NODAL AND LOOP ANALYSIS TECHNIQUES

LEARNING GOALS

NODAL ANALYSIS
LOOP ANALYSIS

Develop systematic techniques to determine all the voltages and currents in a circuit

CIRCUITS WITH OPERATIONAL AMPLIFIERS

Op-amps are very important devices, widely available, that permit the design of very useful circuit…

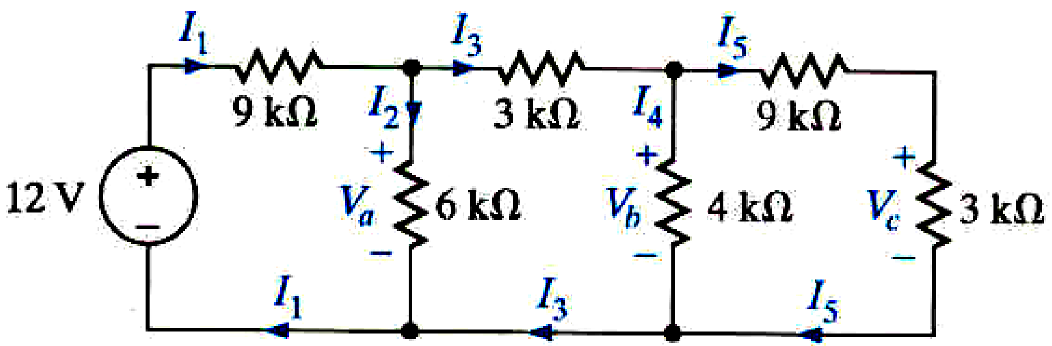
and they can be modeled by circuits with dependent sources

NODE ANALYSIS

- One of the systematic ways to determine every voltage and current in a circuit

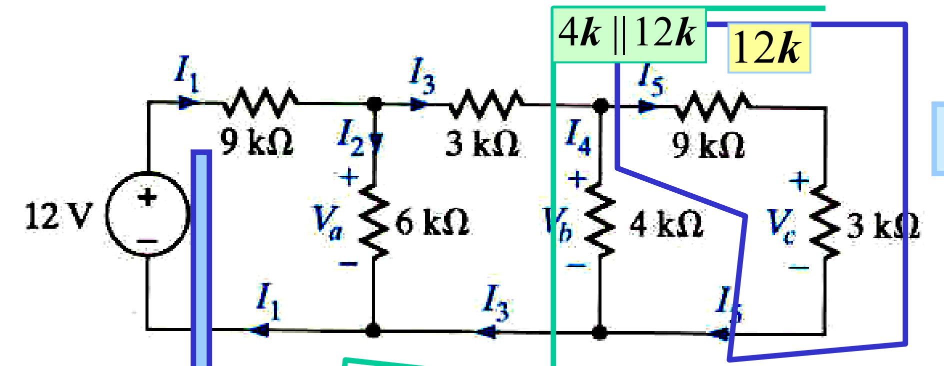
The variables used to describe the circuit will be "Node Voltages"
-- The voltages of each node with respect to a pre-selected reference node

IT IS INSTRUCTIVE TO START THE PRESENTATION WITH A RECAP OF A PROBLEM SOLVED BEFORE USING SERIES/ PARALLEL RESISTOR COMBINATIONS

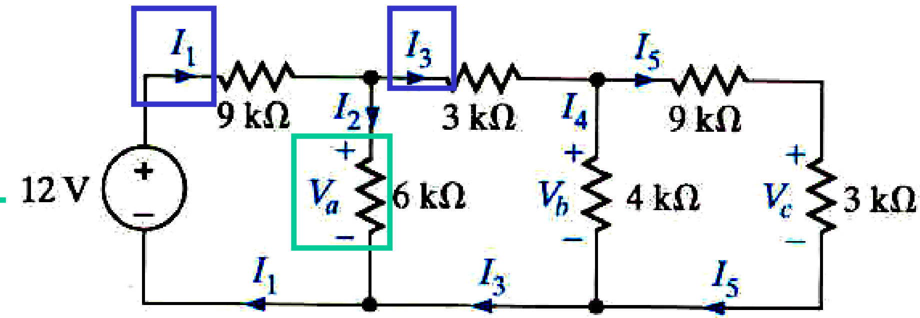


COMPUTE ALL THE VOLTAGES AND CURRENTS IN THIS CIRCUIT

We wish to find all the currents and voltages labeled in the ladder network shown



SECOND: "BACKTRACK" USING KVL, KCL OHM'S



OHM'S: $I_2 = \frac{V_a}{6k}$

KCL: $I_1 - I_2 - I_3 = 0$

OHM'S: $V_b = 3k * I_3$

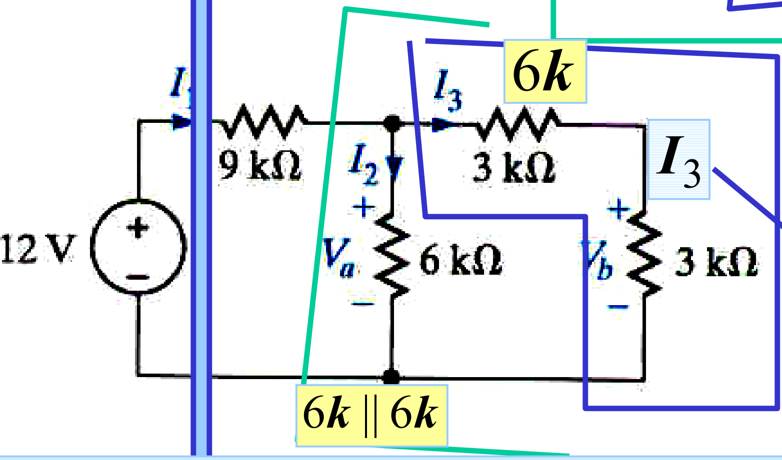
…OTHER OPTIONS...

$$I_4 = \frac{12}{4+12} I_3$$

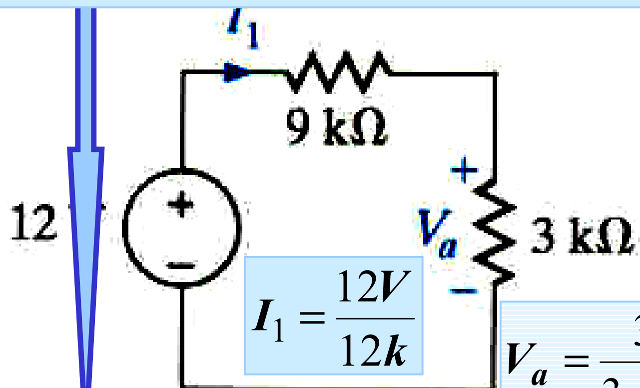
$$V_b = 4k * I_4$$

KCL: $I_5 + I_4 - I_3 = 0$

OHM'S: $V_C = 3k * I_5$



FIRST REDUCE TO A SINGLE LOOP CIRCUIT



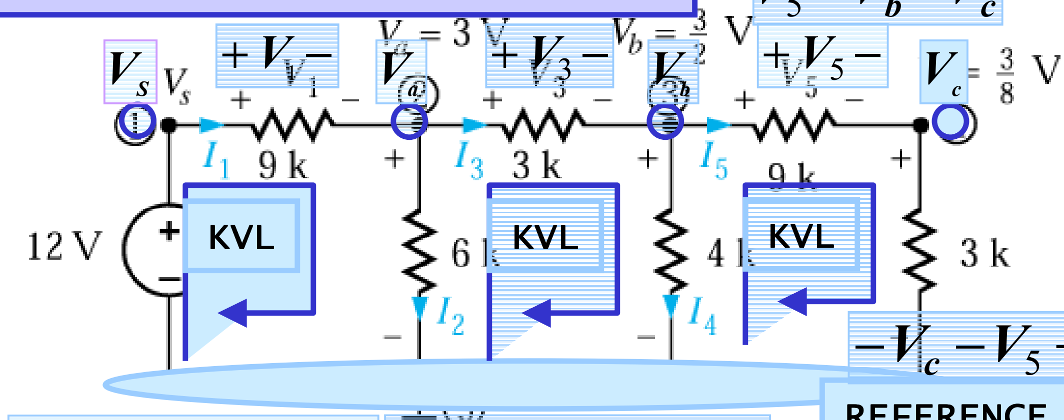
$$I_1 = \frac{12V}{12k}$$

$$V_a = \frac{3}{3+9}(12)$$

GEAUX

THE NODE ANALYSIS PERSPECTIVE

$$V_5 = V_b - V_c$$



$$-V_c - V_5 + V_b = 0$$

REFERENCE

$$-V_s + V_1 + V_a = 0$$

$$V_1 = V_s - V_a$$

$$-V_a + V_3 + V_b = 0$$

$$V_3 = V_a - V_b$$

THERE ARE FIVE NODES.
IF ONE NODE IS SELECTED AS REFERENCE THEN THERE ARE FOUR VOLTAGES WITH RESPECT TO THEREFERENCE NODE

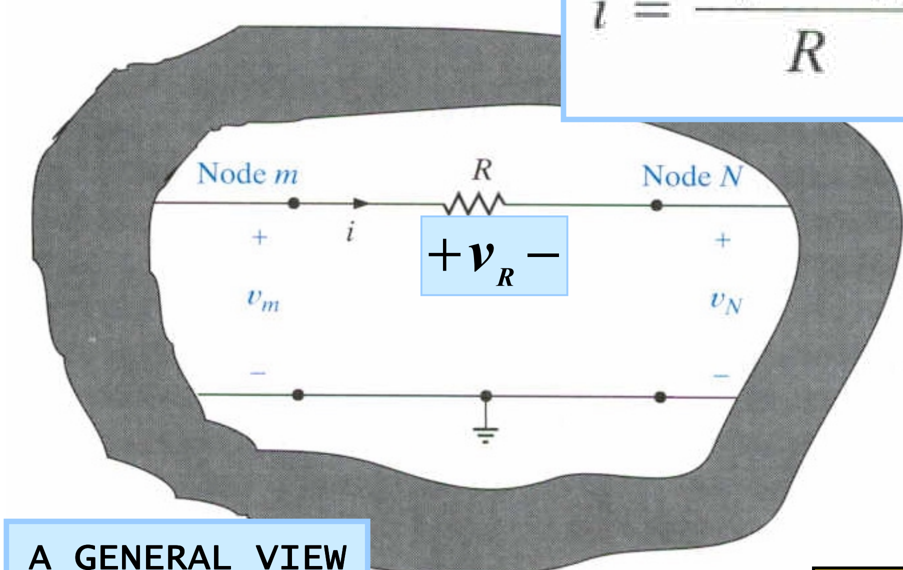
WHAT IS THE PATTERN???

