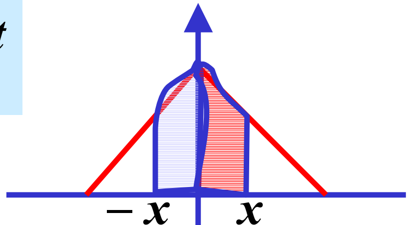
$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\, dt$$

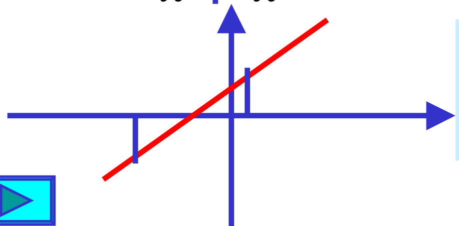
The trigonometric form permits the use of symmetry properties of the function to simplify the computation of coefficients

Even function symmetry $f(t) = f(-t)$



$$\Rightarrow \int_{-x}^{0} f(t)dt = \int_{0}^{x} f(t)dt$$

Odd function symmetry $f(t) = -f(-t)$



$$\Rightarrow \int_{-x}^{0} f(t)dt = -\int_{0}^{x} f(t)dt$$

GEAUX

TRIGONOMETRIC SERIES FOR FUNCTIONS WITH EVEN SYMMETRY

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\, dt$$

$$t_1 = -\frac{T_0}{2}$$

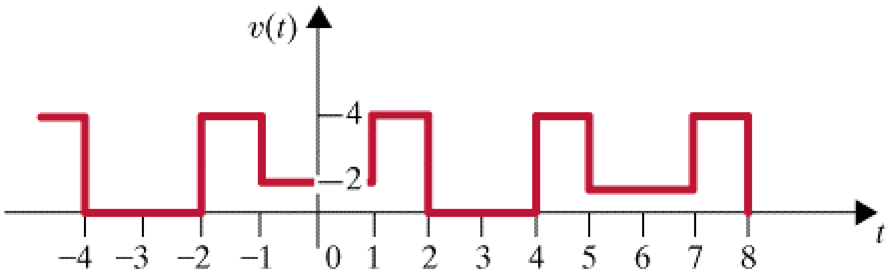
$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\, dt, \;\; n = 1,2,\dots$$

$$a_0 = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} f(t)dt$$

$$2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n \Rightarrow c_n = c_{-n} = a_n$$

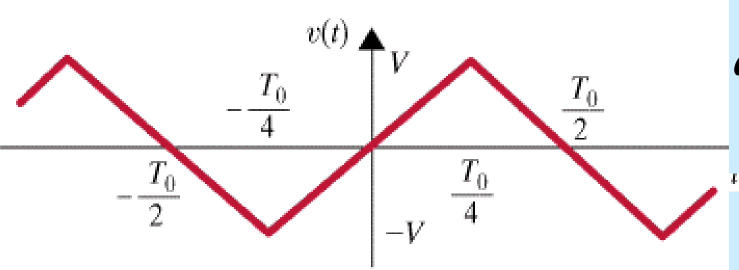
$$b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt = 0, \;\; n = 1,2,\dots$$



$$a_0 = 2$$

$$c_n = n\pi\left[4\sin\frac{2\pi n}{3} - 2\sin\frac{\pi n}{3}\right] = a_n, \;\; n = 1,2,\dots$$

TRIGONOMETRIC SERIES FOR FUNCTIONS WITH ODD SYMMETRY



(a)

$$a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\, dt = 0, \;\; n = 0,1,\dots$$

$$c_0 = 0$$

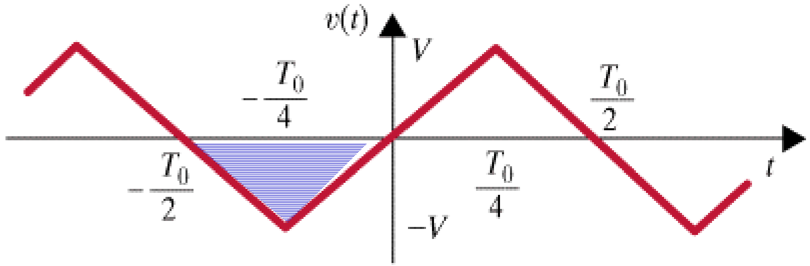
$$c_n = c^*_{-n} = -jb_n, \;\; n = 1,2,\dots$$

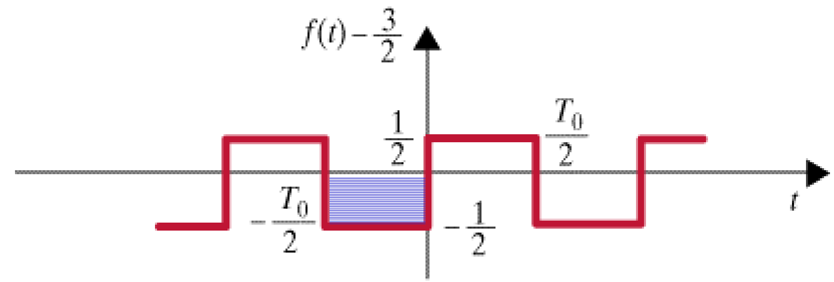
$$b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt, \;\; n = 1,2,\dots$$

GEAUX

FUNCTIONS WITH HALF-WAVE SYMMETRY

$$f(t) = -f\left(t - \frac{T_0}{2}\right), \forall t$$





Examples of signals with half-wave symmetry

Each half cycle is an inverted copy of the adjacent half cycle

$$a_0 = 0$$

$$a_n = b_n = 0 \quad \text{for n even}$$

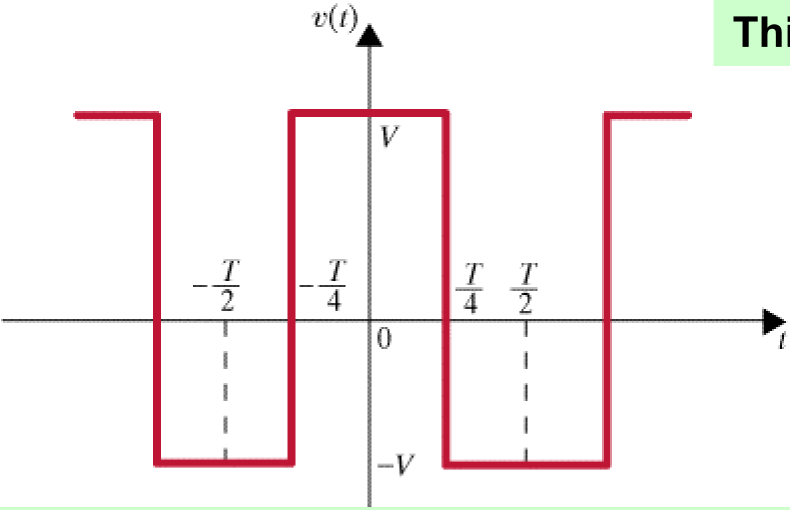
$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\, dt \quad \text{for n odd}$$

$$b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt \quad \text{for n odd}$$

There is further simplification if the function is also odd or even symmetric

GEAUX

LEARNING EXAMPLE Find the trigonometric Fourier series coefficients



This is an even function with half-wave symmetry

$$\text{even} \Rightarrow b_n = 0,\ n = 1,2,\ldots$$

$$\text{half - wave symmetry} \Rightarrow a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\, dt \qquad \text{for n odd}$$

$$a_n = 0 \quad \text{for n even}$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \omega_0 T = 2\pi$$

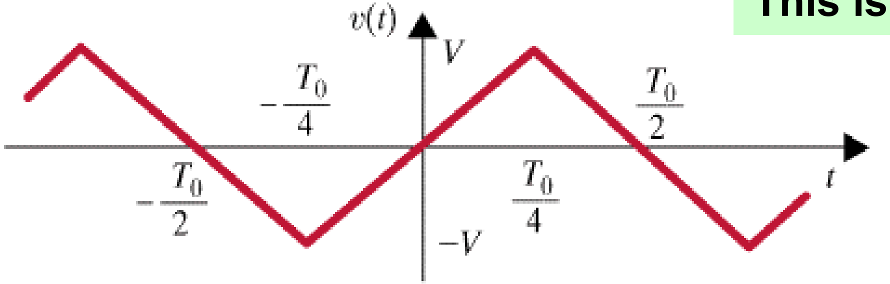
$$a_{2k+1} = \frac{4}{T}\left[\int_0^{\frac{T}{4}} V\cos(2k+1)\omega_0 t\, dt - \int_{\frac{T}{4}}^{\frac{T}{2}} V\cos(2k+1)\omega_0 t\, dt\right] = \frac{8}{T}\int_0^{\frac{T}{4}} V\cos(2k+1)\omega_0 t\, dt$$

$$a_{2k+1} = \frac{8}{T(2k+1)\omega_0}\sin(2k+1)\omega_0\frac{T}{4} = \frac{4V}{(2k+1)\pi}\sin\frac{(2k+1)\pi}{2}$$

GEAUX

LEARNING EXAMPLE

Find the trigonometric Fourier series coefficients



This is an odd function with half-wave symmetry

odd $\Rightarrow \; a_n = 0; \; n = 0,1,2,\ldots$

half-wave symmetry $\Rightarrow b_n = \dfrac{4}{T_0}\displaystyle\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt$ for n odd

$b_n = 0$ for n odd

$$b_{2k+1} = \frac{4}{T_0}\left[\int_0^{\frac{T_0}{4}} \frac{4V}{T_0} t \sin[(2k+1)\omega_0 t]dt + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} -\frac{4V}{T_0}\left(t - \frac{T_0}{2}\right)\sin[(2k+1)\omega_0 t]dt\right]$$

Use change of variable to show that the two integrals have the same value

$$b_{2k+1} = \frac{8\times 4V}{T_0^2}\int_0^{\frac{T_0}{4}} t\sin[(2k+1)\omega_0 t]dt$$

$$\frac{\omega_0 T_0}{4} = \frac{\pi}{2}$$

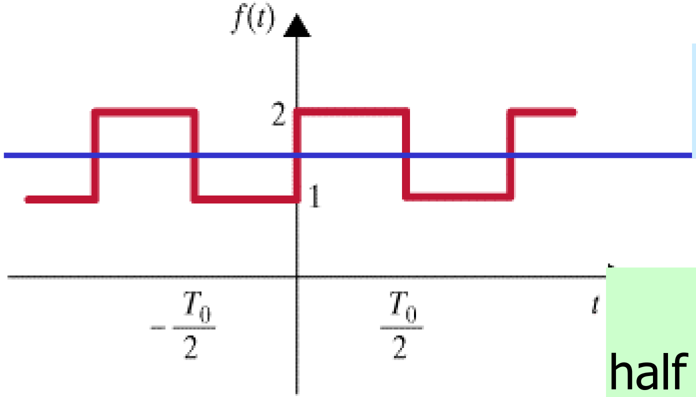
$$\int t \sin\omega t\, dt = -\frac{t\cos\omega t}{\omega} + \frac{\sin\omega t}{\omega^2}$$

$$b_{2k+1} = \frac{32V}{T_0^2}\left[-\frac{t\cos[(2k+1)\omega_0 t}{(2k+1)\omega_0} + \frac{\sin[(2k+1)\omega_0 t]}{[(2k+1)\omega_0]^2}\right]_0^{\frac{T_0}{4}}$$

$$b_{2k+1} = \frac{8V}{[(2k+1)\pi]^2}\sin\frac{(2k+1)\pi}{2}$$

GEAUX

LEARNING EXAMPLE Find the trigonometric Fourier series coefficients

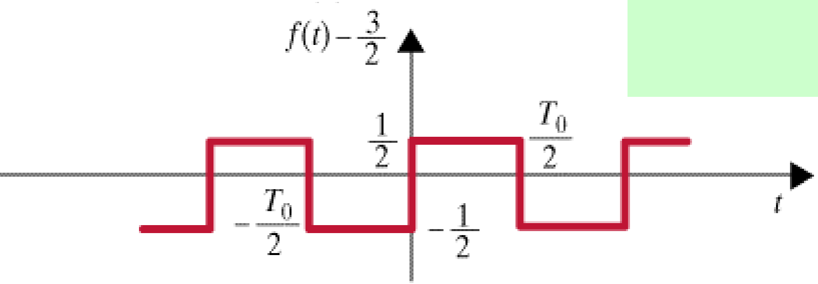


$$f(t)-\frac{3}{2} \text{ is odd with half-wave symmetry}$$

$$\text{odd} \Rightarrow \quad a_n = 0; \quad n = 0,1,2,\ldots$$

$$\text{half-wave symmetry} \Rightarrow b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt \qquad \text{for n odd}$$

$$b_n = 0 \quad \text{for n odd}$$



$$b_{2k+1} = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} \frac{1}{2}\sin[(2k+1)\omega_0 t]dt = \frac{2}{T_0(2k+1)\omega_0}\left[-\cos[(2k+1)\omega_0 t]\right]_0^{\frac{T_0}{2}}$$

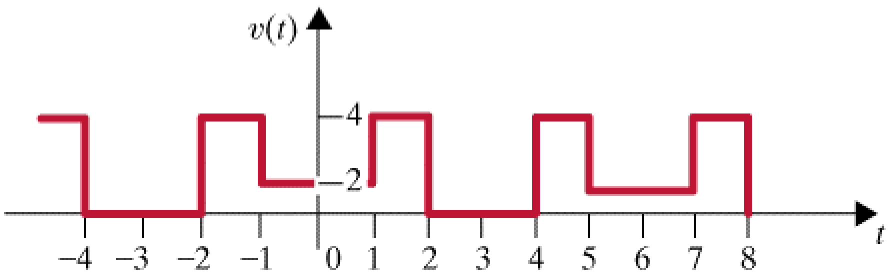
$$b_{2k+1} = \frac{1}{(2k+1)\pi}(1-\cos[(2k+1)\pi]) = \frac{2}{(2k+1)\pi}$$

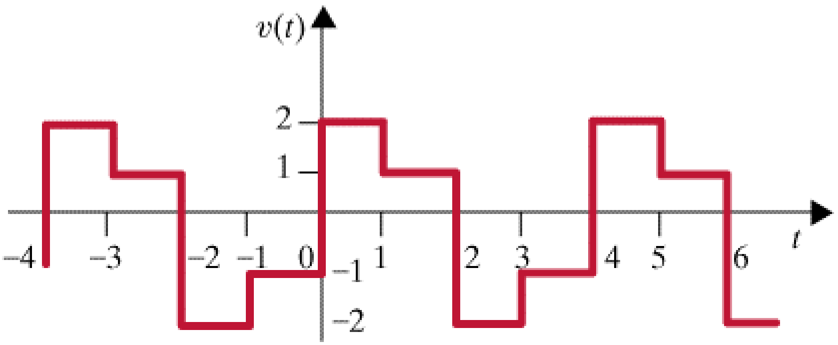
$$\therefore f(t) = \frac{3}{2} + \sum_{k=0}^{\infty}\frac{2}{(2k+1)\pi}\sin(2k+1)\omega_0 t$$

GEAUX

LEARNING EXTENSION

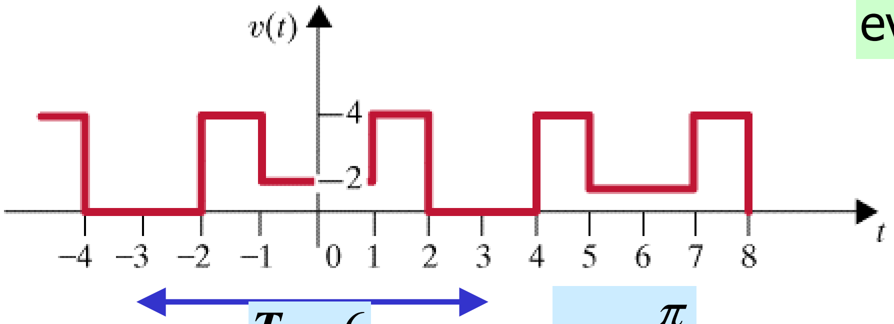
Determine the type of symmetry of the signals