ONCE THE VOLTAGES ARE KNOWN THE CURRENTS CAN BE COMPUTED USING OHM'S LAW

THEOREM: IF ALL NODE VOLTAGES WITH RESPECT TO A COMMON REFERENCE NODE ARE KNOWN THEN ONE CAN DETERMINE ANY OTHER ELECTRICAL VARIABLE FOR THE CIRCUIT

$$v_R = v_m - v_N$$

$$i = \frac{v_m - v_N}{R}$$



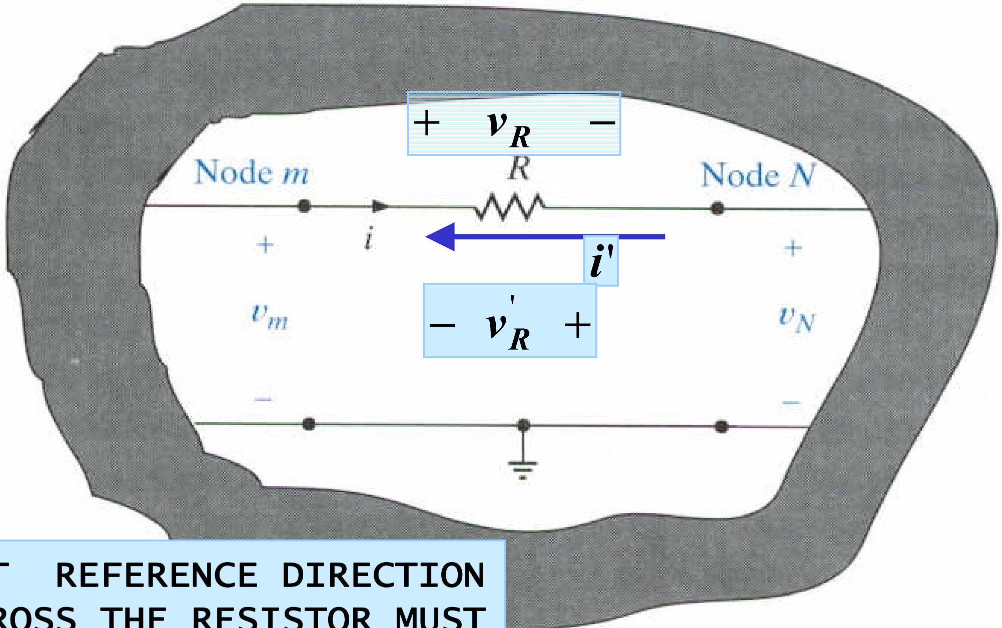
A GENERAL VIEW

DRILL QUESTION

$$V_{ca} = \_\_\_\_\_\_$$

GEAUX

THE REFERENCE DIRECTION FOR CURRENTS IS IRRELEVANT



USING THE LEFT-RIGHT REFERENCE DIRECTION THE VOLTAGE DROP ACROSS THE RESISTOR MUST HAVE THE POLARITY SHOWN

OHM'S LAW $i = \dfrac{v_m - v_N}{R}$

IF THE CURRENT REFERENCE DIRECTION IS REVERSED ...

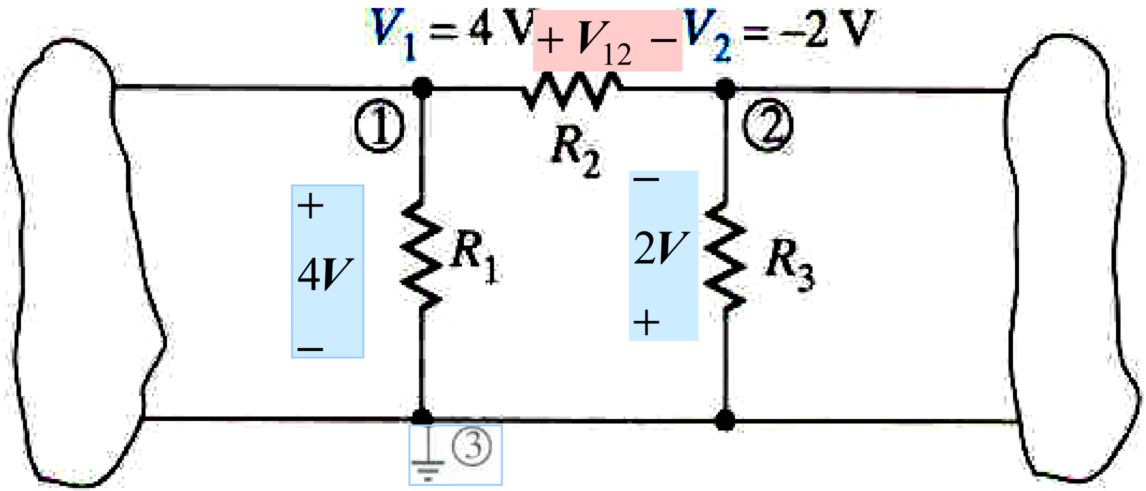
THE PASSIVE SIGN CONVENTION WILL ASSIGN THE REVERSE REFERENCE POLARITY TO THE VOLTAGE ACROSS THE RESISTOR

OHM'S LAW $i' = \dfrac{v_N - v_m}{R}$

$i = -i'$ PASSIVE SIGN CONVENTION RULES!

GEAUX

DEFINING THE REFERENCE NODE IS VITAL



THE STATEMENT $V_1 = 4V$ IS MEANINGLESS

UNTIL THE REFERENCE POINT IS DEFINED

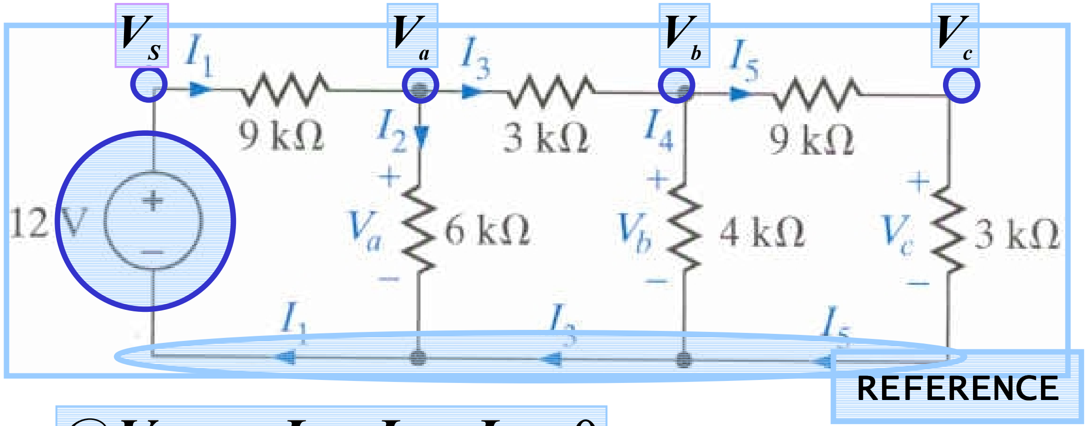
BY CONVENTION THE GROUND SYMBOL SPECIFIES THE REFERENCE POINT.

ALL NODE VOLTAGES ARE MEASURED WITH RESPECT TO THAT REFERENCE POINT

$V_{12} = \_\_\_\_\_\_?$

GEAUX

THE STRATEGY FOR NODE ANALYSIS



1. IDENTIFY ALL NODES AND SELECT A REFERENCE NODE
2. IDENTIFY KNOWN NODE VOLTAGES
3. AT EACH NODE WITH UNKNOWN VOLTAGE WRITE A KCL EQUATION (e.g.,SUM OF CURRENT LEAVING =0)
4. REPLACE CURRENTS IN TERMS OF NODE VOLTAGES

AND GET ALGEBRAIC EQUATIONS IN THE NODE VOLTAGES ...

$$@V_a: \; -I_1 + I_2 + I_3 = 0$$

$$\frac{V_a - V_s}{9k} + \frac{V_a}{6k} + \frac{V_a - V_b}{3k} = 0$$

$$@V_b: \; -I_3 + I_4 + I_5 = 0$$

$$\frac{V_b - V_a}{3k} + \frac{V_b}{4k} + \frac{V_b - V_c}{9k} = 0$$

$$@V_c: \; -I_5 + I_6 = 0$$

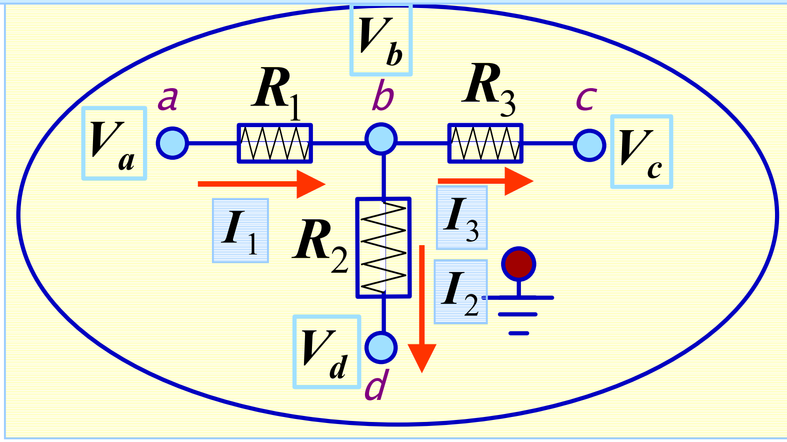
$$\frac{V_c - V_b}{9k} + \frac{V_c}{3k} = 0$$

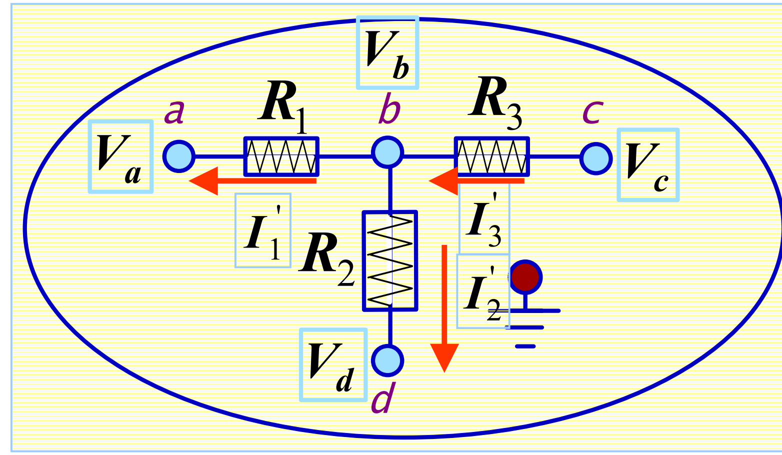
SHORTCUT: SKIP WRITING THESE EQUATIONS...

AND PRACTICE WRITING THESE DIRECTLY

GEAUX

WHEN WRITING A NODE EQUATION...
AT EACH NODE ONE CAN CHOSE ARBITRARY DIRECTIONS FOR THE CURRENTS





AND SELECT ANY FORM OF KCL.
WHEN THE CURRENTS ARE REPLACED IN TERMS OF THE NODE VOLTAGES THE NODE EQUATIONS THAT RESULT ARE THE SAME OR EQUIVALENT

$\sum$ CURRENTS LEAVING $= 0$

$$-I_1 + I_2 + I_3 = 0 \Rightarrow -\frac{V_a - V_b}{R_1} + \frac{V_b - V_d}{R_2} + \frac{V_b - V_c}{R_3} = 0$$

$\sum$ CURRENTS INTO NODE $= 0$

$$I_1 - I_2 - I_3 = 0 \Rightarrow \frac{V_a - V_b}{R_1} - \frac{V_b - V_d}{R_2} - \frac{V_b - V_c}{R_3} = 0$$

$\sum$ CURRENTS LEAVING $= 0$

$$I_1' + I_2' - I_3' = 0 \Rightarrow \frac{V_b - V_a}{R_1} + \frac{V_b - V_d}{R_2} - \frac{V_c - V_b}{R_3} = 0$$

$\sum$ CURRENTS INTO NODE $= 0$

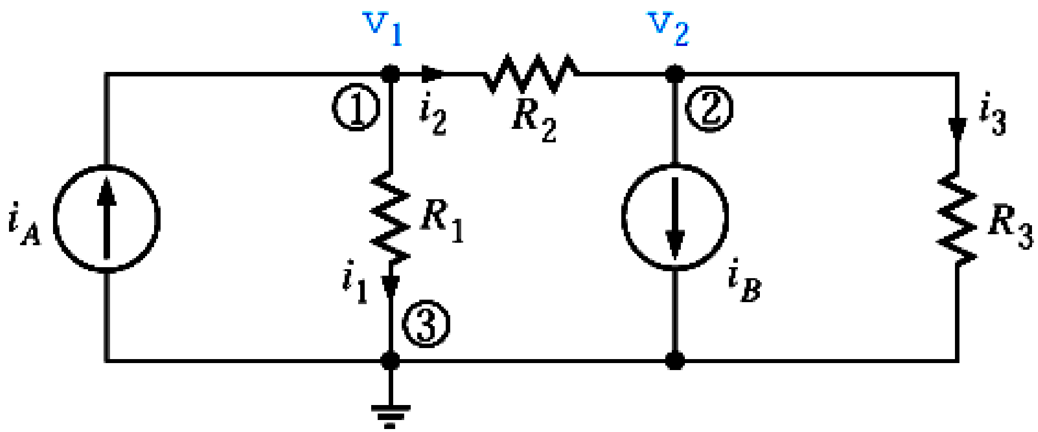
$$-I_1' - I_2' + I_3' = 0 \Rightarrow -\frac{V_b - V_a}{R_1} - \frac{V_b - V_d}{R_2} + \frac{V_c - V_b}{R_3} = 0$$

WHEN WRITING THE NODE EQUATIONS WRITE THE EQUATION DIRECTLY IN TERMS OF THE NODE VOLTAGES.
BY DEFAULT USE KCL IN THE FORM SUM-OF-CURRENTS-LEAVING = 0

THE REFERENCE DIRECTION FOR THE CURRENTS DOES NOT AFFECT THE NODE EQUATION

GEAUX

CIRCUITS WITH ONLY INDEPENDENT SOURCES



HINT: THE FORMAL MANIPULATION OF EQUATIONS MAY BE SIMPLER IF ONE USES CONDUCTANCES INSTEAD OF RESISTANCES.

@ NODE 1 $-i_A + i_1 + i_2 = 0$

USING RESISTANCES $-i_A + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} = 0$

WITH CONDUCTANCES $-i_A + G_1 v_1 + G_2(v_1 - v_2) = 0$

REORDERING TERMS $(G_1 + G_2)v_1 - G_2 v_2 = i_A$

@ NODE 2

$-G_2(v_1 - v_2) + i_B + G_3(v_2 - 0) = 0$

REORDERING TERMS $-G_2 v_1 + (G_2 + G_3)v_2 = -i_B$

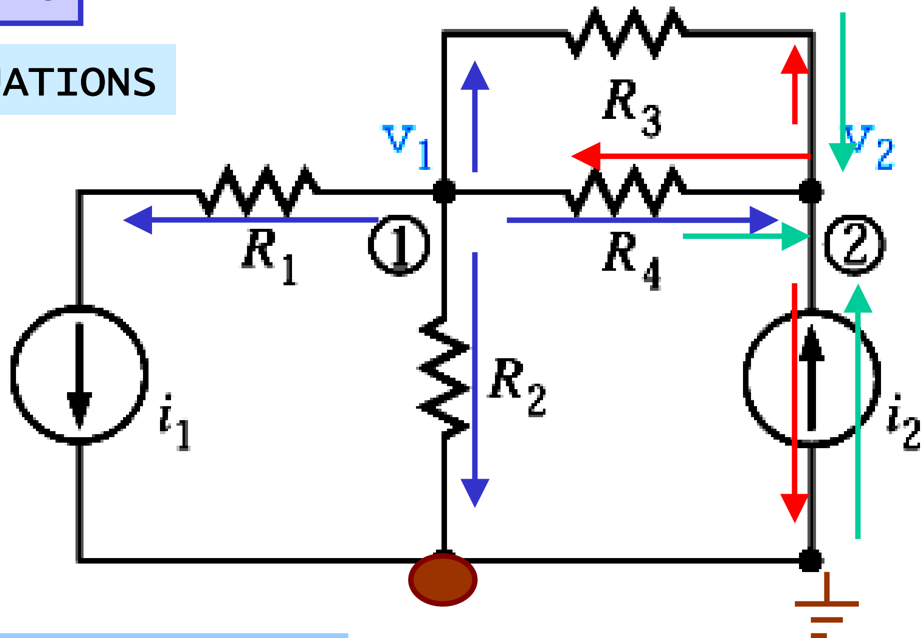
THE MODEL FOR THE CIRCUIT IS A SYSTEM OF ALGEBRAIC EQUATIONS

$$(G_1 + G_2)v_1 - G_2 v_2 = i_A$$
$$-G_2 v_1 + (G_2 + G_3)v_2 = -i_B$$

THE MANIPULATION OF SYSTEMS OF ALGEBRAIC EQUATIONS CAN BE EFFICIENTLY DONE USING MATRIX ANALYSIS

LEARNING BY DOING

WRITE THE KCL EQUATIONS



@ NODE 1 WE VISUALIZE THE CURRENTS LEAVING AND WRITE THE KCL EQUATION

$$i_1 + \frac{v_1}{R_2} + \frac{v_1 - v_2}{R_3} + \frac{v_1 - v_2}{R_4} = 0$$

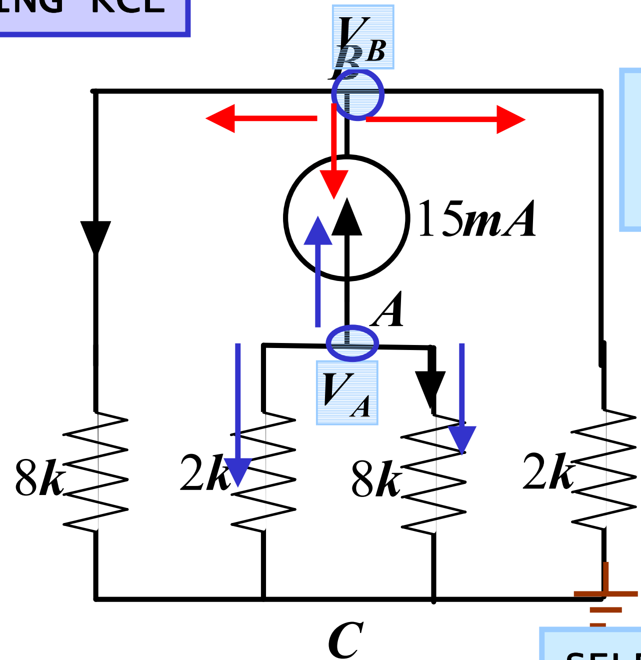
REPEAT THE PROCESS AT NODE 2

$$-i_2 + \frac{v_2 - v_1}{R_4} + \frac{v_2 - v_1}{R_3} = 0$$

OR VISUALIZE CURRENTS GOING INTO NODE

$$i_2 + \frac{v_1 - v_2}{R_3} + \frac{v_1 - v_2}{R_4} = 0$$

ANOTHER EXAMPLE OF WRITING KCL



MARK THE NODES (TO INSURE THAT NONE IS MISSING)