LEARNING EXTENSION

Determine the trigonometric Fourier series expansion



$T_0 = 6$

$\omega_0 = \frac{\pi}{3}$

even function $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} f(t)dt$$

$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\, dt, \ n = 1,2,\ldots$$

$$a_0 = \frac{1}{3}\times(2+4) = 2$$

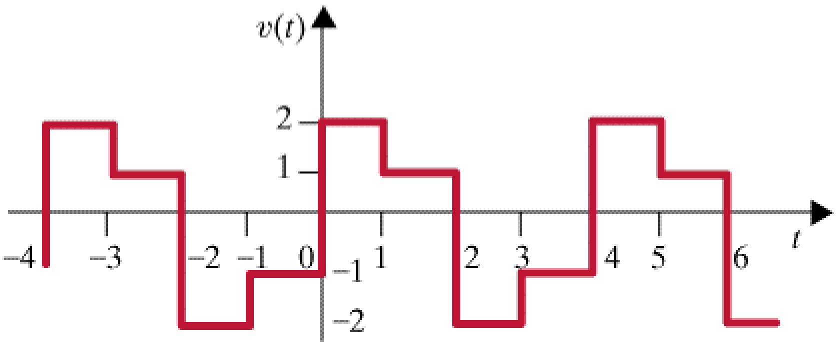
$$a_n = \frac{2}{3}\int_0^1 2\cos n\omega_0 t\, dt + \frac{2}{3}\int_1^2 4\cos n\omega_0 t\, dt$$

$$a_n = \frac{4}{n\pi}\left(2\sin\frac{2\pi n}{3} - \sin\frac{n\pi}{3}\right)$$

GEAUX

LEARNING EXTENSION

Determine the trigonometric Fourier series expansion



$T_0 = 4$

$\omega_0 = \frac{\pi}{2}$

Half-wave symmetry

$a_0 = 0$

$a_n = b_n = 0$ for n even

$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\,dt \quad \text{for n odd}$$

$$b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\,dt \quad \text{for n odd}$$

$$a_n = \int_0^1 2\cos\frac{n\pi}{2}t\,dt + \int_1^2 \cos\frac{n\pi}{2}t\,dt, \text{ for n odd}$$

$$a_{2k+1} = \frac{2}{(2k+1)\pi}\sin\frac{(2k+1)\pi}{2}$$

$$b_n = \int_0^1 2\sin\frac{n\pi}{2}t\,dt + \int_1^2 \sin\frac{n\pi}{2}t\,dt, \text{ for n odd}$$

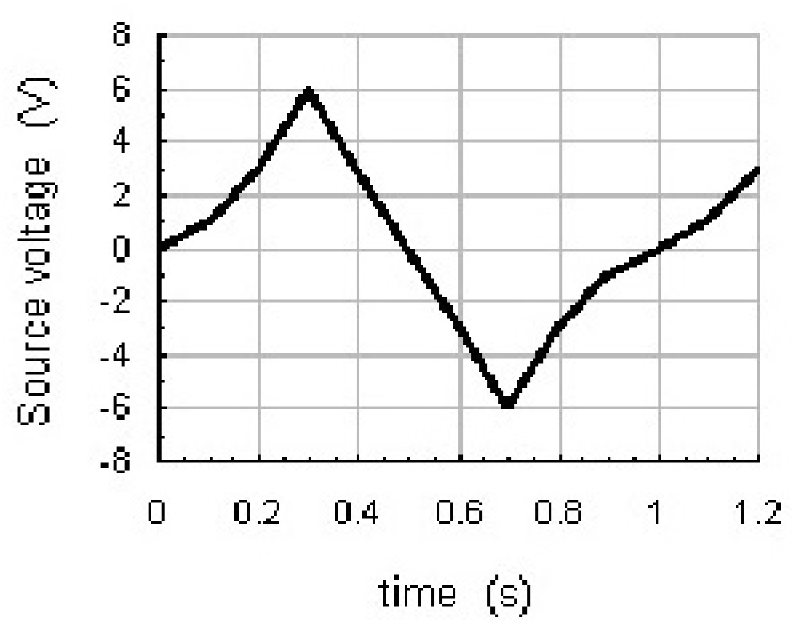
$$b_{2k+1} = \frac{2}{(2k+1)\pi}(2 - \cos(2k+1)\pi)$$

Fourier Series Via PSPICE Simulation

1. Create a suitable PSPICE schematic
2. Create the waveform of interest
3. Set up simulation parameters
4. View the results

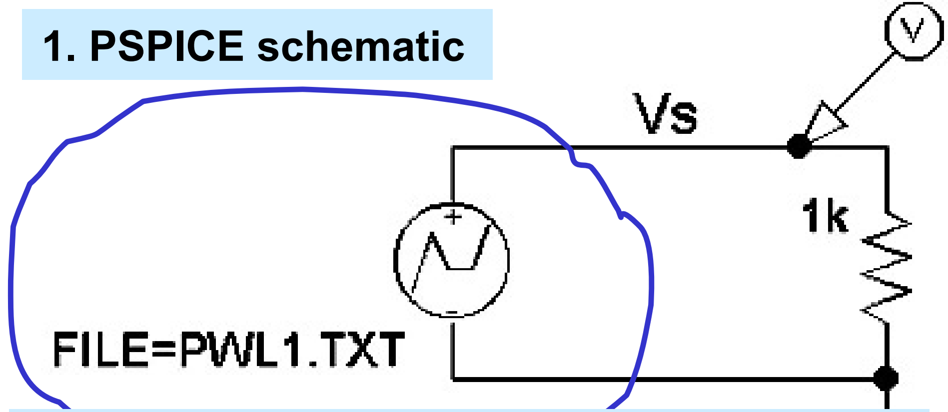
LEARNING EXAMPLE

Determine the Fourier Series of this waveform

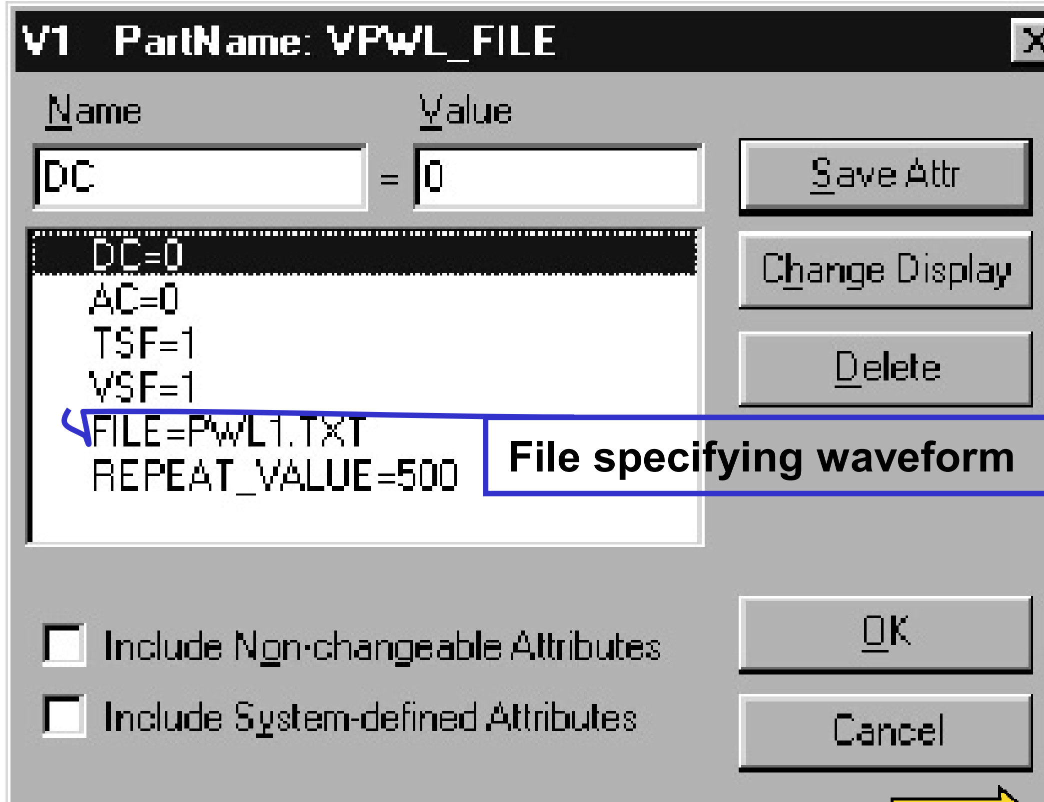


(a)

1. PSPICE schematic



VPWL_FILE in PSPICE library
piecewise linear periodic voltage source



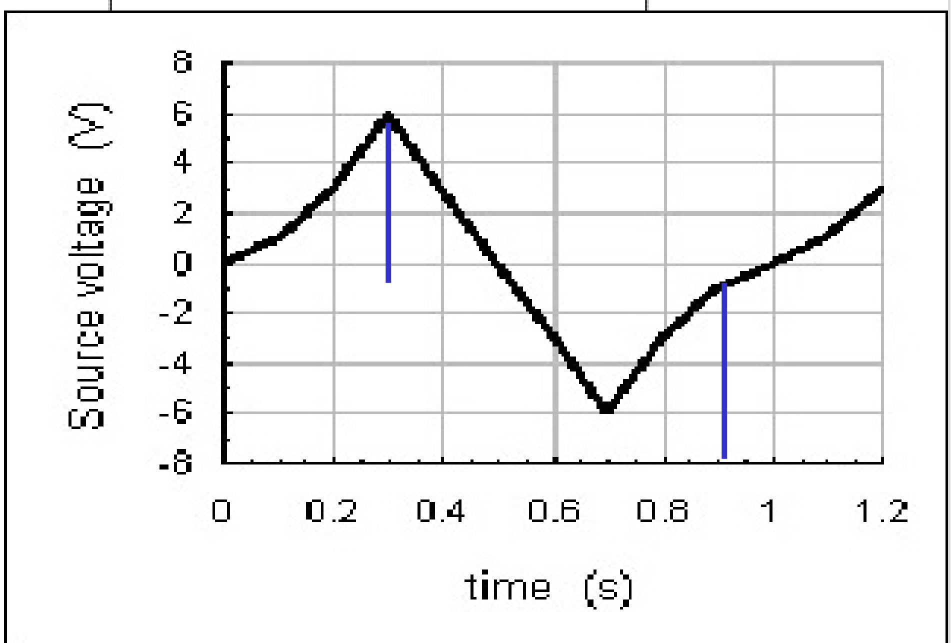
File specifying waveform

GEAUX

Text file defining corners of piecewise linear waveform

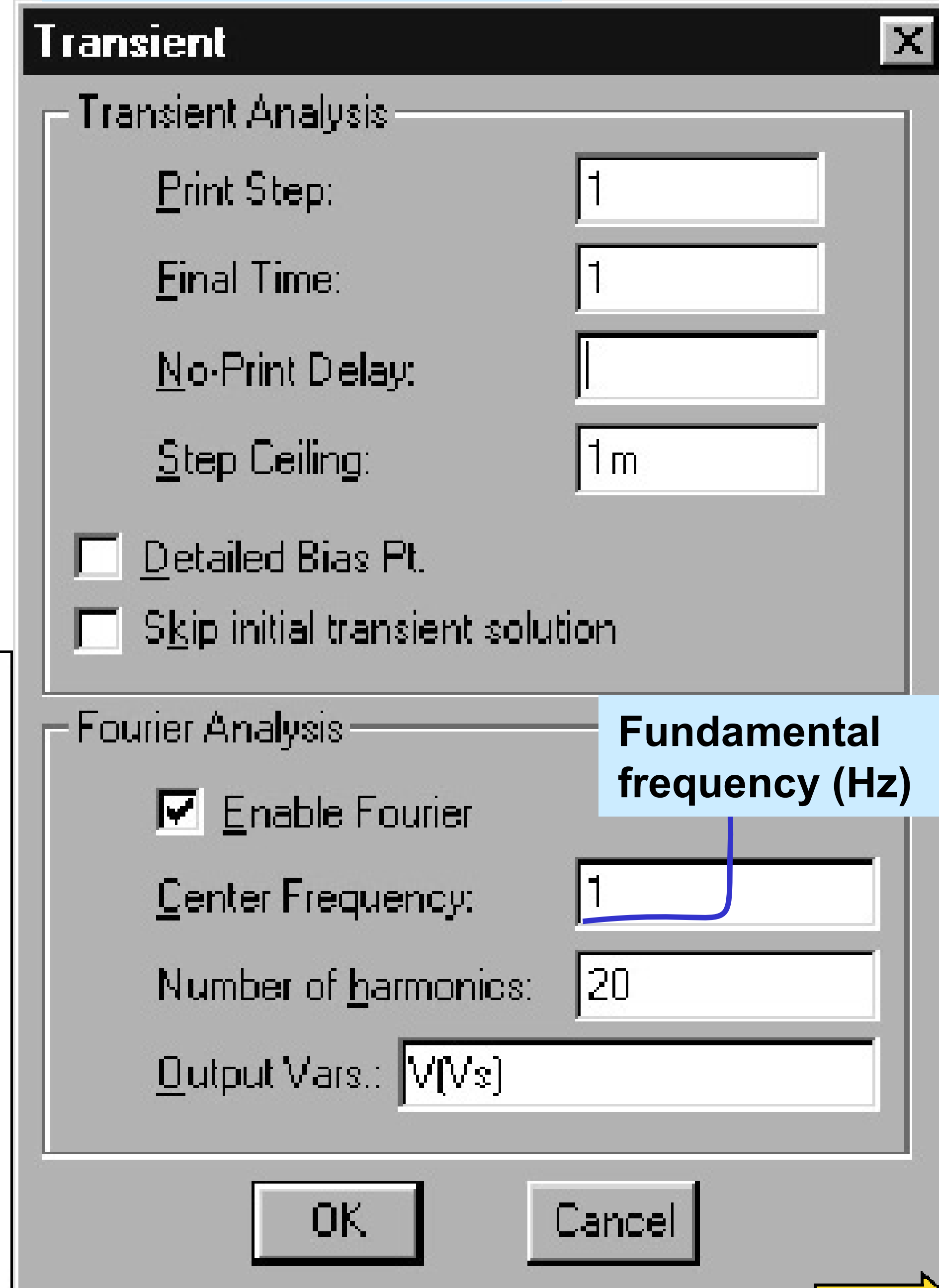
```
* Piecewise Linear File for
* BECA 7 Fourier Series Example
*
0,0
0.1,1
0.2,3
0.3,6
0.4,3
0.5,0
0.6,-3
0.7,-6
0.8,-3
0.9,-1
1.0,0
```

comments



(a)