SELECT AS REFERENCE

WRITE KCL AT EACH NODE IN TERMS OF NODE VOLTAGES

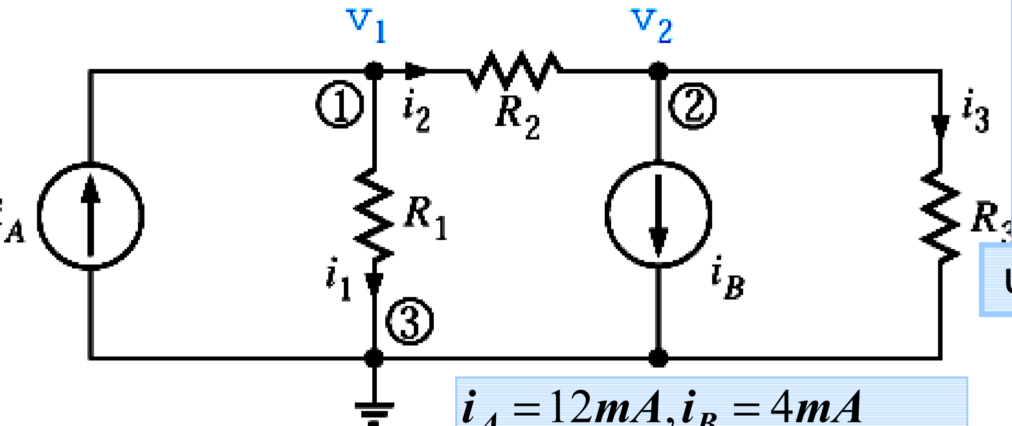
$$@A \quad \frac{V_A}{2k} + \frac{V_A}{8k} + 15mA = 0$$

$$@B \quad \frac{V_B}{8k} + \frac{V_B}{2k} - 15mA = 0$$

GEAUX

A MODEL IS SOLVED BY MANIPULATION OF EQUATIONS AND USING MATRIX ANALYSIS

LEARNING EXAMPLE



$i_A = 12mA, i_B = 4mA$

$R_1 = 12k\Omega, R_2 = R_3 = 6k\Omega$

THE NODE EQUATIONS

$$-i_A + G_1(v_1 - 0) + G_2(v_1 - v_2) = 0$$

$$-G_2(v_1 - v_2) + i_B + G_3(v_2 - 0) = 0$$

THE MODEL

$$(G_1 + G_2)v_1 - G_2 v_2 = i_A$$

$$-G_2 v_1 + (G_2 + G_3)v_2 = -i_B$$

REPLACE VALUES AND SWITCH NOTATION TO UPPER CASE

$$V_1\left[\frac{1}{12k} + \frac{1}{6k}\right] - V_2\left[\frac{1}{6k}\right] = 1 \times 10^{-3}$$

$$-V_1\left[\frac{1}{6k}\right] + V_2\left[\frac{1}{6k} + \frac{1}{6k}\right] = -4 \times 10^{-3}$$

NUMERICAL MODEL

$$\frac{V_1}{4k} - \frac{V_2}{6k} = 1 \times 10^{-3}$$

$$-\frac{V_1}{6k} + \frac{V_2}{3k} = -4 \times 10^{-3}$$

USE GAUSSIAN ELIMINATION

$$V_1 = V_2\left(\frac{2}{3}\right) + 4$$

$$\frac{-1}{6k}\left(\frac{2}{3}V_2 + 4\right) + \frac{V_2}{3k} = -4 \times 10^{-3}$$

$$V_2 = -15\text{ V}$$

$$V_1 = \frac{2}{3}V_2 + 4 = -6\text{ V}$$

ALTERNATIVE MANIPULATION

$$\frac{V_1}{4k} - \frac{V_2}{6k} = 1 \times 10^{-3} \quad */12k$$

$$-\frac{V_1}{6k} + \frac{V_2}{3k} = -4 \times 10^{-3} \quad */6k$$

RIGHT HAND SIDE IS VOLTS. COEFFS ARE NUMBERS

$$3V_1 - 2V_2 = 12$$

$$-V_1 + 2V_2 = -24$$

$*/3$ (and add equations)

$4V_2 = -60[V]$

ADD EQS $2V_1 = -12[V]$

GEAUX

SOLUTION USING MATRIX ALGEBRA

$$\frac{V_1}{4k} - \frac{V_2}{6k} = 1 \times 10^{-3}$$

$$-\frac{V_1}{6k} + \frac{V_2}{3k} = -4 \times 10^{-3}$$

PLACE IN MATRIX FORM

$$\begin{bmatrix} \frac{1}{4k} & \frac{-1}{6k} \\ \frac{-1}{6k} & \frac{1}{3k} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix}$$

USE MATRIX ANALYSIS TO SHOW SOLUTION

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4k} & \frac{-1}{6k} \\ \frac{-1}{6k} & \frac{1}{3k} \end{bmatrix}^{-1} \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix}$$

PERFORM THE MATRIX MANIPULATIONS

$$A^{-1} = \frac{Adj(A)}{|A|}$$

FOR THE ADJOINT REPLACE EACH ELEMENT BY ITS COFACTOR

$$\text{Adj}\,\mathbf{A} = \begin{bmatrix} \frac{1}{3k} & \frac{1}{6k} \\ \frac{1}{6k} & \frac{1}{4k} \end{bmatrix}$$

$$|\mathbf{A}| = \left(\frac{1}{3k}\right)\left(\frac{1}{4k}\right) - \left(\frac{-1}{6k}\right)\left(\frac{-1}{6k}\right) = \frac{1}{18k^2}$$

AND DO THE MATRIX ALGEBRA ...

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 18k^2 \begin{bmatrix} \frac{1}{3k} & \frac{1}{6k} \\ \frac{1}{6k} & \frac{1}{4k} \end{bmatrix} \begin{bmatrix} 1 \times 10^{-3} \\ -4 \times 10^{-3} \end{bmatrix}$$

$$= 18k^2 \begin{bmatrix} \frac{1}{3k^2} - \frac{4}{6k^2} \\ \frac{1}{6k^2} - \frac{1}{k^2} \end{bmatrix}$$

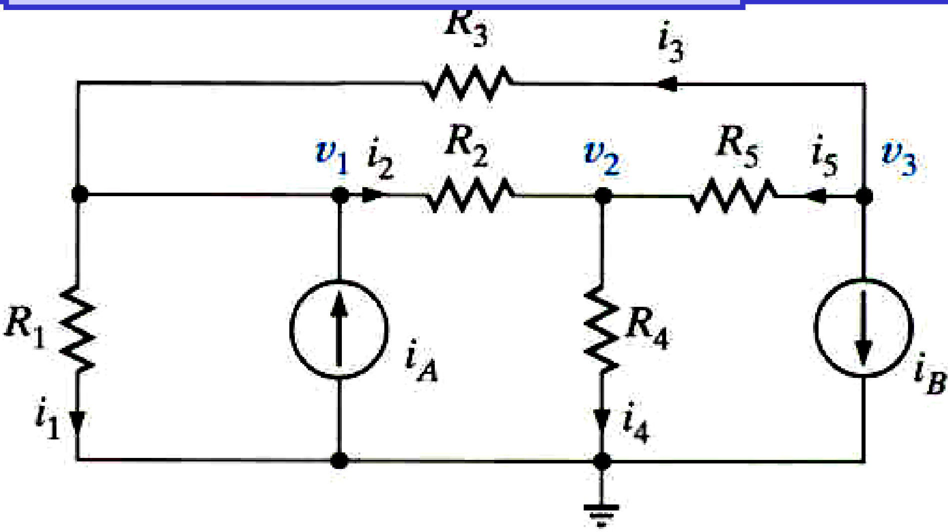
$$= \begin{bmatrix} -6 \\ -15 \end{bmatrix}$$

SAMPLE

$$V_1 = 18k^2\left(\frac{1\times10^{-3}}{3k} + \frac{-4\times10^{-3}}{6k}\right)$$

GEAUX

AN EXAMPLE OF NODE ANALYSIS



Rearranging terms ...

$$v_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-v_2\frac{1}{R_2}-v_3\frac{1}{R_3}=i_A$$

$$-v_1\frac{1}{R_2}+v_2\left(\frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5}\right)-v_3\frac{1}{R_5}=0$$

$$-v_1\frac{1}{R_3}-v_2\frac{1}{R_5}+v_3\left(\frac{1}{R_3}+\frac{1}{R_5}\right)=-i_B$$

@ v_1

$$\frac{v_1}{R_1}-i_A+\frac{v_1-v_2}{R_2}-\frac{v_3-v_1}{R_3}=0$$

$$v_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-v_2\frac{1}{R_2}-v_3\frac{1}{R_3}=i_A$$

@ v_2

$$-\frac{v_1-v_2}{R_2}+\frac{v_2}{R_4}-\frac{v_3-v_2}{R_5}=0$$

$$-v_1\frac{1}{R_2}+v_2\left(\frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5}\right)-v_3\frac{1}{R_5}=0$$

@ v_3

$$\frac{v_3-v_1}{R_3}+\frac{v_3-v_2}{R_5}+i_B=0$$

$$-v_1\frac{1}{R_3}-v_2\frac{1}{R_5}+v_3\left(\frac{1}{R_3}+\frac{1}{R_5}\right)=-i_B$$

$$\begin{bmatrix}\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3}\\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_5}\\ -\frac{1}{R_3} & -\frac{1}{R_5} & \frac{1}{R_3}+\frac{1}{R_5}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}i_A\\ 0\\ -i_B\end{bmatrix}$$

COULD WRITE EQUATIONS BY INSPECTION

(blue) $= \sum$ CONDUCTANCES CONNECTED TO NODE

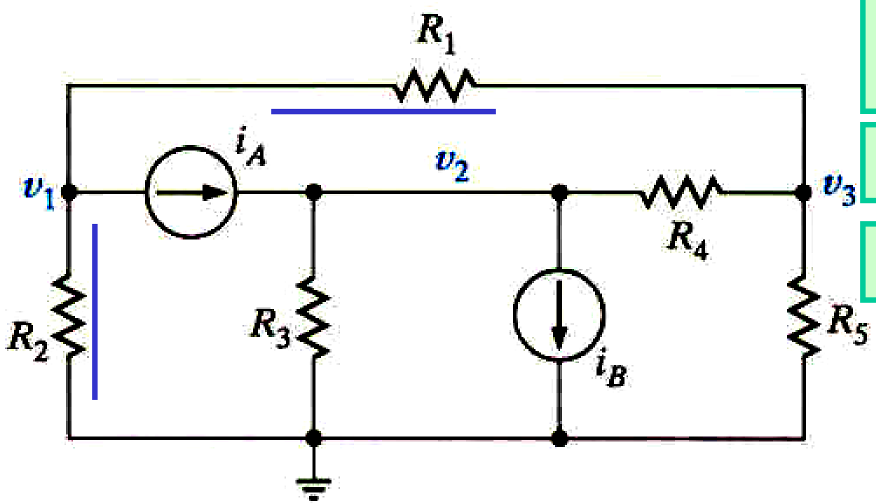
(red) $= \sum$ CONDUCTANCES BETWEEN 1 & 2

(green) $= \sum$ CONDUCTANCES BETWEEN 1 & 3

(magenta) $= \sum$ CONDUCTANCES BETWEEN 2 & 3

GEAUX

WRITING EQUATIONS "BY INSPECTION"



FOR CIRCUITS WITH ONLY INDEPENDENT SOURCES THE MATRIX IS ALWAYS SYMMETRIC

THE DIAGONAL ELEMENTS ARE POSITIVE

THE OFF-DIAGONAL ELEMENTS ARE NEGATIVE

Conductances connected to node 1

Conductances between 1 and 2

Conductances between 1 and 3

Conductances between 2 and 3

$$v_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-v_2(0)-v_3\left(\frac{1}{R_1}\right)=-i_A$$

$$-v_1(0)+v_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right)-v_3\left(\frac{1}{R_4}\right)=i_A-i_B$$

$$-v_1\left(\frac{1}{R_1}\right)-v_2\left(\frac{1}{R_4}\right)+v_3\left(\frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5}\right)=0$$

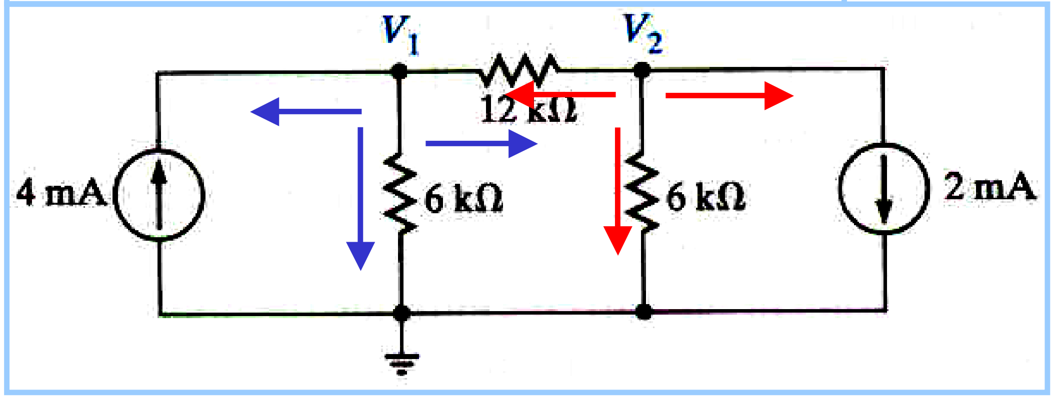
VALID ONLY FOR CIRCUITS WITHOUT DEPENDENT SOURCES

$$\begin{bmatrix}\frac{1}{R_1}+\frac{1}{R_2} & 0 & -\frac{1}{R_1}\\ 0 & \frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4}\\ -\frac{1}{R_1} & -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}-i_A\\ i_A-i_B\\ 0\end{bmatrix}$$

GEAUX

LEARNING EXTENSION

Write the node equations



$$@V_1: \ -4mA + \frac{V_1}{6k} + \frac{V_1 - V_2}{12k}$$

USING KCL

$$@V_2: \ 2mA + \frac{V_2}{6k} + \frac{V_2 - V_1}{12k} = 0$$

BY “INSPECTION”

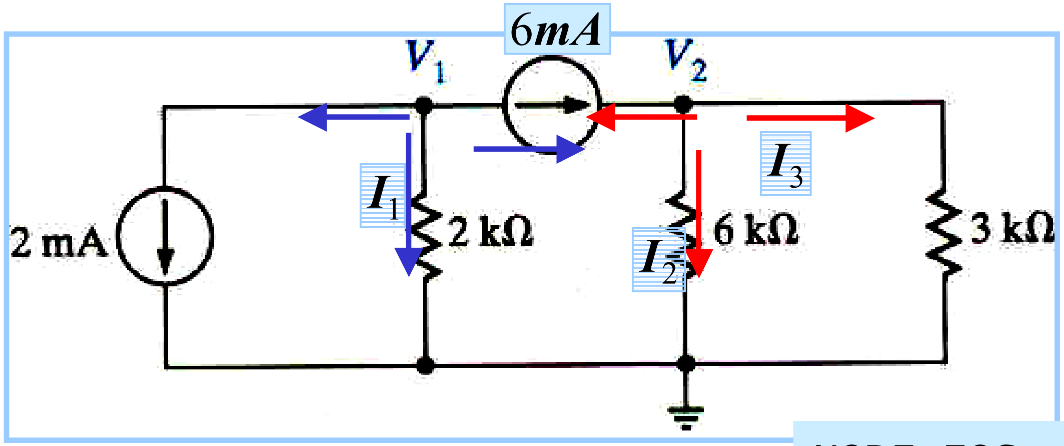
$$\left(\frac{1}{6k} + \frac{1}{12k}\right)V_1 - \frac{1}{12k}V_2 = 4mA$$

$$-\frac{1}{12k} + \left(\frac{1}{6k} + \frac{1}{12k}\right)V_2 = -2mA$$

GEAUX

LEARNING EXTENSION

Find all the branch currents



NODE EQS. BY INSPECTION

$$KCL@V_1: \; I_1 + 2mA + 6mA = 0$$

$$@V_2: \quad -6mA + \frac{V_2}{6k} + \frac{V_2}{3k} = 0 \Rightarrow$$

$$V_2 = 12V$$
$$I_2 = 2mA$$
$$I_3 = 4mA$$

$$\frac{1}{2k}V_1 + (0)V_2 = -(2+6)mA$$

$$(0)V_1 + \left(\frac{1}{6k} + \frac{1}{3k}\right)V_2 = 6mA$$

$$I_1 = \frac{V_1}{2k} \qquad I_2 = \frac{V_2}{6k} \qquad I_3 = \frac{V_2}{3k}$$

IN MOST CASES THERE ARE SEVERAL DIFFERENT WAYS OF SOLVING A PROBLEM

$$I_2 = \frac{3k}{3k+6k}(6mA) = 2mA$$

$$I_3 = \frac{6k}{3k+6k}(6mA) = 4mA$$

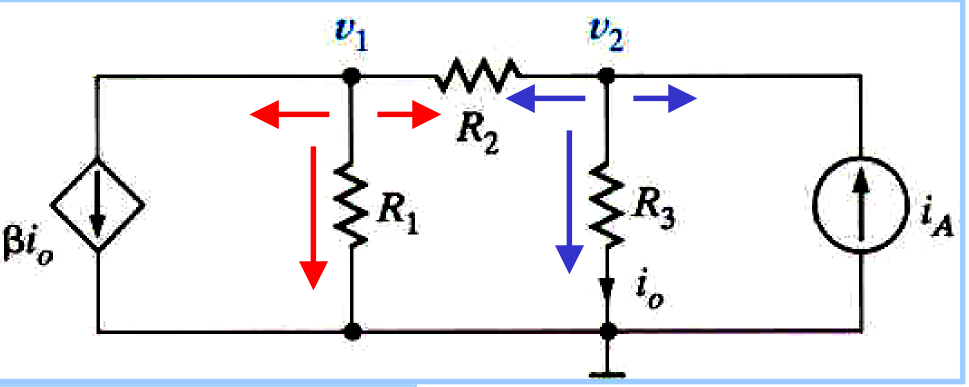
CURRENTS COULD BE COMPUTED DIRECTLY USING CURRENT DIVIDER!!

CIRCUITS WITH DEPENDENT SOURCES

CIRCUITS WITH DEPENDENT SOURCES CANNOT BE MODELED BY INSPECTION. THE SYMMETRY IS LOST.

A PROCEDURE FOR MODELING

- WRITE THE NODE EQUATIONS USING DEPENDENT SOURCES AS REGULAR SOURCES.
- FOR EACH DEPENDENT SOURCE WE ADD ONE EQUATION EXPRESSING THE CONTROLLING VARIABLE IN TERMS OF THE NODE VOLTAGES



$$\beta i_o + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} = 0$$

$$-i_A + \frac{v_2}{R_3} + \frac{v_2 - v_1}{R_2} = 0$$

MODEL FOR CONTROLLING VARIABLE

$$i_o = \frac{v_2}{R_3}$$

REPLACE AND REARRANGE

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_1 + \left(\frac{\beta}{R_3} - \frac{1}{R_2}\right)v_2 = 0$$

$$-\frac{1}{R_2}v_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 = i_A$$

LEARNING EXAMPLE

NUMERICAL EXAMPLE

$\beta = 2$ $\quad R_2 = 6\,\text{k}\Omega$ $\quad i_A = 2\,\text{mA}$

$R_1 = 12\,\text{k}\Omega$ $\quad R_3 = 3\,\text{k}\Omega$

$$\left(\frac{1}{12k} + \frac{1}{6k}\right)v_1 + \left(\frac{2}{3k} - \frac{1}{6k}\right)v_2 = 0$$

$$-\frac{1}{6k}v_1 + \left(\frac{1}{12k} + \frac{1}{3k}\right)v_2 = 2mA$$

$$\frac{1}{4\text{k}}V_1 + \frac{1}{2\text{k}}V_2 = 0 \qquad */4k$$

$$-\frac{1}{6\text{k}}V_1 + \frac{1}{2\text{k}}V_2 = 2 \times 10^{-3} \qquad */6k$$