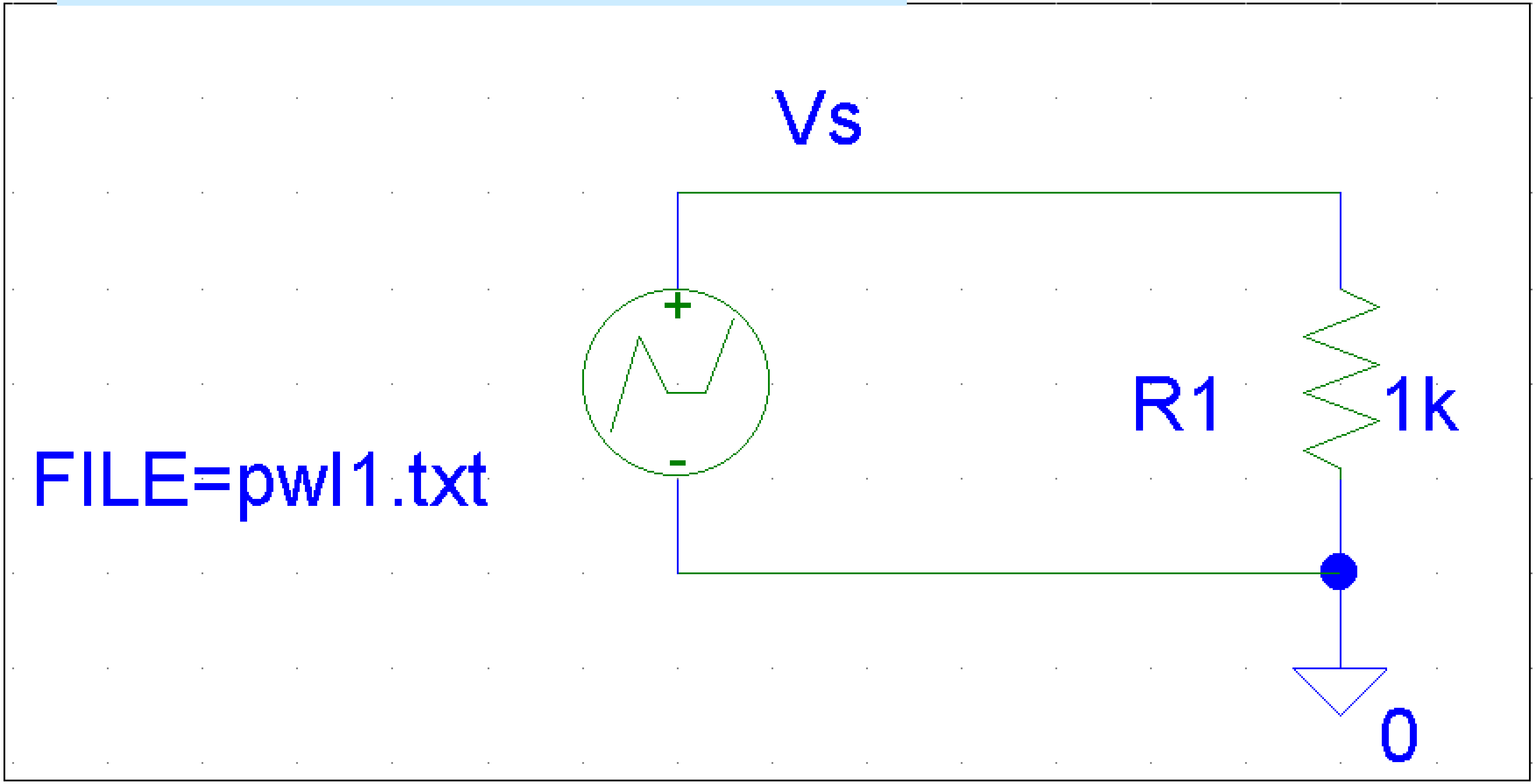
Use transient analysis



Fundamental frequency (Hz)

GEAUX

Schematic used for Fourier series example



To view result:
From PROBE menu
View/Output File
and search until you find the Fourier analysis data

Accuracy of simulation is affected by setup parameters.
Decreases with number of cycles
increases with number of points

GEAUX

FOURIER COMPONENTS OF TRANSIENT RESPONSE V(V_Vs)

RELEVANT SEGMENT OF OUTPUT FILE

DC COMPONENT = -1.353267E-08

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+00	4.222E+00	1.000E+00	2.969E-07	0.000E+00
2	2.000E+00	1.283E+00	3.039E-01	1.800E+02	1.800E+02
3	3.000E+00	4.378E-01	1.037E-01	-1.800E+02	-1.800E+02
4	4.000E+00	3.838E-01	9.090E-02	4.620E-07	-7.254E-07
5	5.000E+00	1.079E-04	2.556E-05	1.712E-03	1.711E-03
6	6.000E+00	1.703E-01	4.034E-02	1.800E+02	1.800E+02
7	7.000E+00	8.012E-02	1.898E-02	-9.548E-06	-1.163E-05
8	8.000E+00	8.016E-02	1.899E-02	5.191E-06	2.816E-06
9	9.000E+00	5.144E-02	1.218E-02	-1.800E+02	-1.800E+02
10	1.000E+01	1.397E-04	3.310E-05	1.800E+02	1.800E+02
11	1.100E+01	3.440E-02	8.149E-03	-1.112E-04	-1.145E-04
12	1.200E+01	3.531E-02	8.364E-03	1.800E+02	1.800E+02
13	1.300E+01	2.343E-02	5.549E-03	1.800E+02	1.800E+02
14	1.400E+01	3.068E-02	7.267E-03	-3.545E-05	-3.960E-05
15	1.500E+01	3.379E-04	8.003E-05	-3.208E-03	-3.212E-03
16	1.600E+01	2.355E-02	5.579E-03	-1.800E+02	-1.800E+02
17	1.700E+01	1.309E-02	3.101E-03	2.905E-04	2.854E-04
18	1.800E+01	1.596E-02	3.781E-03	-5.322E-05	-5.856E-05
19	1.900E+01	1.085E-02	2.569E-03	-1.800E+02	-1.800E+02
20	2.000E+01	2.994E-04	7.092E-05	1.800E+02	1.800E+02

```
* file pwl1.txt
* example 14.5
* BECA 7
* ORCAD 9.1
* By J.L. Aravena
0,0
0.1,1
0.2,3
0.3,6
0.4,3
0.5,0
0.6,-3
0.7,-6
0.8,-3
0.9,-1
1.0,0
```

TOTAL HARMONIC DISTORTION = 3.378352E+01 PERCENT

TIME-SHIFTING

It is easier to study the effect of time-shift with the exponential series expansion

$$f(t)=\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$$

$$f(t-t_0)=\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 (t-t_0)}=\sum_{n=-\infty}^{n=\infty}\left(c_n e^{-jn\omega_0 t_0}\right)e^{jn\omega_0 t}=$$

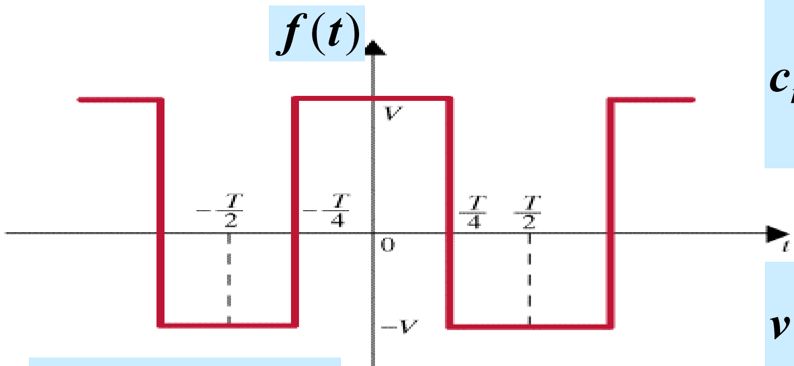
Time shifting the function only changes the phase of the coefficients

t_0 = time shift for $f(t)$

$-n\omega_0 t_0$ = phase shift for c_n

$$\omega_0 T_0 = 2\pi$$

LEARNING EXAMPLE



$$c_n=\begin{cases} 0 & n \quad even \\ \frac{2V}{\pi n}\sin\frac{n\pi}{2} & n \quad odd \end{cases}$$

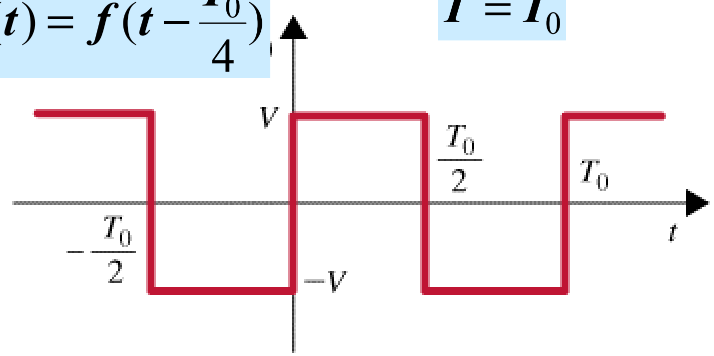
$$-n\omega_0 t_0 = -n\omega_0\frac{T_0}{4}=-\frac{n\pi}{2}$$

$$v(t)=\sum_{n=-\infty}^{n=\infty} c_{vn} e^{jn\omega_0 t}$$

$$c_{vn}=\begin{cases} 0 & n \quad even \\ \frac{2V}{\pi n}\sin\frac{n\pi}{2}e^{-j\frac{n\pi}{2}} & n \quad odd \end{cases}$$

$$v(t)=f(t-\frac{T_0}{4})$$

$$T=T_0$$



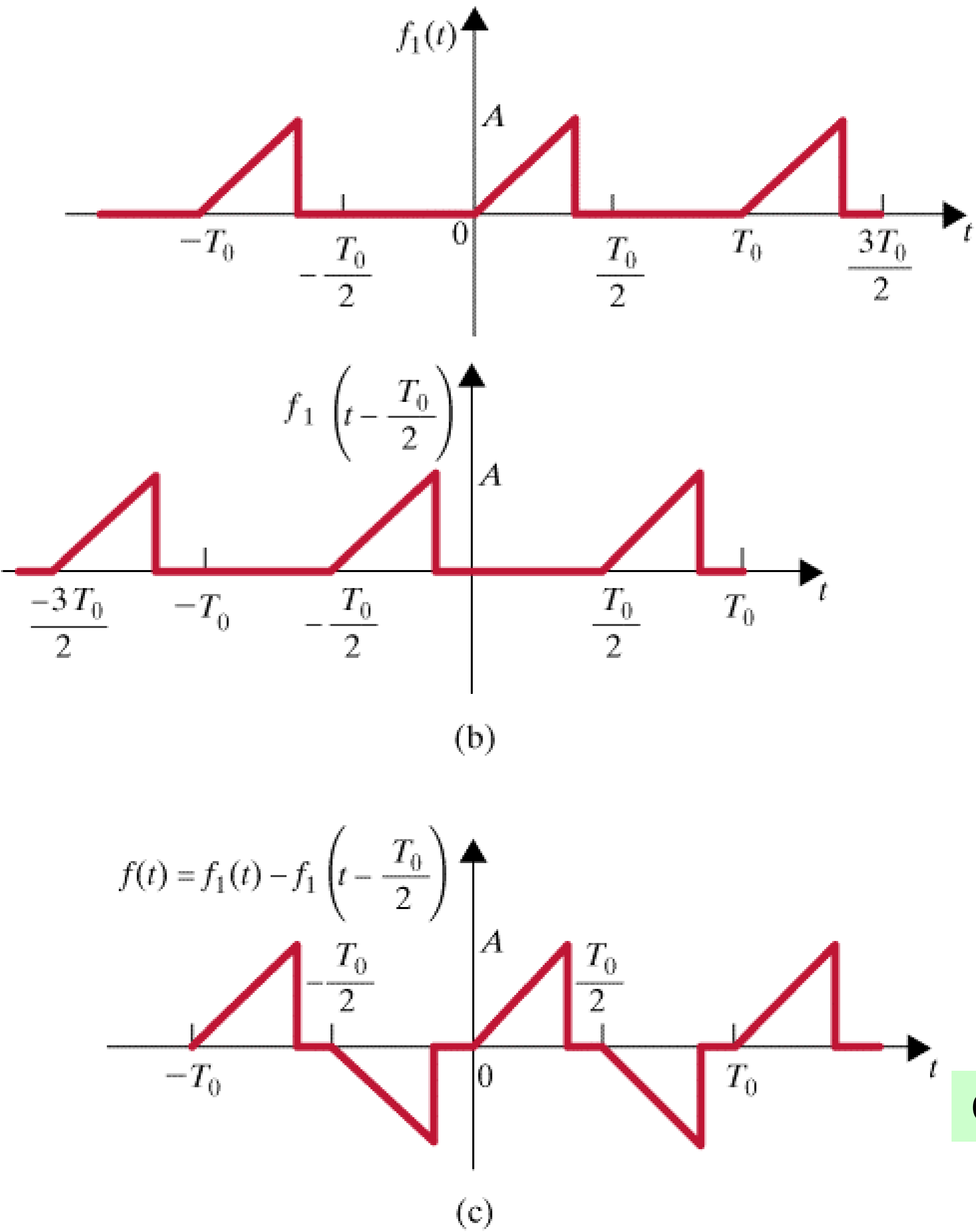
$$e^{-jn\frac{\pi}{2}}=\cos n\frac{\pi}{2}-j\sin n\frac{\pi}{2}$$

$$\left(\sin n\frac{\pi}{2}\right)^2=1$$

$$c_{v,2k+1}=-j\frac{2V}{\pi n}$$

GEAUX

Time shifting and half-wave periodic signals



Every half - wave periodic signal, $f(t)$, with period, T_0, can be expressed as

$$f(t)=f_1(t)-f_1\left(t-\frac{T_0}{2}\right)$$

with $f_1(t)=\begin{cases} f(t) & 0\le t<\frac{T_0}{2} \\ 0 & \frac{T_0}{2}\le t<T_0 \end{cases}$

Assume $f_1(t)=\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$

$\omega_0 T_0=2\pi$

$$f_1\left(t-\frac{T_0}{2}\right)=\sum_{n=-\infty}^{n=\infty}\left(c_n e^{-jn\omega_0\frac{T_0}{2}}\right)e^{jn\omega_0 t}=$$

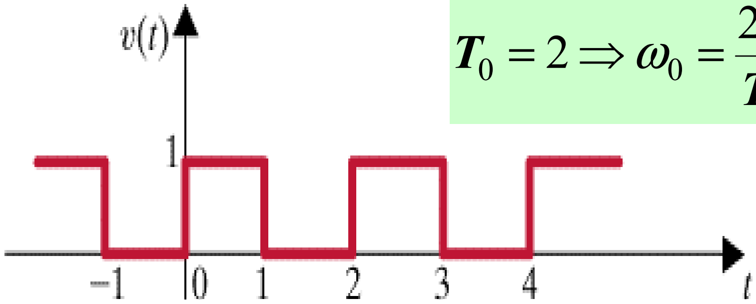
$e^{-jn\pi}=(-1)^n,\ \forall n$

$$\therefore\ f(t)=\sum_{k=-\infty}^{\infty}2c_{2k+1}e^{-j(2k+1)\omega_0 t}$$

Only the odd coefficients of *f1* are used

GEAUX

LEARNING EXTENSION



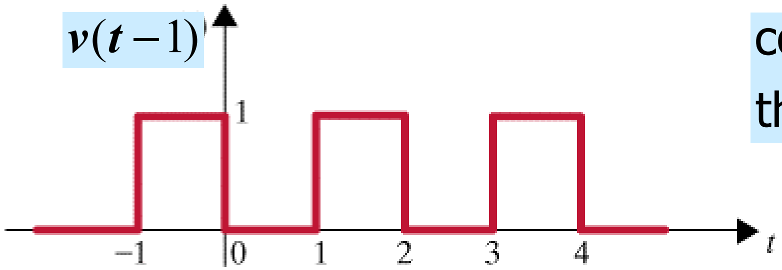
$$T_0 = 2 \Rightarrow \omega_0 = \frac{2\pi}{T_0} = \pi$$

It was shown before that for *v(t)*

$$c_0 = \frac{1}{2}$$

$$c_n = \frac{1 - e^{-jn\pi}}{j2\pi n}, n \neq 0$$

$v(t-1)$



compute the coefficients for $v(t\text{-}1)$ and show that they differ by a πn angle from those of $v(t)$.