$$V_1 + 2V_2 = 0$$

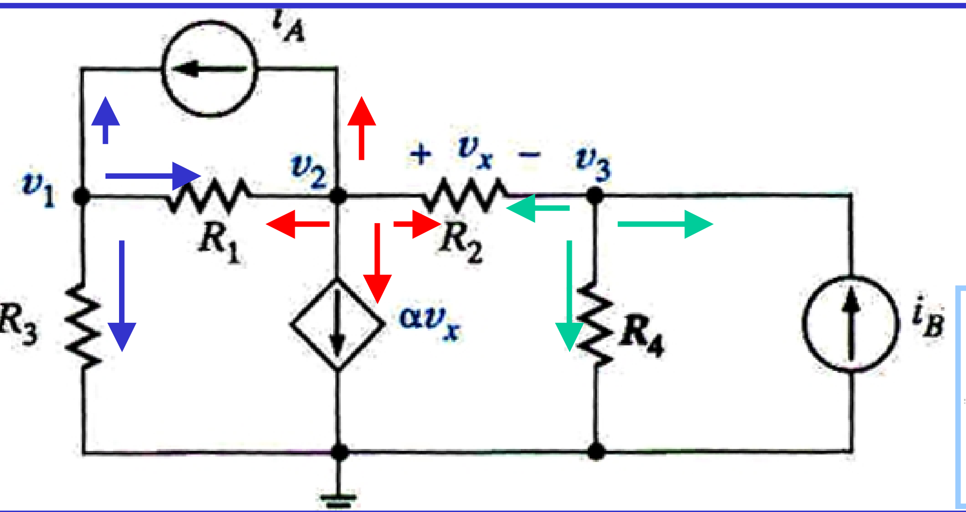
$$-V_1 + 3V_2 = 12[V]$$

ADDING THE EQUATIONS $\quad 5V_2 = 12[V]$

$$V_1 = -\frac{24}{5}V$$

GEAUX

LEARNING EXAMPLE: CIRCUIT WITH VOLTAGE-CONTROLLED CURRENT



WRITE NODE EQUATIONS. TREAT DEPENDENT SOURCE AS REGULAR SOURCE

$$G_3 v_1 + G_1(v_1 - v_2) - i_A = 0$$

$$i_A + G_1(v_2 - v_1) + \alpha v_x + G_2(v_2 - v_3) = 0$$

$$G_2(v_3 - v_2) + G_4 v_3 - i_B = 0$$

EXPRESS CONTROLLING VARIABLE IN TERMS OF NODE VOLTAGES

$$v_x = v_2 - v_3$$

FOUR EQUATIONS IN OUR UNKNOWNS. SOLVE USING FAVORITE TECHNIQUE

REPLACE AND REARRANGE

$$(G_1 + G_3)v_1 - G_1 v_2 = i_A$$
$$-G_1 v_1 + (G_1 + \alpha + G_2)v_2 - (\alpha + G_2)v_3 = -i_A$$
$$-G_2 v_2 + (G_2 + G_4)v_3 = i_B$$

CONTINUE WITH GAUSSIAN ELIMINATION...

OR USE MATRIX ALGEBRA

$$\begin{bmatrix} (G_1 + G_3) & -G_1 & 0 \\ -G_1 & (G_1 + \alpha + G_2) & -(\alpha + G_2) \\ 0 & -G_2 & (G_2 + G_4) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} i_A \\ -i_A \\ i_B \end{bmatrix}$$

GEAUX

USING MATLAB TO SOLVE THE NODE EQUATIONS

$$\begin{bmatrix} (G_1+G_3) & -G_1 & 0 \\ -G_1 & (G_1+\alpha+G_2) & -(\alpha+G_2) \\ 0 & -G_2 & (G_2+G_4) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} i_A \\ -i_A \\ i_B \end{bmatrix}$$

$R_1 = 1k\Omega,\ R_2 = R_3 = 2k\Omega,$
$R_4 = 4k\Omega, i_A = 2mA, i_B = 4mA,$
$\alpha = 2[A/V]$

DEFINE THE COMPONENTS OF THE CIRCUIT

```
» R1=1000;R2=2000;R3=2000;
R4=4000; %resistances in Ohm
» iA=0.002;iB=0.004; %sources in Amps
» alpha=2; %gain of dependent source
```

DEFINE THE MATRIX G

Entries in a row are separated by commas (or plain spaces). Rows are separated by semi colon

```
» G=[(1/R1+1/R2),   -1/R1,   0; %first row of the matrix
-1/R1,    (1/R1+alpha+1/R2),   -(alpha+1/R2);  %second row
0,   -1/R2,    (1/R2+1/R4)], %third row. End in comma to have the echo

G =

   0.0015   -0.0010         0
  -0.0010    2.0015   -2.0005
        0   -0.0005    0.0008
```

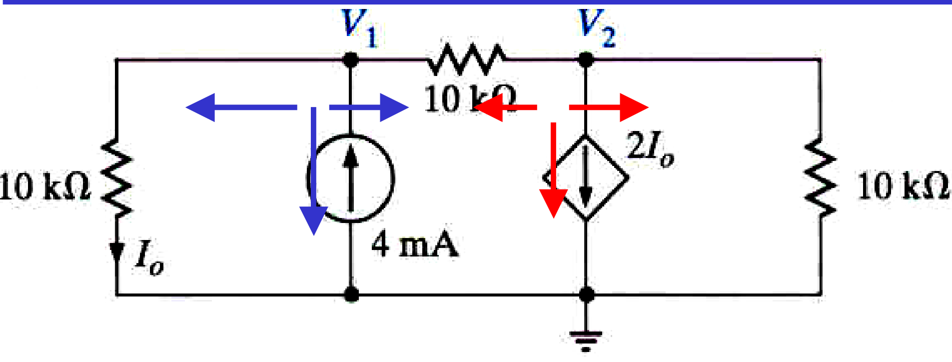
DEFINE RIGHT HAND SIDE VECTOR

```
» I=[iA;-iA;iB]; %end in ";" to skip echo
```

SOLVE LINEAR EQUATION

```
» V=G\I  % end with carriage return and get the echo
V =
11.9940
15.9910
15.9940
```

LEARNING EXTENSION: FIND NODE VOLTAGES



NODE EQUATIONS

$$@V_1: \ \frac{V_1}{10k} - 4mA + \frac{V_1 - V_2}{10k} = 0$$

$$@V_2: \ \frac{V_2 - V_1}{10k} + 2I_O + \frac{V_2}{10k} = 0$$

CONTROLLING VARIABLE (IN TERMS ON NODE VOLTAGES)

$$I_O = \frac{V_1}{10k}$$

REPLACE

$$\frac{V_1}{10k} - 4mA + \frac{V_1 - V_2}{10k} = 0$$

$$\frac{V_2 - V_1}{10k} + 2\frac{V_1}{10k} + \frac{V_2}{10k} = 0$$

REARRANGE AND MULTIPLY BY 10k

$$2V_1 - V_2 = 40[V]$$ $*/2$ and add eqs.

$$V_1 + 2V_2 = 0$$

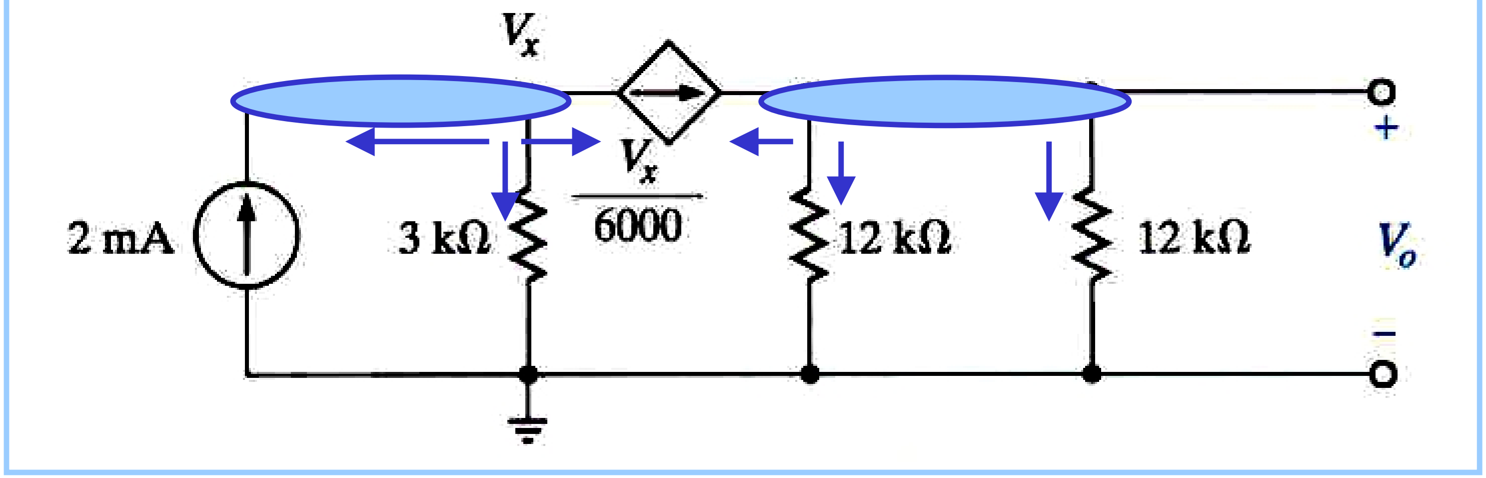
$$5V_1 = 80V \Rightarrow V_1 = 16V$$

$$V_2 = -\frac{V_1}{2} \Rightarrow V_2 = -8V$$

GEAUX

FIND THE VOLTAGE V_O

LEARNING EXTENSION



NODE EQUATIONS

NOTICE REPLACEMENT OF DEPENDENT SOURCE IN TERMS OF NODE VOLTAGE

$$-2mA + \frac{V_x}{3k} + \frac{V_x}{6k} = 0 \quad */6k$$

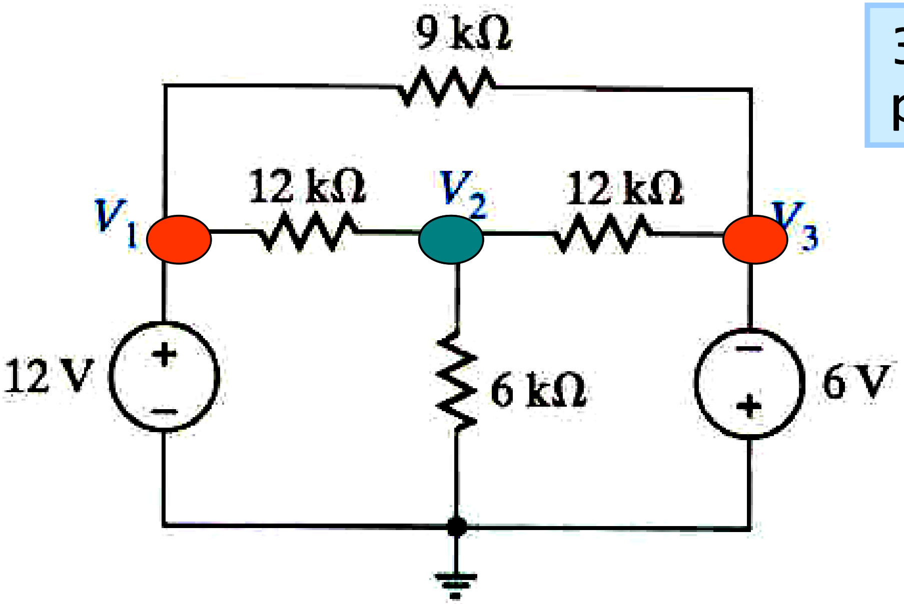
$$-\frac{V_x}{6k} + \frac{V_O}{12k} + \frac{V_O}{12k} = 0 \quad */12k$$

$$3V_x = 12[V] \Rightarrow V_x = 4[V]$$

$$2V_O - 2V_x = 0 \Rightarrow V_O = 4[V]$$

GEAUX

CIRCUITS WITH INDEPENDENT VOLTAGE SOURCES



3 nodes plus the reference. In principle one needs 3 equations...