For the delayed function

$$c_{0d} = \frac{1}{2}\int_0^2 v(t-1)dt = \frac{1}{2}$$

For $n \neq 0$

$$c_{nd} = \frac{1}{2}\int_0^2 v(t-1)e^{-jn\pi t}dt = \frac{1}{2}\int_1^2 e^{-jn\pi t}dt = \frac{1}{j2\pi n}\left[e^{-jn\pi} - e^{-j2n\pi}\right] = \frac{e^{-jn\pi}}{j2\pi n}\left[1 - e^{-jn\pi}\right]$$

$$c_{nd} = e^{-jn\pi}c_n$$

GEAUX

WAVEFORM GENERATION

Assume $f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$

$$f(\alpha t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\alpha n\omega_0 t}$$

Time scaling does not change the values of the series expansion

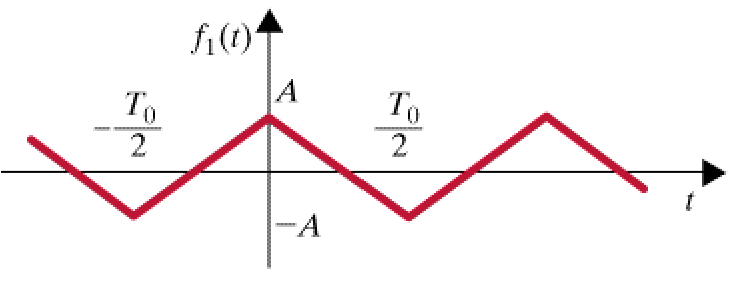
$$f(t-t_0) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 (t-t_0)} = \sum_{n=-\infty}^{n=\infty} \left(c_n e^{-jn\omega_0 t_0}\right) e^{jn\omega_0 t}$$

Time-shifting modifies the phase of the coefficients

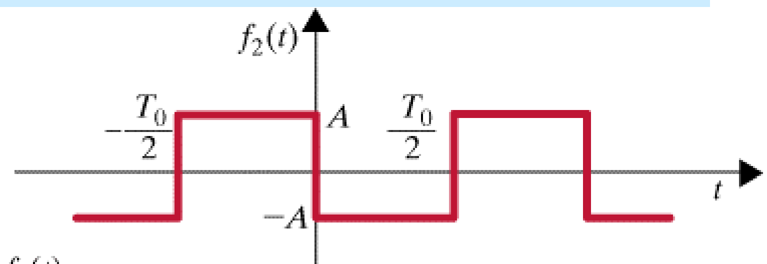
$$f(\alpha t - t_0) = \sum_{n=-\infty}^{n=\infty} \left(c_n e^{-jn\omega_0 t_0}\right) e^{j\alpha n\omega_0 t}$$

If the Fourier series for *f(t)* is known then one can easily determine the expansion for any time-shifted and time-scaled version of *f(t)*

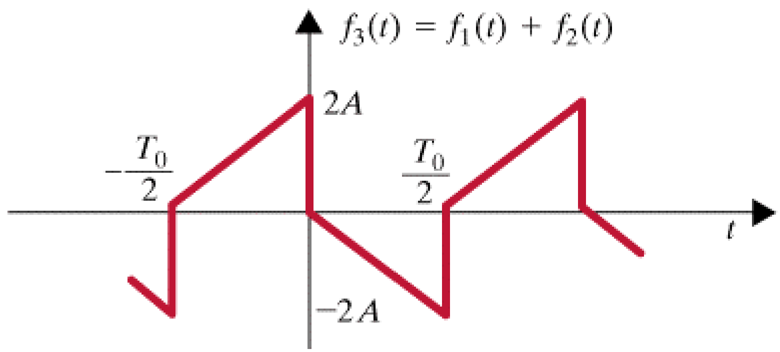
The coefficients of a linear combination of signals are the linear combination of the coefficients



(a)



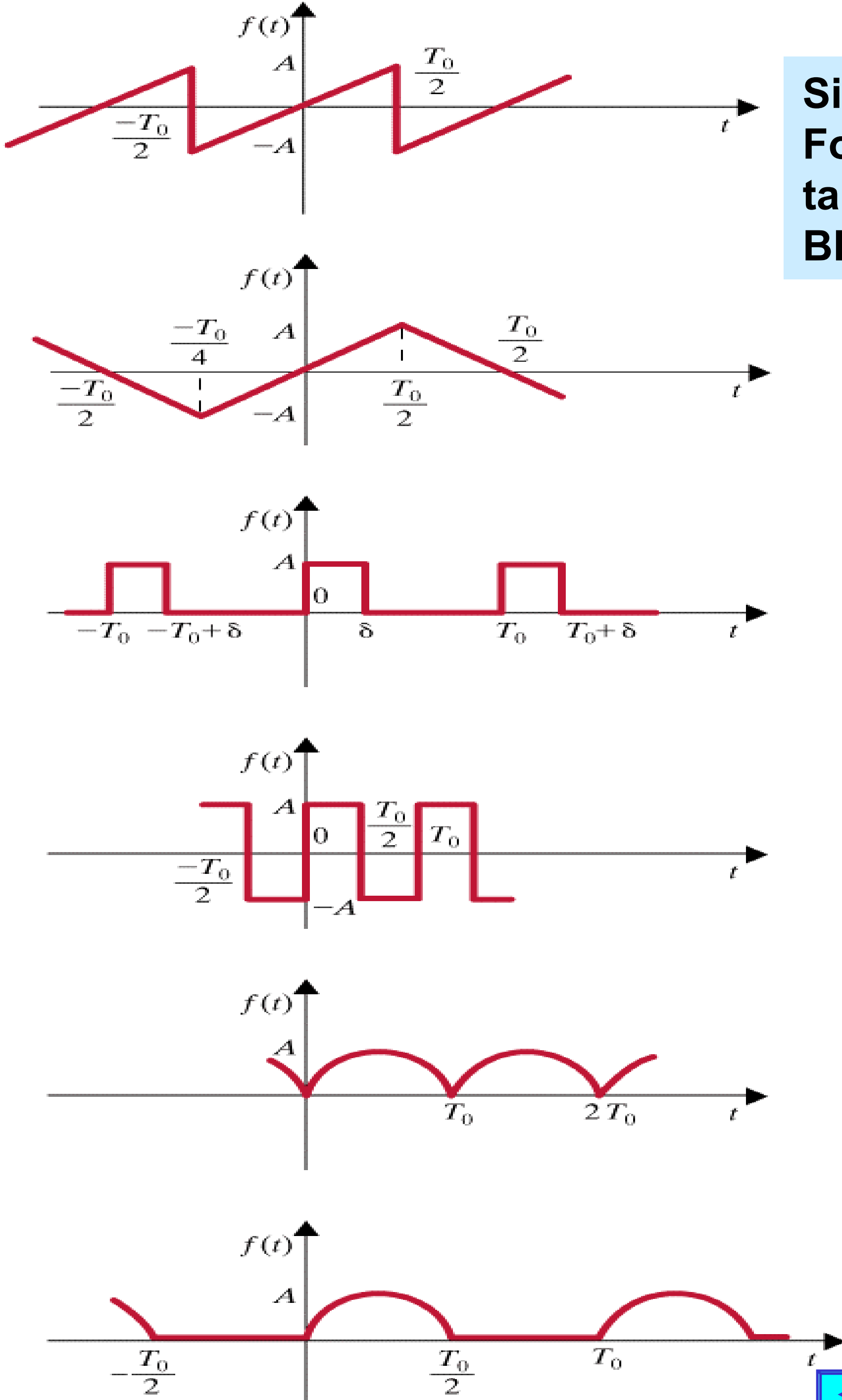
(b)

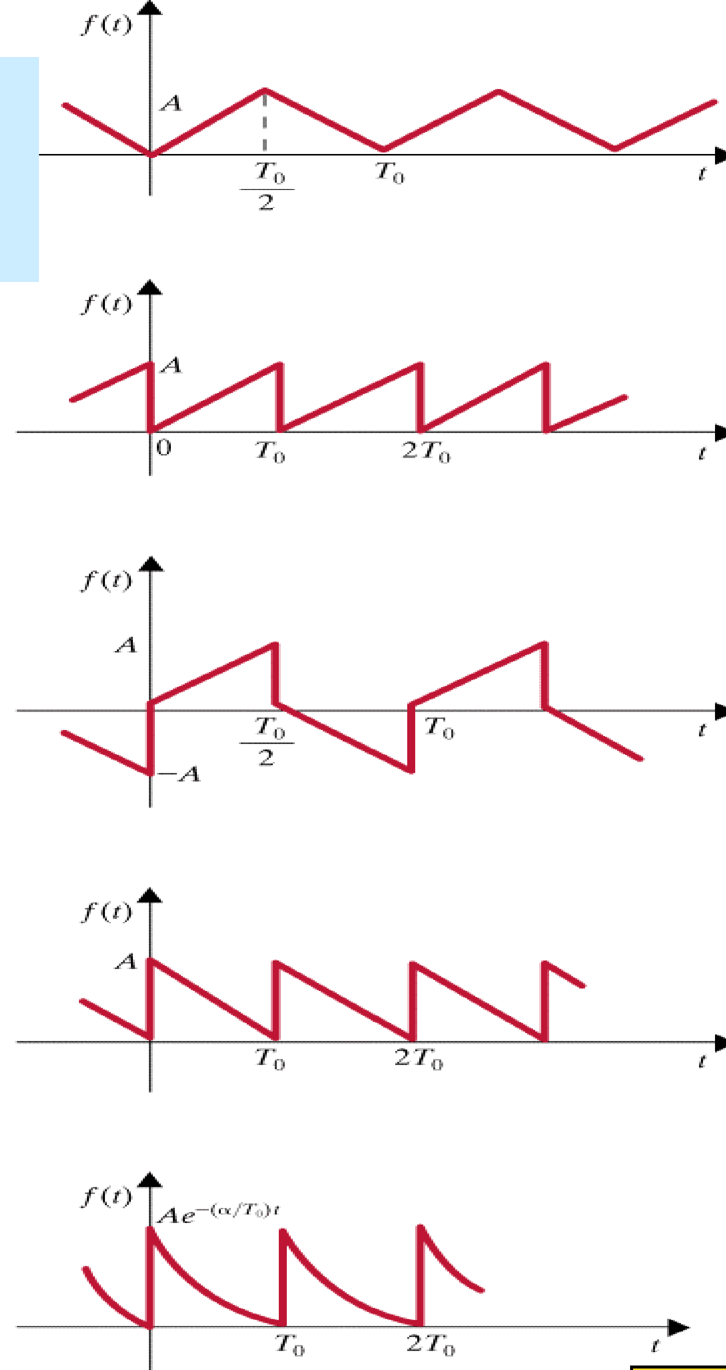


$$c_{n3} = c_{n1} + c_{n2} \quad \forall n$$

One can tabulate the expansions for some basic waveforms and use them to determine the expansions or other signals

Signals with Fourier series tabulated in BECA 7

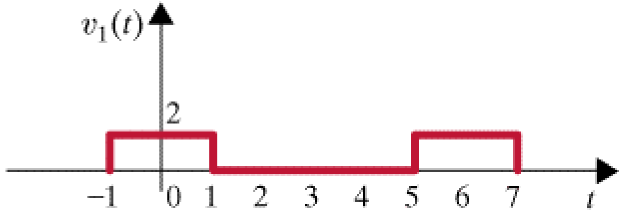


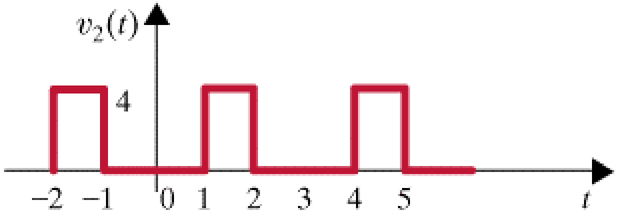


GEAUX

LEARNING EXTENSION

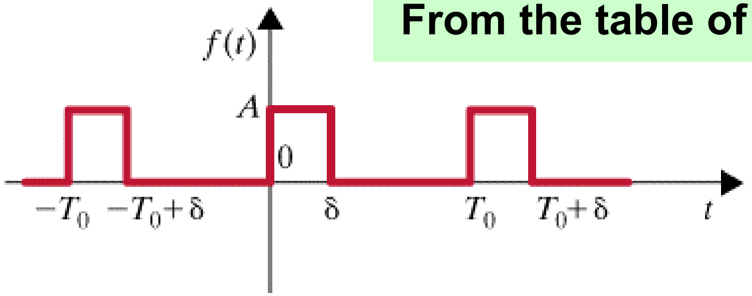
Use the table of Fourier series to determine the expansions of these functions





$$c_0 = \frac{4}{3}$$

From the table of Fourier series



$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A}{n\pi}\sin\frac{n\pi\delta}{T_0} e^{jn\omega_0\left(t-\frac{\delta}{2}\right)}$$

$$c_0 = \frac{A\delta}{T_0}$$

For $v_1(t)$

$$v_1(t) = f(t+\delta/2)$$

$$A=2,\ T_0=6,\ \delta=2$$

$$v_1(t) = \sum_{n=-\infty}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{3} e^{jn\frac{\pi}{3}t}$$

$$c_0 = \frac{2}{3}$$

Strictly speaking the value for n=0 must be computed separately.

$$e^{-jn\pi} = (-1)^n$$

For $v_2(t)$

$$v_2(t) = f\left(t - \frac{T_0-\delta}{2}\right)$$

$$A=4,\ T_0=3,\ \delta=1, \omega_0 = \frac{2\pi}{3}$$

$$v_2(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi}{3} e^{jn\frac{2\pi}{3}\left(t-\frac{3-1}{2}-\frac{1}{2}\right)}$$

$$v_2(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi}{3} e^{-jn\pi} e^{jn\frac{2\pi}{3}t}$$

GEAUX

FREQUENCY SPECTRUM

The spectrum is a graphical display of the coefficients of the Fourier series.
The one-sided spectrum is based on the representation

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_o t + \theta_n) = a_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[D_n \angle \theta_n e^{jn\omega_o t}\right]$$

The amplitude spectrum displays *Dn* as the function of the frequency.
The phase spectrum displays the angle $\angle\theta_n$ function of the frequency.
The frequency axis is usually drawn in units of fundamental frequency

The two-sided spectrum is based on the exponential representation

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

In the two-sided case, the amplitude spectrum plots |*cn*| while the phase spectrum plots $\angle c_n$ versus frequency (in units of fundamental frequency)

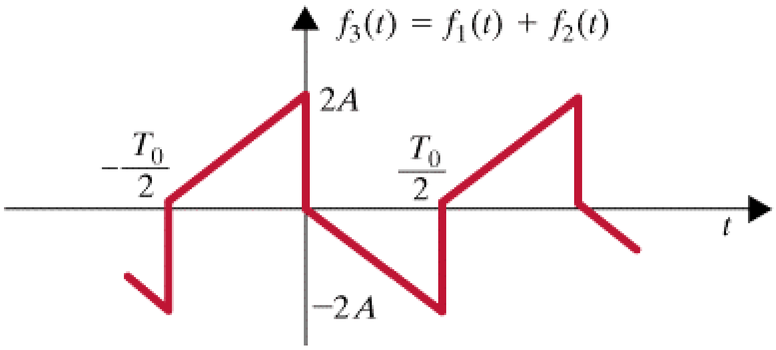
$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$c_{-n} = c_n^*$$

Both spectra display equivalent information

GEAUX

LEARNING EXAMPLE



(c)

The Fourier series expansion, when A=5, is given by

$$v(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{20}{n\pi}\sin n\omega_0 t - \frac{40}{n^2\pi^2}\cos n\omega_0 t)$$

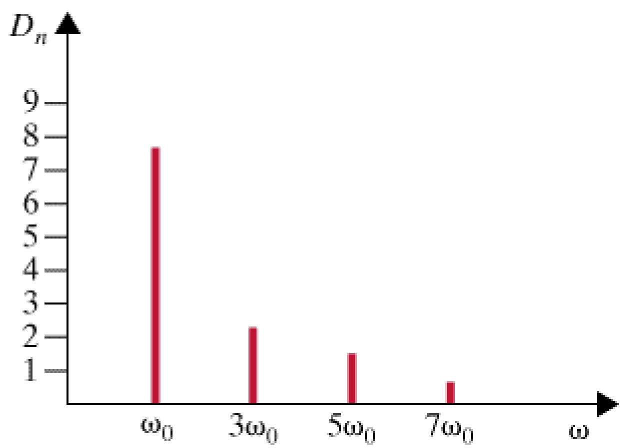
Determine and plot the first four terms of the spectrum

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

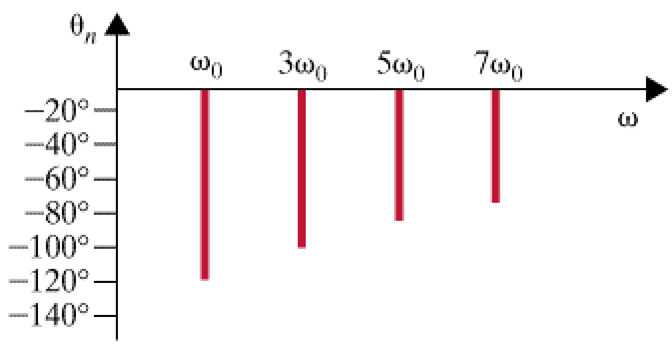
$$D_1 \angle \theta_1 = -\frac{40}{\pi^2} - j\frac{20}{\pi} = 7.5\angle -122^\circ$$

$$D_3 = -\frac{40}{9\pi^2} - j\frac{20}{3\pi} = 2.2\angle -102^\circ$$

$$D_5 \angle \theta_5 = 1.3\angle -97^\circ, D_7 \angle \theta_7 = 0.92\angle -95^\circ$$



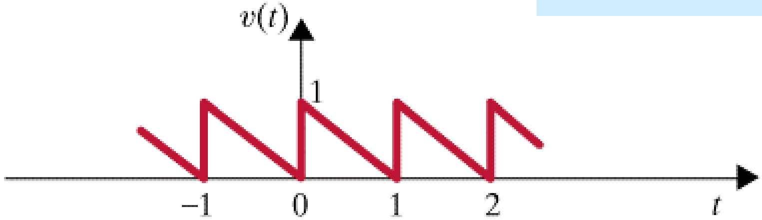
Amplitude spectrum



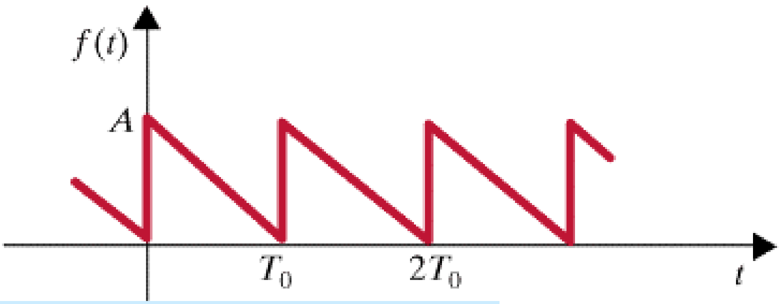
Phase spectrum

LEARNING EXTENSION

Determine the trigonometric Fourier series and plot the first four terms of the amplitude and phase spectra



$$v(t) = \frac{1}{2} + \sum_{1}^{\infty} \frac{1}{\pi n} \sin 2\pi n t$$



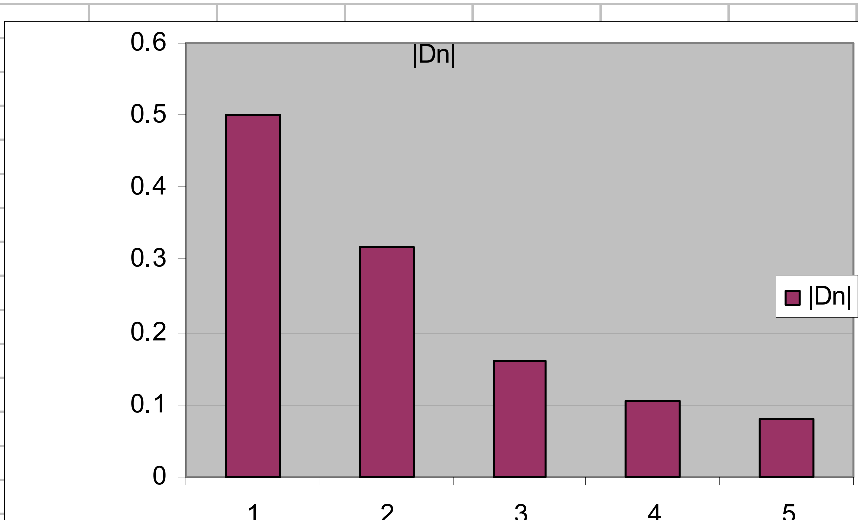
$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{\pi n} \sin n\omega_0 t$$

From the table of series

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$D_0 = \frac{1}{2};\ D_n = -j\frac{1}{\pi n};\ n > 0$$

n	|Dn|	Arg(Dn)
0	0.5	0
1	0.318309	-90
2	0.159155	-90
3	0.106103	-90
4	0.079577	-90

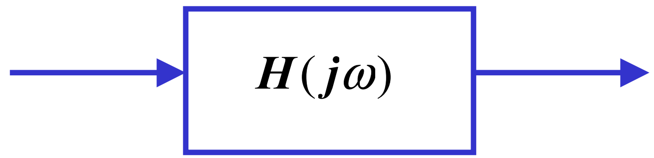


GEAUX

STEADY STATE NETWOK RESPONSE TO PERIODIC INPUTS

1. Replace the periodic signal by its Fourier series
2. Determine the steady state response to each harmonic
3. Add the steady state harmonic responses

$$H(j\omega) = |H(j\omega)| e^{j\phi(\omega)} = |H(j\omega)| \angle \phi(\omega)^{\circ}$$



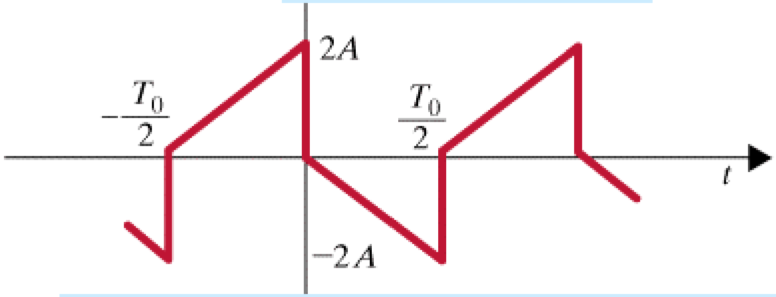
$$|c| e^{j(\omega_1 t + \theta)} \longrightarrow |c| |H(j\omega_1)| e^{j(\omega_1 t + \theta + \phi(\omega_1))}$$

$$|D| \cos(\omega_1 t + \theta) \longrightarrow |D| |H(j\omega_1)| \cos(\omega_1 t + \theta + \phi(\omega_1))$$

$$\mathrm{Re}\{|D| \angle \theta\, e^{j\omega_1 t}\} \longrightarrow \mathrm{Re}\{|D| \angle \theta\, |H(j\omega_1)| \angle \phi(\omega_1) e^{\omega_1 t}\}$$

LEARNING EXAMPLE

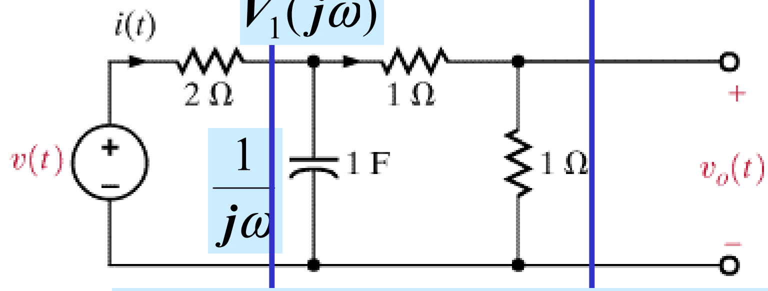
$v(t):\; A=5,\; T_0=\pi$



$I(j\omega)$

$V_1(j\omega)$

$(2 \| \frac{1}{j\omega}) = \frac{2}{1+2j\omega}$



$\frac{1}{j\omega}$

$V_0 = \frac{1}{2}V_1$

$$v(t)=\sum_{\substack{n=1\\ n\ odd}}^{\infty}\left(\frac{20}{n\pi}\sin 2nt-\frac{40}{n^2\pi^2}\cos 2nt\right)$$

$$V_1(j\omega)=\frac{\frac{2}{1+2j\omega}}{2+\frac{2}{1+2j\omega}}V(j\omega)=\frac{2}{4+4j\omega}V(j\omega)$$

$$|4+j4\omega|=\sqrt{16+16\omega^2}$$

$$\angle(4+j4\omega)=\tan^{-1}\omega$$

$$\omega_n=2n$$

$$H(j\omega)=\frac{1}{4+j4\omega}=\frac{1}{\sqrt{16+16\omega^2}}\angle-\tan^{-1}\omega$$

$$D_n=a_n-jb_n=-\frac{40}{n^2\pi^2}-j\frac{20}{n\pi}$$

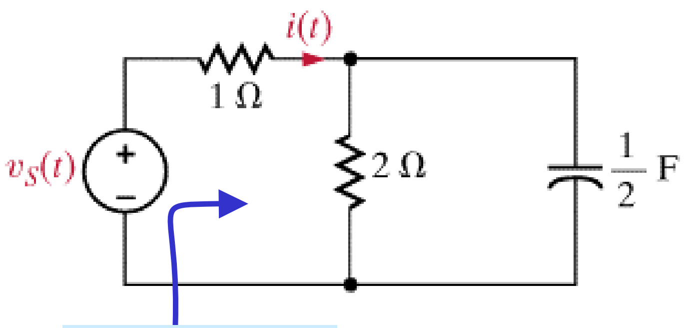
$$|D_n|=\sqrt{\left(\frac{40}{n^2\pi^2}\right)^2+\left(\frac{20}{n\pi}\right)^2}$$

$$\angle D_n=180^\circ+\tan^{-1}\frac{n\pi}{2}=-(180^\circ-\tan^{-1}\frac{n\pi}{2})$$

$$v_{on}(t)=\sqrt{\left(\frac{40}{n^2\pi^2}\right)^2+\left(\frac{20}{n\pi}\right)^2}\frac{1}{\sqrt{16+64n^2}}\cos(2nt+\tan^{-1}\frac{n\pi}{2}-\tan^{-1}2n-180^\circ)$$

GEAUX

LEARNING EXTENSION Find the steady state expression for the current $i(t)$



$$Z = 1 + 2 \,||\, \frac{2}{j\omega}$$

$$v_S(t) = \frac{20}{\pi} + \sum_{n=1}^{\infty} \frac{-40}{\pi(4n^2-1)} \cos 2nt$$

$$D_0 = \frac{20}{\pi};\ D_n = \frac{40}{\pi(4n^2-1)} \angle 180°,\ n \ge 1;\ \omega_n = 2n$$

$$Z(j\omega) = 1 + \frac{\frac{4}{j\omega}}{2 + \frac{2}{j\omega}} = 1 + \frac{4}{2 + j2\omega} = \frac{6 + j2\omega}{2 + j2\omega} = \frac{3 + j\omega}{1 + j\omega}$$

$$I(j\omega) = \frac{V_S(j\omega)}{Z(j\omega)} = Y(j\omega)V_S(j\omega)$$

$$Y(j\omega) = \frac{1 + j\omega}{3 + j\omega}$$

$$I(j2n) = Y(j2n)D_n = \frac{1 + j2n}{3 + j2n} D_n$$ phasor for n-th harmonic

$$Y(0) = \frac{1}{3}$$

$$Y(j2n) = \sqrt{\frac{1 + 4n^2}{9 + 4n^2}} (\angle \tan^{-1} 2n - \angle \tan^{-1}(2n/3))$$

$$i(t) = Y(0)D_0 - \sum_{n=1}^{\infty} |Y_n||D_n| \cos(2nt - \angle Y_n)$$

GEAUX

AVERAGE POWER

In a network with periodic sources (of the same period) the steady state voltage across any element and the current through are all of the form

$$v(t) = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn})$$

$$i(t) = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in})$$

The average power is the sum of the average powers for each harmonic

$$\text{Average Power } P = \frac{1}{T}\int_{t_o}^{t_o+T} v(t)i(t)dt$$

$$\int_{t_0}^{t_0+T}$$

$$v(t)i(t) = V_{dc}I_{dc} + V_{dc}\sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in}) + I_{dc}\sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn})$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} V_{n_1} I_{n_2} \cos(n_1\omega_0 t - \theta_{vn_1})\cos(n\omega_0 t - \theta_{in_2})$$

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} V_{n_1} I_{n_2} \cos(n_1\omega_0 t - \theta_{vn_1})\cos(n\omega_0 t - \theta_{in_2}) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \frac{V_{n1}I_{n2}}{2}\left[\begin{array}{l}\cos((n_1+n_2)\omega_0 t - (\theta_{vn_1}+\theta_{in_2})) \\ +\cos((n_1-n_2)\omega_0 t - (\theta_{vn_1}-\theta_{in_2}))\end{array}\right]$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \int_{t_0}^{t_0+T}\cos(k\omega_0 t + \theta)dt = \begin{cases} 0 & k \neq 0 \\ T & k = 0 \end{cases}$$

$$= T\cos(\theta_{vn} - \theta_{in}) \quad for\ n_1 = n_2$$

$$\text{Average Power } P = V_{dc}I_{dc} + \sum_{n=1}^{\infty}\frac{V_n I_n}{2}\cos(\theta_{vn} - \theta_{in})$$

GEAUX

LEARNING EXTENSION

Determine the average power

$$v(t) = 64 + 36\cos(377t + 60°) \underline{- 24}\cos(754t + 102°)[V]$$

$$i(t) = 1.8\cos(377t + 45°) + 1.2\cos(754t + 100°)[A]$$

$$-\cos\alpha = \cos(\alpha - 180°)$$

$$P = 0.5(36 \times 1.8\cos(15°) + 24 \times 1.2\cos(102° - 180° - 100°))$$

$$P = \frac{62.59 - 28.78}{2} = 16.91[W]$$

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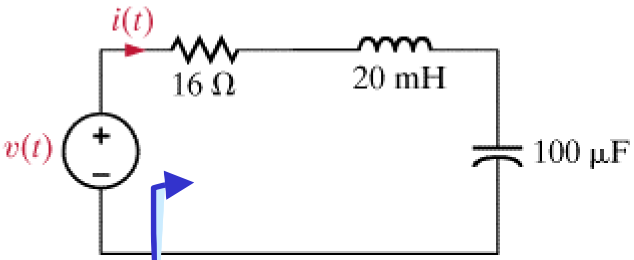
LEARNING EXAMPLE

Determine the current, $i(t)$, and the average power absorbed by the network



$$v(t) = 42 + 16\cos(377t + 30°) + 12\cos(754t - 20°)[V]$$

$$Z(j\omega) = R + jL\omega + \frac{1}{jC\omega}$$

$$I(j\omega) = \frac{V(j\omega)}{Z(j\omega)}$$

$\omega = 377$

$V(j377) = 16\angle 30°,$

$$Z(j377) = 16 + j0.020 \times 377 - j\frac{1}{10^{-4}377}$$

$$I(j377) = \frac{16\angle 30°}{16 + j7.54 - j26.53} = 0.64\angle 79.88°$$

$\omega = 0$

capacitor acts as open circuit ($Z = \infty$)