…but two nodes are connected to the reference through voltage sources. Hence those node voltages are known!!!

…Only one KCL is necessary

$$\frac{V_2}{6k} + \frac{V_2 - V_3}{12k} + \frac{V_2 - V_1}{12k} = 0$$

$$V_1 = 12[V]$$

$$V_3 = -6[V]$$

THESE ARE THE REMAINING TWO NODE EQUATIONS

Hint: Each voltage source connected to the reference node saves one node equation

SOLVING THE EQUATIONS

$$2V_2 + (V_2 - V_3) + (V_2 - V_1) = 0$$

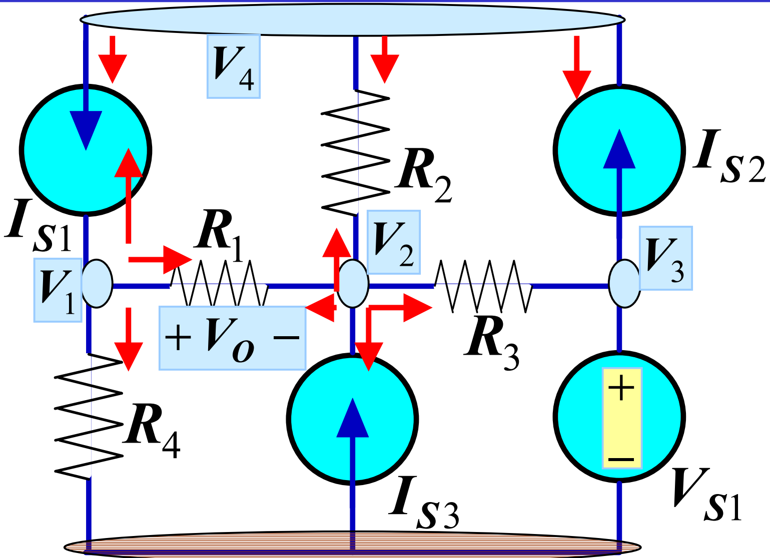
$$4V_2 = 6[V] \Rightarrow V_2 = 1.5[V]$$

One more example ….

GEAUX

Problem 3.67 (6th Ed) Find V_0

R1 = 1k; R2 = 2k, R3 = 1k, R4 = 2k
Is1 =2mA, Is2 = 4mA, Is3 = 4mA,
Vs = 12 V



IDENTIFY AND LABEL ALL NODES

WRITE THE NODE EQUATIONS

NOW WE LOOK WHAT IS BEING ASKED TO DECIDE THE SOLUTION STRATEGY.

$$V_0 = V_1 - V_2$$

ONLY V_1, V_2 ARE NEEDED FOR V_O

KNOWN NODE VOLTAGE

$$@V_3: \quad V_3 = V_{VS} = 12[V]$$

$$@V_1: \; -I_{S1} + \frac{V_1 - V_2}{R_1} + \frac{V_1}{R_4} = 0$$

$$-2[mA] + \frac{V_1 - V_2}{1k} + \frac{V_1}{2k} = 0$$

$$@V_2: \; -I_{S3} + \frac{V_2 - V_1}{R_1} + \frac{V_2 - V_3}{R_3} + \frac{V_2 - V_4}{R_2} = 0$$

$$-4[mA] + \frac{V_2 - V_1}{1k} + \frac{V_2 - 12}{1k} + \frac{V_2 - V_4}{2k} = 0$$

$$@V_4: \; I_{S1} - I_{S2} + \frac{V_4 - V_2}{R_2} = 0$$

$$2[mA] - 4[mA] + \frac{V_4 - V_2}{2k} = 0$$

GEAUX

TO SOLVE BY HAND ELIMINATE DENOMINATORS

$$-2[mA]+\frac{V_1-V_2}{1k}+\frac{V_1}{2k}=0 \quad \xrightarrow{*/2k} \quad 3V_1-2V_2=4[V] \quad (1)$$

$$-4[mA]+\frac{V_2-V_1}{1k}+\frac{V_2-12}{1k}+\frac{V_2-V_4}{2k}=0 \quad \xrightarrow{*/2k} \quad -2V_1+5V_2-V_4=32[V] \quad (2)$$

$$2[mA]-4[mA]+\frac{V_4-V_2}{2k}=0 \quad \xrightarrow{*/2k} \quad -V_2+V_4=4[V] \quad (3)$$

Add 2+3 $\quad -2V_1+4V_2=36[V]$

$3V_1-2V_2=4[V]$ $*/2$ and add

$$4V_1=40[V]\Rightarrow V_1=10[V]$$

$$4V_2=56[V]\Rightarrow V_2=14[V]$$

FINALLY!! $\quad V_0=V_1-V_2=-4[V]$

ALTERNATIVE: USE LINEAR ALGEBRA

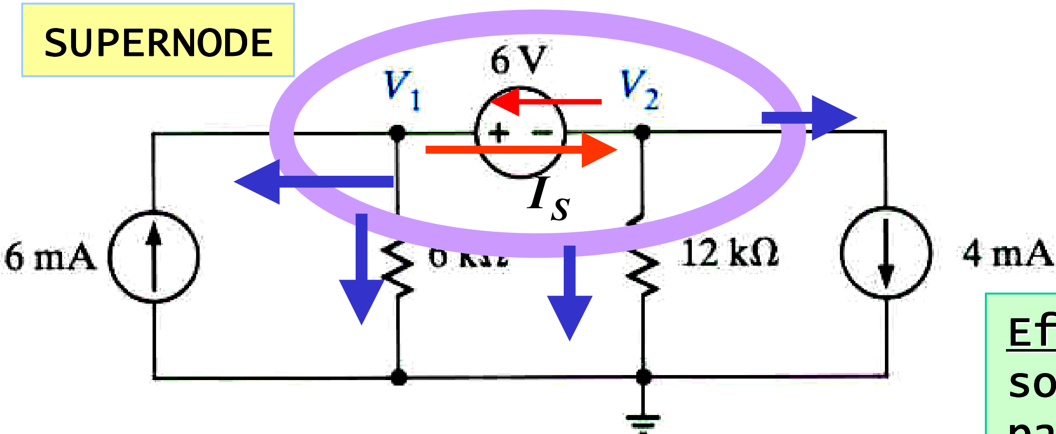
$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 32 \\ 4 \end{bmatrix}$$

So. What happens when sources are connected between two non reference nodes?

THE SUPERNODE TECHNIQUE

We will use this example to introduce the concept of a SUPERNODE

SUPERNODE



Conventional node analysis requires all currents at a node

@V_1

$$-6mA + \frac{V_1}{6k} + I_S = 0$$

@V_2

$$-I_S + 4mA + \frac{V_2}{12k} = 0$$

2 eqs, 3 unknowns...Panic!!
The current through the source is not related to the voltage of the source

Math solution: add one equation

$$V_1 - V_2 = 6[V]$$

Efficient solution: enclose the source, and all elements in parallel, inside a surface.

Apply KCL to the surface!!!

$$-6mA + \frac{V_1}{6k} + \frac{V_2}{12k} + 4mA = 0$$

The source current is interior to the surface and is not required

We STILL need one more equation

$$V_1 - V_2 = 6[V]$$

Only 2 eqs in two unknowns!!!

ALGEBRAIC DETAILS

The Equations

$$(1)\quad \frac{V_1}{6k} + \frac{V_2}{12k} - 6mA + 4mA = 0$$

$* / 12\,k$

$$(2)\quad V_1 - V_2 = 6[V]$$

Solution

1. Eliminate denominators in Eq(1). Multiply by ...

$$2V_1 + V_2 = 24[V]$$

$$V_1 - V_2 = 6[V]$$

2. Add equations to eliminate V_2

$$3V_1 = 30[V] \Rightarrow V_1 = 10[V]$$

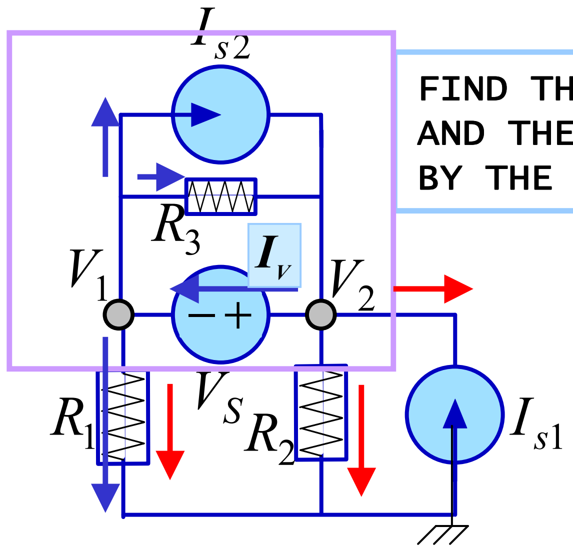
3. Use Eq(2) to compute V_2

$$V_2 = V_1 - 6[V] = 4[V]$$

GEAUX

The supernode technique

- Used when a branch between two nonreference nodes contains a voltage source.
- First encircle the voltage source and the two connecting nodes to form the supernode.
- Write the equation that defines the voltage relationship between the two nonreference nodes as a result of the presence of the voltage source.
- Write the KCL equation for the supernode.
- If the voltage source is dependent, then the controlling equation for the dependent source is also needed.