$I(j0) = 0$

$$i(t) = 0.64\sin(377t + 79.88°) + 0.75\cos(754t - 26.49°)$$

Average Power $P = V_{dc}I_{dc} + \sum_{n=1}^{\infty}\frac{V_n I_n}{2}\cos(\theta_{vn} - \theta_{in})$

$\omega = 754$

$V(j754) = 12\angle -20°,$

$$Z(j754) = 16 + j0.020 \times 754 - j\frac{1}{10^{-4} \times 754}$$

$$I(j754) = \frac{12\angle -20°}{16 + j15.08 - j13.26} = 0.75\angle -26.49°$$

$$P = 42(0) + \frac{16 \times 0.64\cos(49.88°) + 12 \times 0.75\cos(6.49°)}{2}$$

FOURIER TRANSFORM

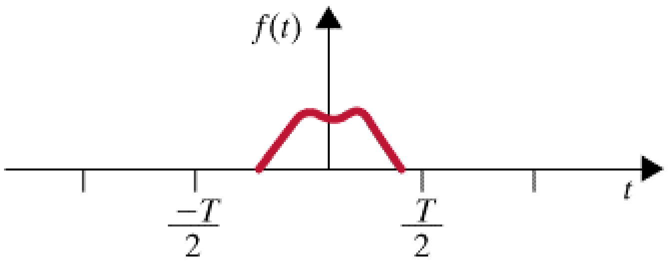
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

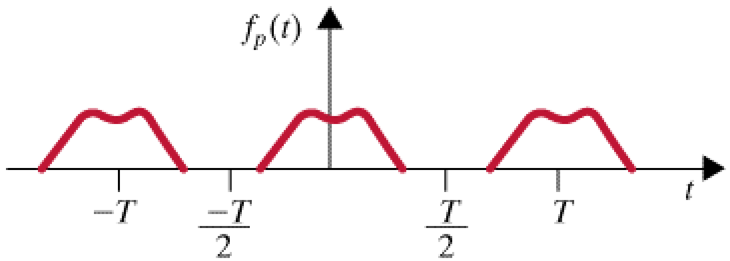
$$F(\omega) = \mathrm{F}[f(t)]$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$f(t) = \mathrm{F}^{-1}[F(\omega)]$$

A heuristic view of the Fourier transform





A non-periodic function can be viewed as the limit of a periodic function when the period approaches infinity

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\frac{2\pi}{T}t} dt$$

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n T e^{jn\frac{2\pi}{T}t} \frac{1}{T} \rightarrow \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$c_n T = \int_{-T/2}^{T/2} f(t) e^{-jn\frac{2\pi}{T}t} dt \rightarrow \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$$

$$\frac{2\pi}{T} = \Delta\omega$$

$$n\Delta\omega \approx \omega_n$$

$$T \rightarrow \infty$$

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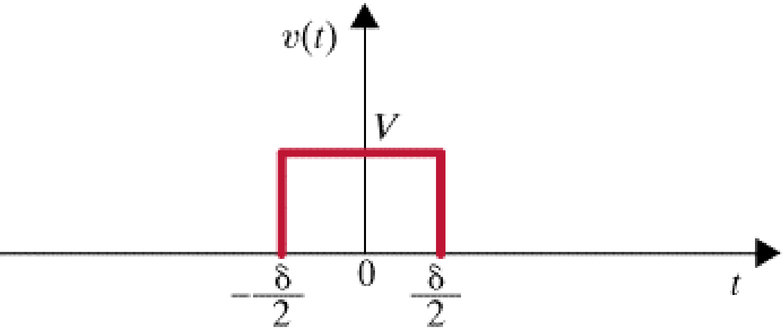
LEARNING EXAMPLE

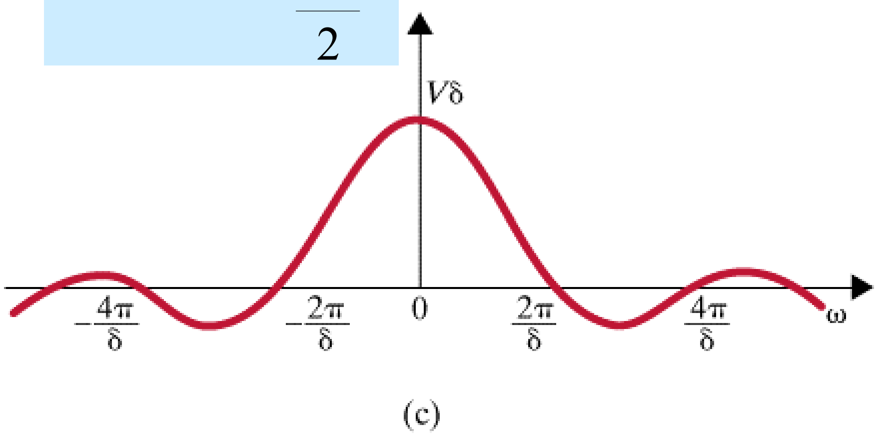
Determine the Fourier transform



$$V(\omega)=\int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt = V\int_{-\delta/2}^{\delta/2} e^{-j\omega t}dt$$

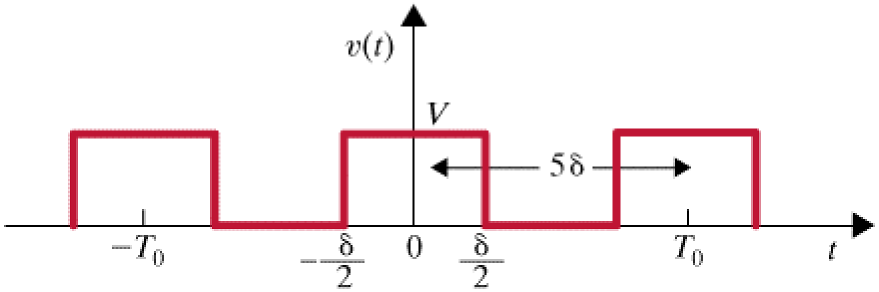
$$V(\omega)=V\left[-\frac{1}{j\omega}e^{-j\omega t}\right]_{-\delta/2}^{\delta/2} = V\frac{e^{j\omega\delta/2}-e^{-j\omega\delta/2}}{j\omega}$$

$$V(\omega)=V\delta\frac{\sin\frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$

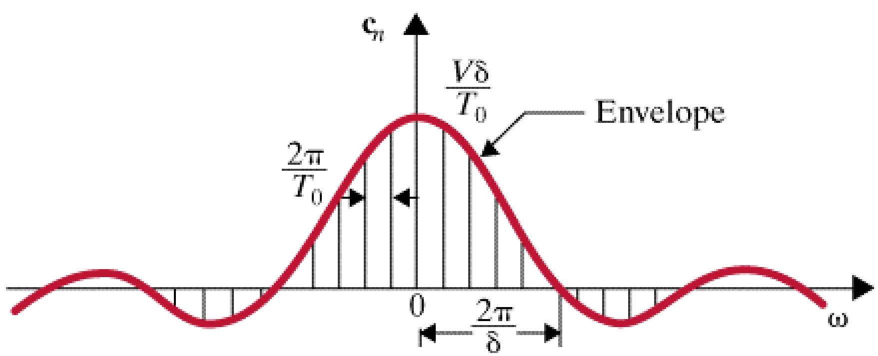


(c)

For comparison we show the spectrum of a related periodic function



$$c_n=\frac{V\delta}{T_0}\frac{\sin\frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$



Spectrum for $T_0=5\delta$

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LEARNING EXAMPLES

Determine the Fourier transform of the unit impulse function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \mathrm{F}[f(t)]$$

Sifting or sampling property of the impulse

$$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$$

$$\mathrm{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t-a) e^{-j\omega t} dt = e^{-j\omega a}$$

Determine the Fourier transform of

$$f(t) = e^{j\omega_o t}$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

Consider $F(\omega) = 2\pi\delta(\omega - \omega_0)$ and find the correspond ing function of time

$$f(t) = \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$f(t) = e^{j\omega_0 t}$$

$$F(\omega) = \mathrm{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$$

LEARNING EXTENSION

Determine $F(\omega) = \mathrm{F}[\sin\omega_0 \mathrm{t}]$

$$f(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \Rightarrow F(\omega) = \frac{2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)}{2j}$$

$$F(\omega) = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Table 14.3 Fourier transform pairs

f(t)	F(ω)
$\delta(t-a)$	$e^{-j\omega a}$
A	$2\pi A\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin\omega_0 t$	$j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0)$
$e^{-at}u(t), a>0$	$\frac{1}{a+j\omega}$
$e^{-\alpha\|t\|}, a>0$	$\frac{2a}{a^2+\omega^2}$
$e^{-at}\cos\omega_0 t u(t), a>0$	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$
$e^{-at}\sin\omega_0 t u(t), a>0$	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$

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Table 14.4 Properties of the Fourier transform

$f(t)$	$\mathbf{F}(\omega)$	Property
$Af(t)$	$A\mathbf{F}(\omega)$	Linearity
$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\omega) \pm \mathbf{F}_2(\omega)$	
$f(at)$	$\frac{1}{a}\mathbf{F}\left(\frac{\omega}{a}\right), a > 0$	Time-scaling
$f(t - t_0)$	$e^{-j\omega t_0}\mathbf{F}(\omega)$	Time-shifting
$e^{j\omega t_0}f(t)$	$\mathbf{F}(\omega - \omega_0)$	Modulation
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n\mathbf{F}(\omega)$	
$t^n f(t)$	$(j)^n \frac{d^n\mathbf{F}(\omega)}{d\omega^n}$	Differentiation
$\int_{-\infty}^{\infty} f_1(x)f_2(t - x)\, dx$	$\mathbf{F}_1(\omega)\mathbf{F}_2(\omega)$	Convolution
$f_1(t)f_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{F}_1(x)\mathbf{F}_2(\omega - x)\, dx$	

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Proof of the convolution property

$$f(t) = f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x)\,dx$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(t-x)\,dx \right] e^{-j\omega t}\,dt$$

Exchanging orders of integration

$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(t-x) e^{-j\omega t}\,dt \right] dx$$

Change integration variable

$$u = t - x \Rightarrow t = u + x$$

And limits of integration remain the same

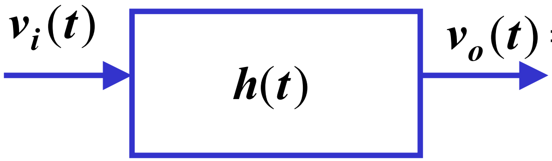
$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(u) e^{-j\omega(u+x)}\,du \right] dx$$

$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(u) e^{-j\omega u} e^{-j\omega x}\,du \right] dx$$

$$F(\omega) = \int_{-\infty}^{\infty} f_1(x) e^{-j\omega x}\,dx \left[\int_{-\infty}^{\infty} f_2(u) e^{-j\omega u}\,du \right]$$

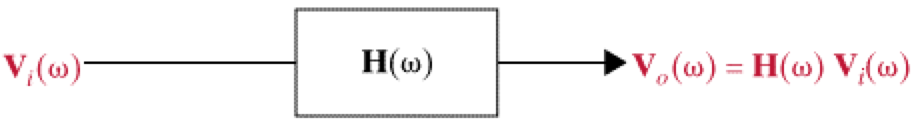
$$F(\omega) = F_1(\omega) F_2(\omega)$$

A Systems application of the convolution property



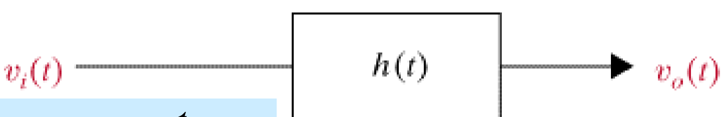
$$v_o(t) = \int_{-\infty}^{\infty} h(t-x) v_i(x)\,dx$$

The output (response) of a network can be computed using the Fourier transform



LEARNING EXTENSION

Use Fourier transform to determine $v_o(t)$



$v_i(t) = e^{-t}u(t)$

$h(t) = e^{-2t}u(t)$

(And all initial conditions are zero)

$$V_o(\omega) = H(\omega)V_i(\omega) = \frac{1}{j\omega+2}\frac{1}{j\omega+1}$$

Use partial fraction expansion!

$$\frac{1}{(s+2)(s+1)} = \frac{A_1}{s+2} + \frac{A_2}{s+1}$$

From the table of transforms

$e^{-at}u(t),\ a>0$	$\dfrac{1}{a+j\omega}$

$$A_1 = (s+2)V_o(s)\,|_{s=-2} = -1$$
$$A_2 = (s+1)V_o(s)\,|_{s=-1} = 1$$

$$V_o(\omega) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2}$$

$$v_o(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

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PARSEVAL'S THEOREM

Think of *f(t)* as a voltage applied to a one Ohm resistor

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \frac{d\omega}{2\pi}$$

$$p(t) = v(t)i(t) = |f(t)|^2$$

By definition, the left hand side is the energy of the signal

$|f(t)|^2 =$ power (or energy density in time)

$|F(\omega)|^2 =$ Energy density in the frequency domain

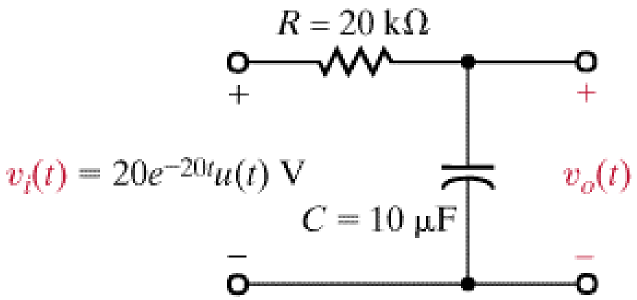
Parseval's theorem permits the determination of the energy of a signal in a given frequency range

Intuitively, if the Fourier transform has a large magnitude over a frequency range then the signal has significant energy over that range

And if the magnitude of the Fourier transform is zero (or very small) then the signal has no significant energy in that range

LEARNING BY APPLICATION

Examine the effect of this low-pass filter in the quality of the input signal

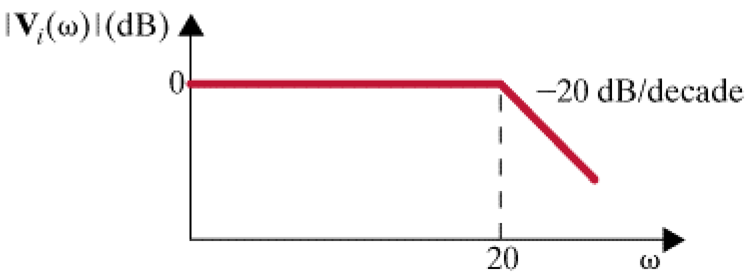


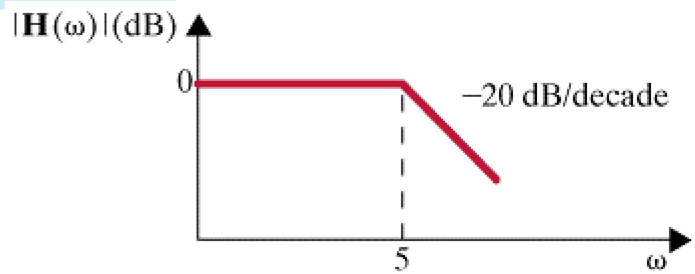
$$V_o(\omega) = H(\omega)V_i(\omega)$$

$$V_i(\omega) = \frac{20}{j\omega + 20}$$

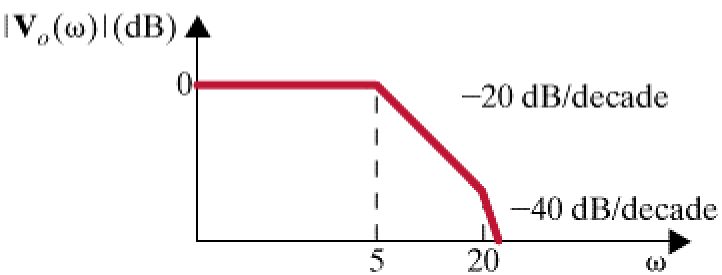
$$H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{j\omega RC + 1}$$

One can use Bode plots to visualize the effect of the filter





(c)



(d)

High frequencies in the input signal are attenuated in the output

The effect is clearly visible in the time domain

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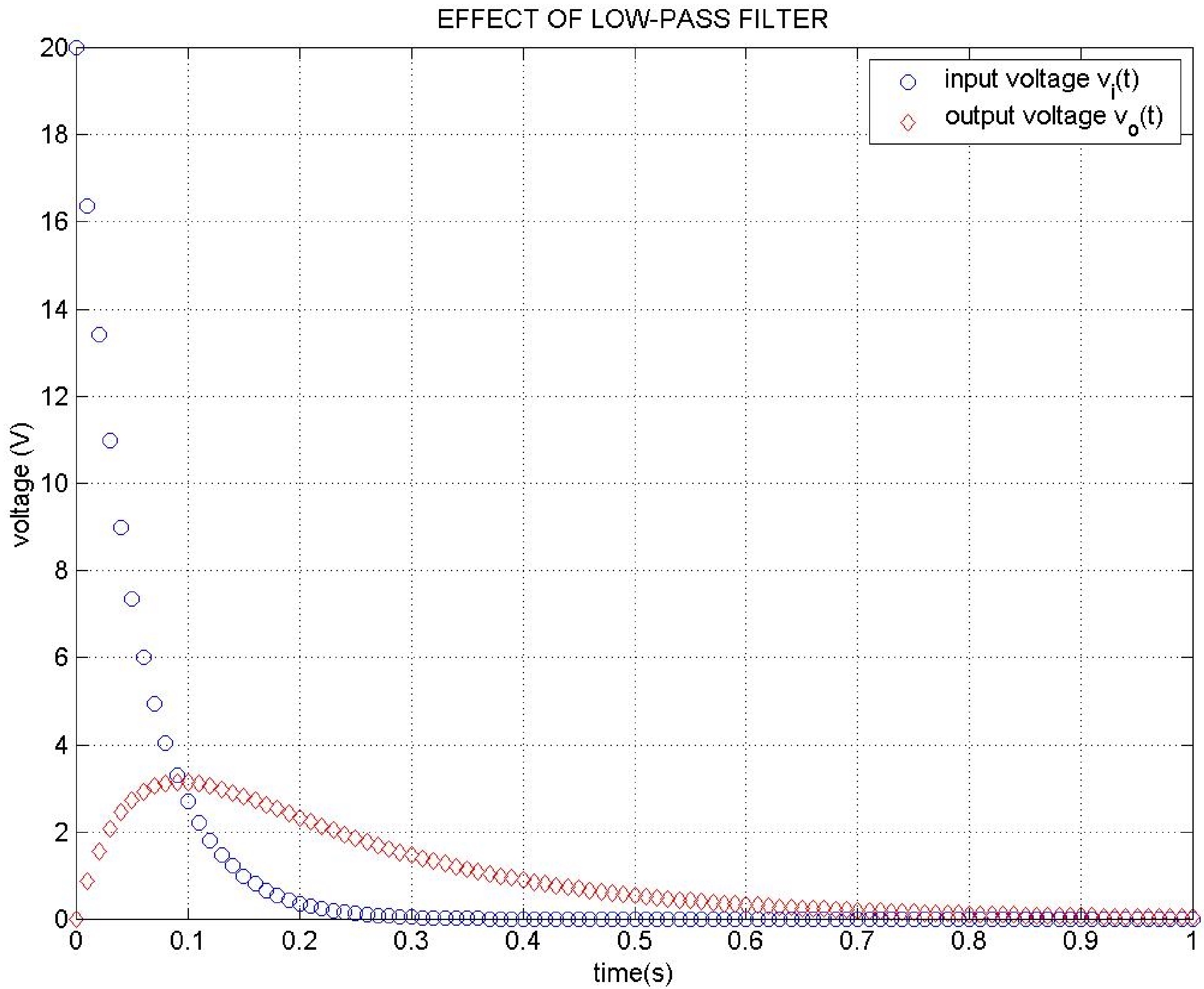
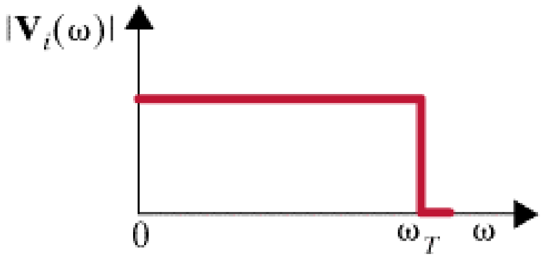


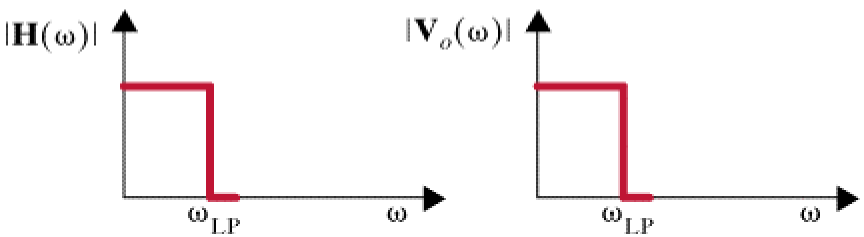
The output signal is slower and with less energy than the input signal

EFFECT OF IDEAL FILTERS



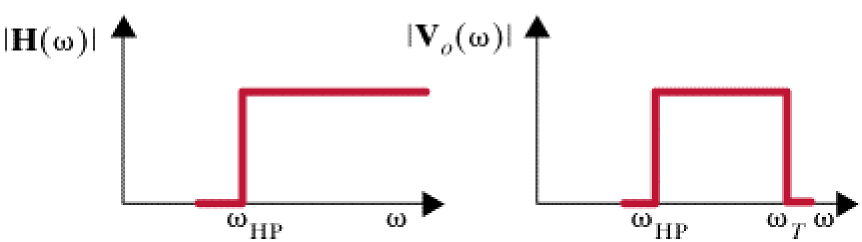
(a)

Effect of low-pass filter



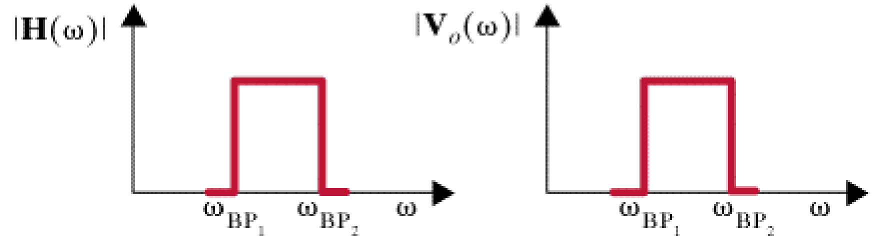
(b)

Effect of high-pass filter



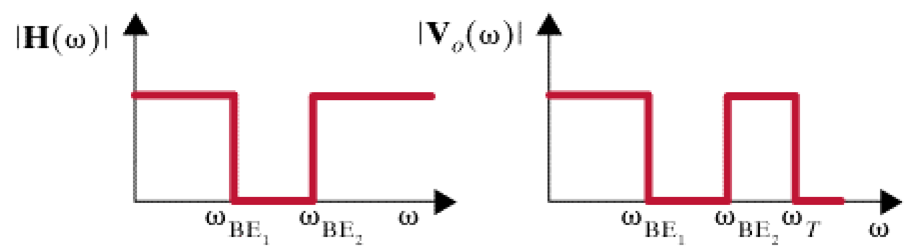
(c)

Effect of band-pass filter



(d)

Effect of band-stop filter



(e)

LEARNING BY DESIGN

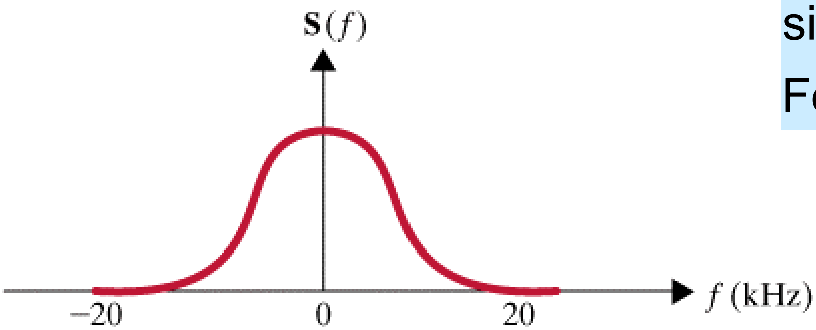
"Tuning-out" an AM radio station

signal from station 1: $v_1(t) = [1 + s_1(t)]\cos\omega_1 t$

signal from station 2: $v_2(t) = [1 + s_2(t)]\cos\omega_2 t$

For simplicity assume $s_1(t) = s_2(t)$

$$f_1 = 900kHz,\quad f_2 = 960kHz$$

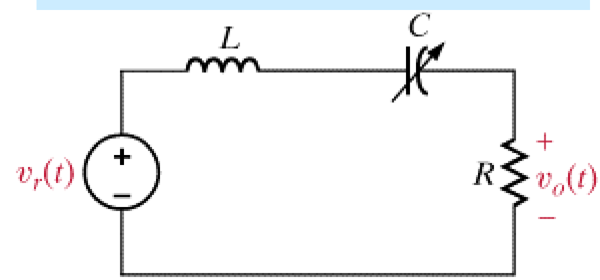


Fourier transform of signal broadcast by two AM stations

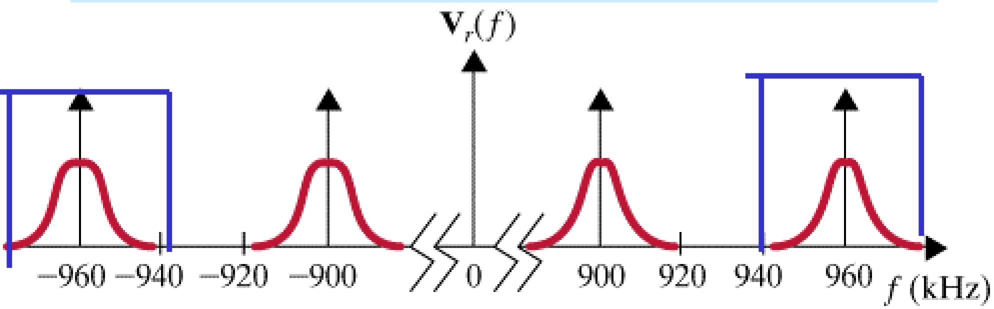
Assuming both stations reach the receiver with equal strength;

signal at antenna : $v_r(t) = K[v_1(t) + v_2(t)]$

Proposed tuning circuit



Ideal filter to tune out one AM station



Fourier transform of received signal

$$G_v(s) = \frac{v_o(s)}{v_r(s)} = \frac{R}{R + Ls + \frac{1}{Cs}} \times \frac{(s/L)}{(s/L)}$$

$$G_v(s) = \frac{s\frac{R}{L}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

Next we show how to design the tuning circuit by selecting suitable R,L,C

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Designing the tuning circuit

$$G_v(s) = \frac{s\frac{R}{L}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

Design equations

center frequency : $f_o = \frac{1}{2\pi\sqrt{LC}}$

bandwidth : $BW = \frac{1}{2\pi}\frac{R}{L}$

Design specifications

$BW \le 60kHz$

$f_o = 900kHz,\ or\ 960kHz$

More unknowns than equations. Make some choices

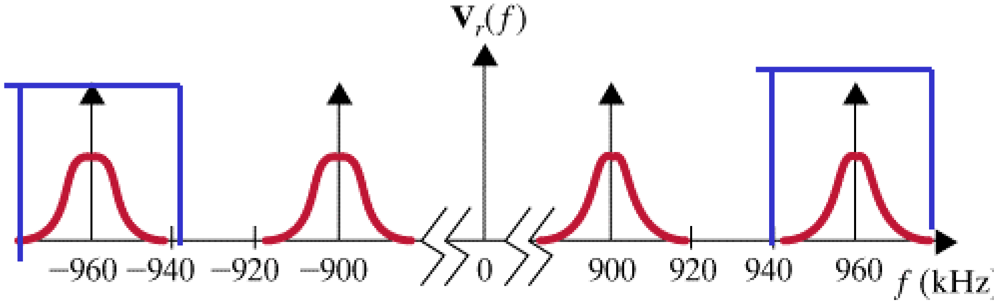
$BW = 10kHz$

$R = 10\Omega$

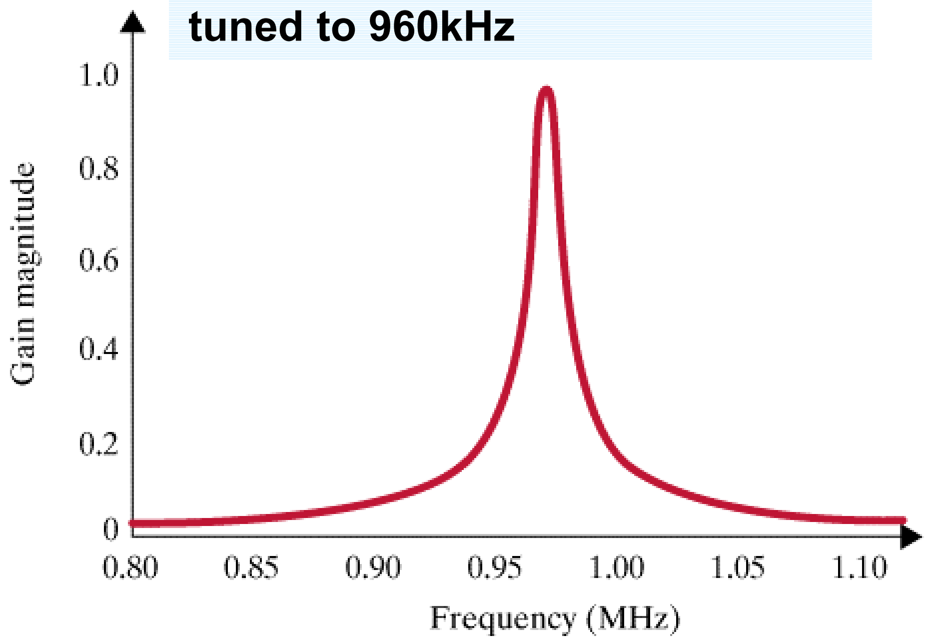
$$BW = \frac{1}{2\pi}\frac{R}{L} \Rightarrow L = 159.2\mu H$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow \begin{cases} C = 196.4pF & f_o = 900kHz \\ C = 172.6pF & f_o = 960kHz \end{cases}$$

Ideal filter to tune out one AM station



Fourier transform of received signal



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LEARNING EXAMPLE

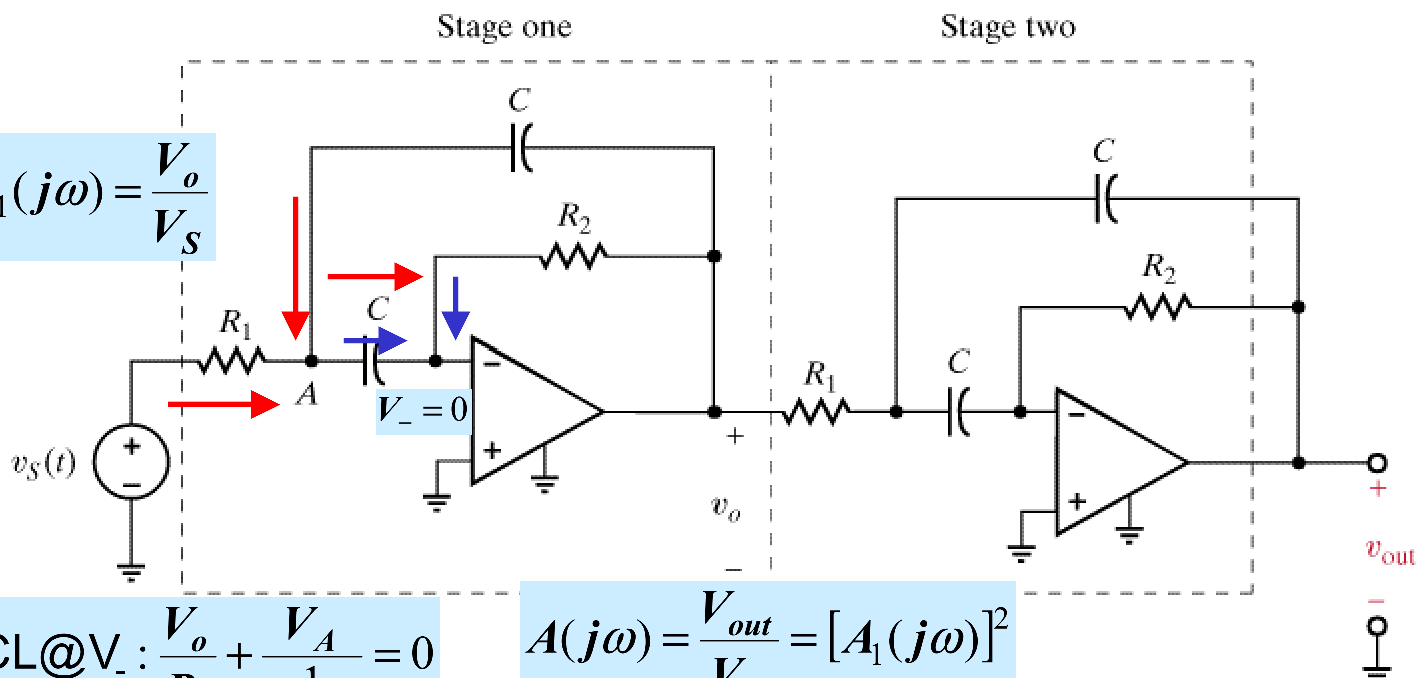
An example of band-pass filter

$$v_S(t) = 0.1\sin[2\pi\times 10^3 t] + 0.01\sin[2\pi\times 10^4 t] + 0.1\sin[2\pi\times 10^5 t]$$

noise — signal — noise

Design requirement: Make the signal 100 times stronger than the noise

$$A_1(j\omega) = \frac{V_o}{V_S}$$



Proposed two-stage band-pass with two identical stages

$$A(j\omega) = \frac{V_{out}}{V_S} = [A_1(j\omega)]^2$$

$$\text{KCL@V}_- : \frac{V_o}{R_2} + \frac{V_A}{\frac{1}{j\omega C}} = 0$$

$$\text{KCL@V}_A : \frac{V_S - V_A}{R_1} + \frac{V_o - V_A}{\frac{1}{j\omega C}} - \frac{V_A}{\frac{1}{j\omega C}} = 0$$

$$A_1(j\omega) = \frac{V_o}{V_S} = \frac{-\left[\frac{1}{CR_1}\right] j\omega}{-\omega^2 + \left[\frac{2}{CR_2}\right] j\omega + \frac{1}{C^2 R_1 R_2}}$$

GEAUX

General form of a band-pass

$$G = \frac{A_o\left[\frac{\omega_0}{Q}\right]j\omega}{-\omega^2 + \left[\frac{\omega_0}{Q}\right]j\omega + \omega_o^2}$$

ω_o = center frequency

A_o = gain at center frequency

Q = quality factor of the filter

In a log scale, this filter is symmetric around the center frequency. Hence, focus on 1kHz

$$A_1(j\omega) = \frac{V_o}{V_S} = \frac{-\left[\frac{1}{CR_1}\right]j\omega}{-\omega^2 + \left[\frac{2}{CR_2}\right]j\omega + \frac{1}{C^2 R_1 R_2}}$$

Gain at $\omega = \frac{\omega_o}{10}$: $\dfrac{j\frac{A_o\omega_o^2}{10Q}}{\omega_o^2(1-\frac{1}{100}) + j\frac{\omega_o^2}{10Q}}$

Design constraint

$$\left|\frac{j\frac{\omega_o^2}{10Q}}{\omega_o^2(1-\frac{1}{100}) + j\frac{\omega_o^2}{10Q}}\right| \le \frac{1}{\sqrt{1000}}$$

After two stages the noise gain is 1000 times smaller

Equating terms one gets a set of equations that can be used for design

Design equations

$$\omega_o = \frac{1}{C\sqrt{R_1 R_2}} \qquad \frac{\omega_o}{Q} = \frac{2}{CR_2}$$

$$Q = \frac{1}{2}\sqrt{\frac{R_2}{R_1}} \qquad A_o = -\frac{R_2}{2R_1}$$

Since center frequency is given this equation constrains the quality factor

Solving for Q

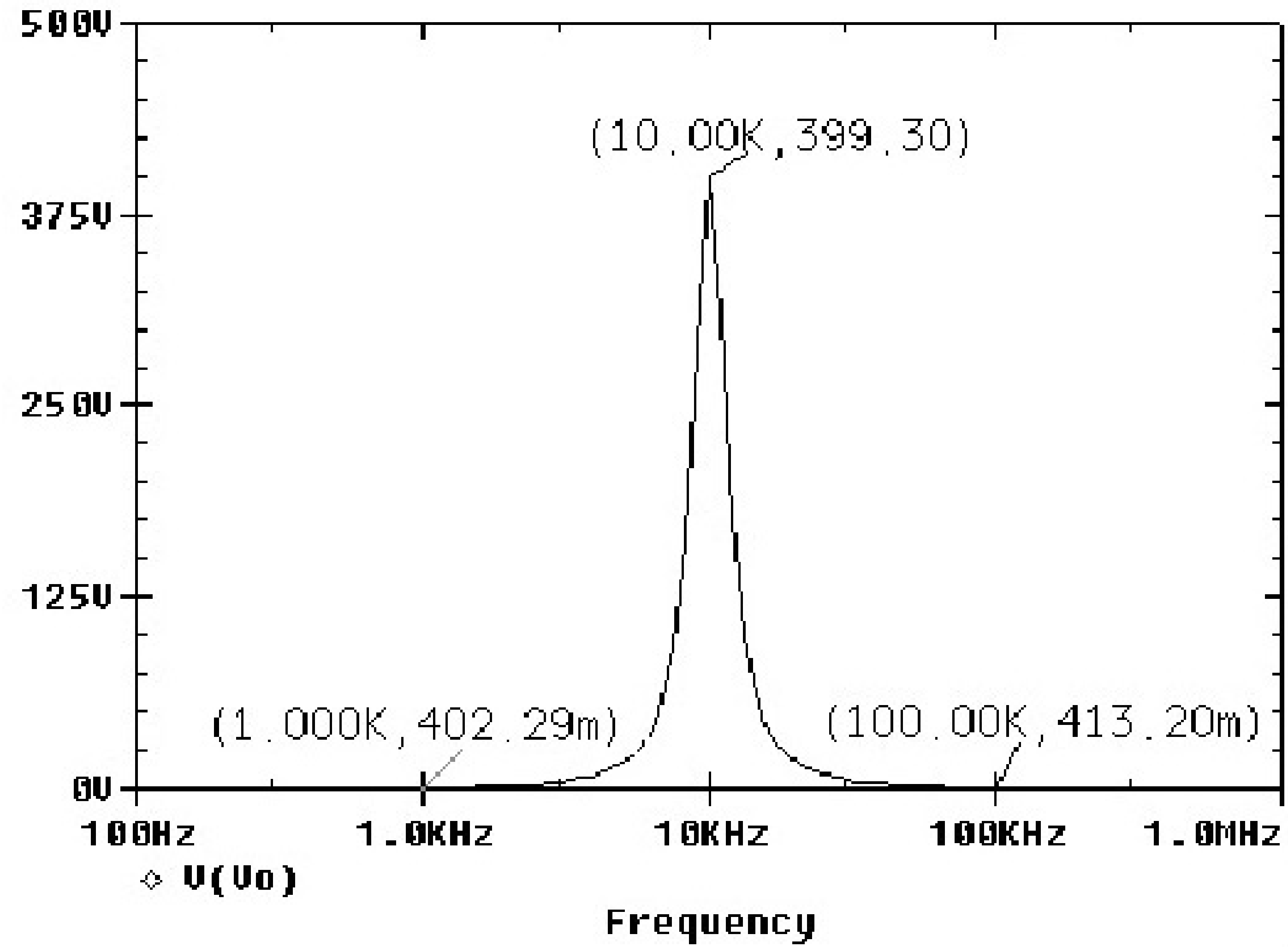
$$\sqrt{1+100Q^2} \ge \sqrt{1000} \Rightarrow Q \approx \sqrt{10} \qquad \Rightarrow \frac{R_2}{R_1} \approx 40$$

Picking $R_1 = 1k\Omega \Rightarrow R_2 = 40k\Omega$

$$\omega_o = 2\pi \times 10^4 \Rightarrow C = 2.5nF$$

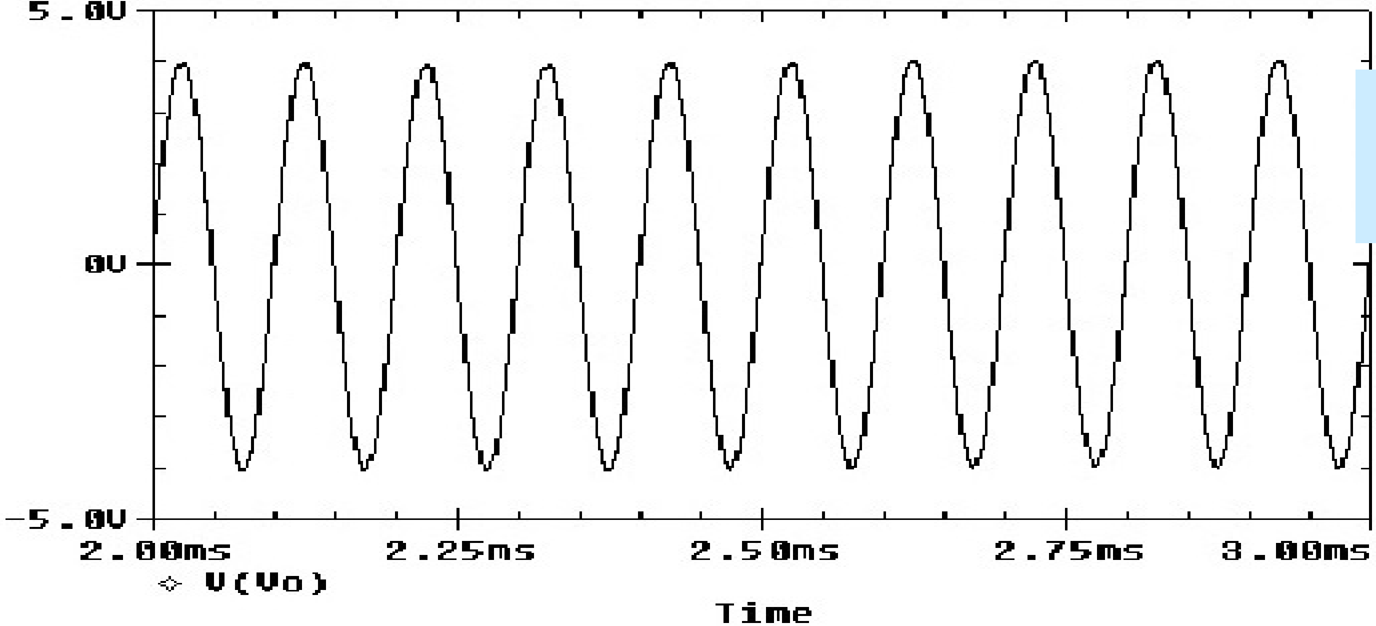
1kHz, 100kHz are noise frequencies
10kHz is the signal frequency

We use the requirements to constraint Q

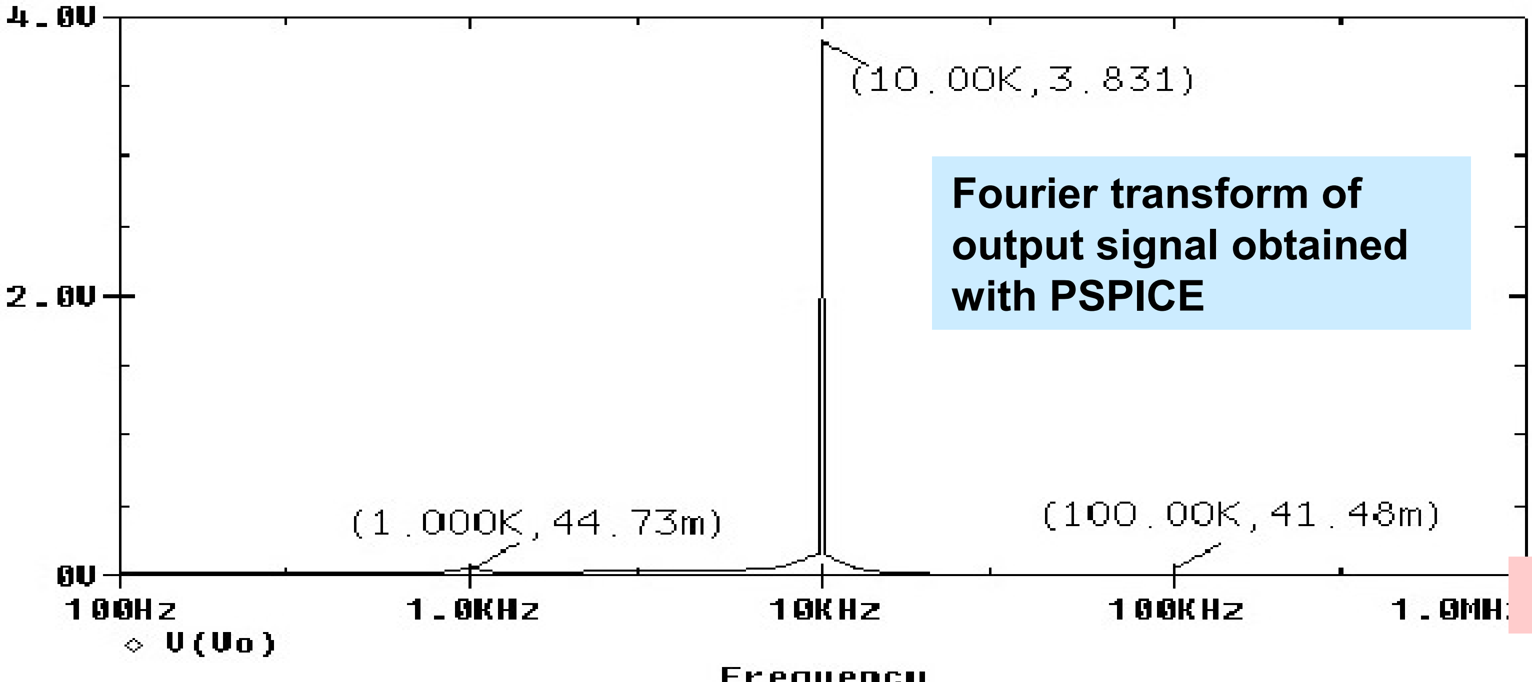


(a)

Resulting Bode plot obtained with PSPICE AC sweep



Filter output obtained with PSPICE



(c)

Fourier transform of output signal obtained with PSPICE

Fourier

STOP