FIND THE NODE VOLTAGES AND THE POWER SUPPLIED BY THE VOLTAGE SOURCE

$$R_1 = R_2 = 10k\Omega, R_3 = 4k\Omega$$

$$V_S = 20[V], I_{s1} = 10[mA], I_{s2} = 6[mA]$$

$$V_2 - V_1 = 20 \quad \Rightarrow -V_1 + V_2 = 20[V]$$

$$\frac{V_1}{10k} + \frac{V_2}{10k} - 10mA = 0 \quad */10k \Rightarrow V_1 + V_2 = 100[V]$$

adding: $V_2 = 60[V]$

$V_1 = 100 - V_2 = 40[V]$

TO COMPUTE THE POWER SUPPLIED BY VOLTAGE SOURCE WE MUST KNOW THE CURRENT THROUGH IT

$$I_V = \frac{V_1}{10k} + 6mA + \frac{V_1 - V_2}{10k} = 8mA$$

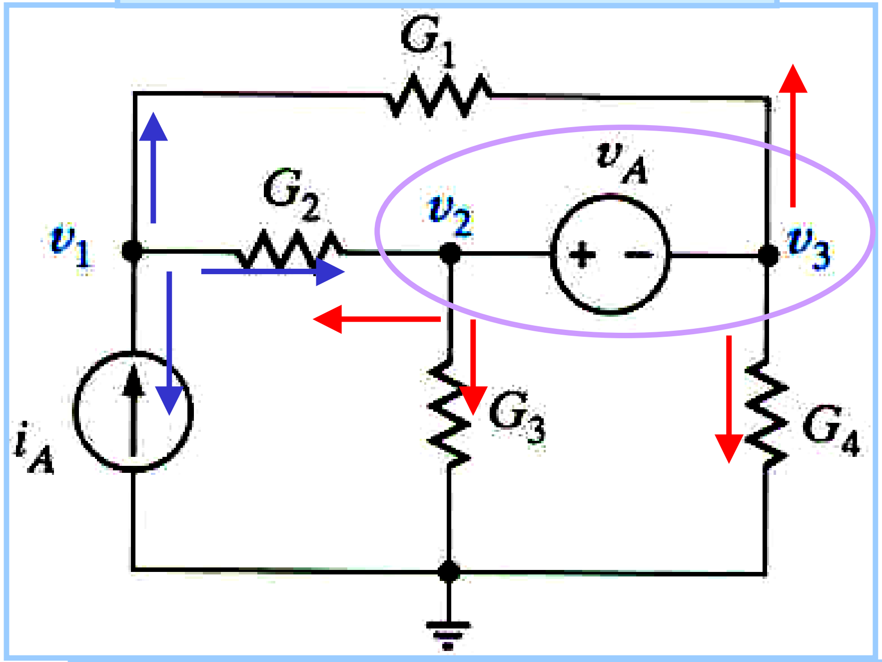
$$P = 20[V] \times 8[mA] = 160mW$$

BASED ON PASSIVE SIGN CONVENTION THE POWER IS RECEIVED BY THE SOURCE!!

GEAUX

LEARNING EXAMPLE

WRITE THE NODE EQUATIONS



@v_1 $(v_1 - v_3)G_1 + (v_1 - v_2)G_2 - i_A = 0$

@ SUPERNODE

CONSTRAINT : $v_2 - v_3 = v_A$

KCL (leaving supernode) :

$$(v_2 - v_1)G_2 + v_2G_3 + (v_3 - v_1)G_1 + v_3G_4 = 0$$

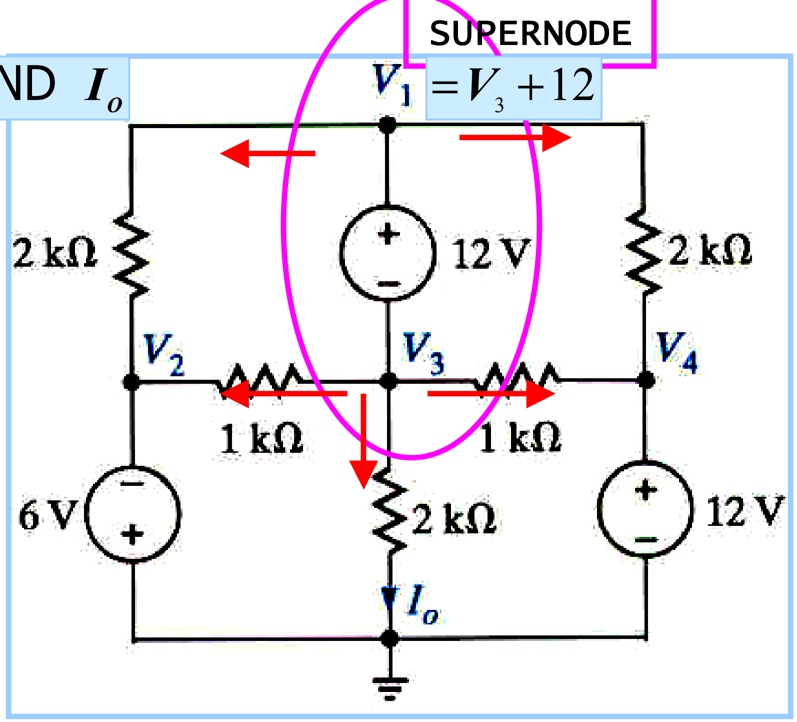
THREE EQUATIONS IN THREE UNKNOWNS

GEAUX

LEARNING EXAMPLE

FIND I_o

$V_1 = V_3 + 12$



$V_2 = -6V, V_4 = 12V$ KNOWN NODE VOLTAGES

SUPERNODE CONSTRAINT $\Rightarrow V_1 - V_3 = 12$

$$\frac{V_3 + 12 - (-6)}{2k} + \frac{V_3 + 12 - 12}{2k} + \frac{V_3 - (-6)}{1k} + \frac{V_3 - 12}{1k} + \frac{V_3}{2k} = 0$$

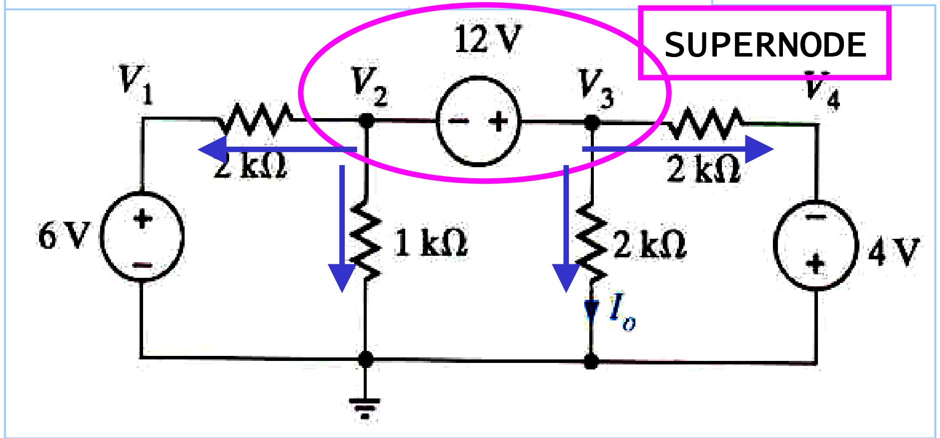
KCL @ SUPERNODE

$$V_3 = -\frac{6}{7}\text{V} \Rightarrow I_o = \frac{-\frac{6}{7}}{2k} = -\frac{3}{7}\text{mA}$$

GEAUX

LEARNING EXTENSION

Use nodal analysis to find I_o



$V_1 = 6V$

$V_4 = -4V$

SOURCES CONNECTED TO THE REFERENCE

CONSTRAINT EQUATION $V_3 - V_2 = 12V$

KCL @ SUPERNODE

$$\frac{V_2 - 6}{2k} + \frac{V_2}{1k} + \frac{V_3}{2k} + \frac{V_3 - (-4)}{2k} = 0 \quad */2k$$

$$3V_2 + 2V_3 = 2V$$

$$-V_2 + V_3 = 12V \quad */3 \text{ and add}$$

$$5V_3 = 38V$$

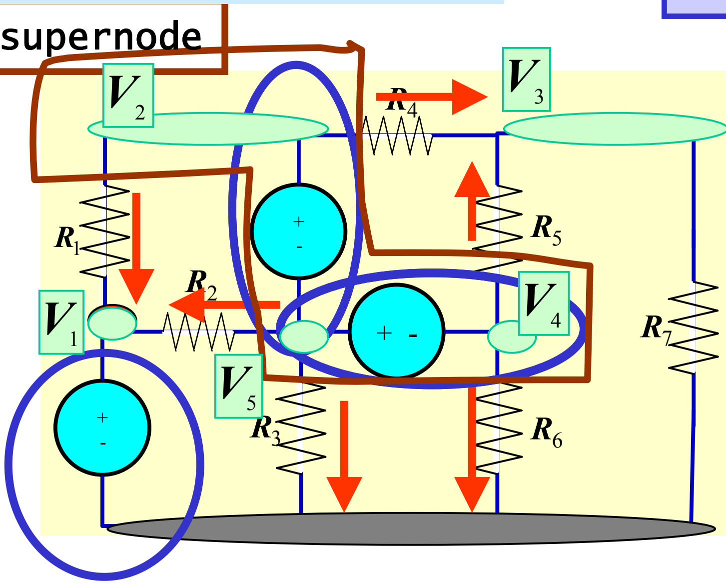
V_2 IS NOT NEEDED FOR I_o

OHM'S LAW $I_o = \frac{V_3}{2k} = 3.8mA$

GEAUX

WRITE THE NODE EQUATIONS

Supernodes can be more complex



KCL@V_3

$$\frac{V_3 - V_2}{R_4} + \frac{V_3 - V_4}{R_5} + \frac{V_3}{R_7} = 0$$

KCL @SUPERNODE
(Careful not to omit any current)

$$\frac{V_2 - V_1}{R_1} + \frac{V_5 - V_1}{R_2} + \frac{V_5}{R_3} + \frac{V_4}{R_6} + \frac{V_4 - V_3}{R_5} + \frac{V_2 - V_3}{R_4} = 0$$

CONSTRAINTS DUE TO VOLTAGE SOURCES

$$V_1 = V_{S1}$$

$$V_2 - V_5 = V_{S2}$$

$$V_5 - V_4 = V_{S3}$$

5 EQUATIONS IN FIVE UNKNOWNS.

Identify all nodes, select a reference and label nodes

Nodes connected to reference through a voltage source

Voltage sources in between nodes and possible supernodes

EQUATION BOOKKEEPING:
KCL@ V_3, KCL@ supernode, 2 constraints equations and one known node

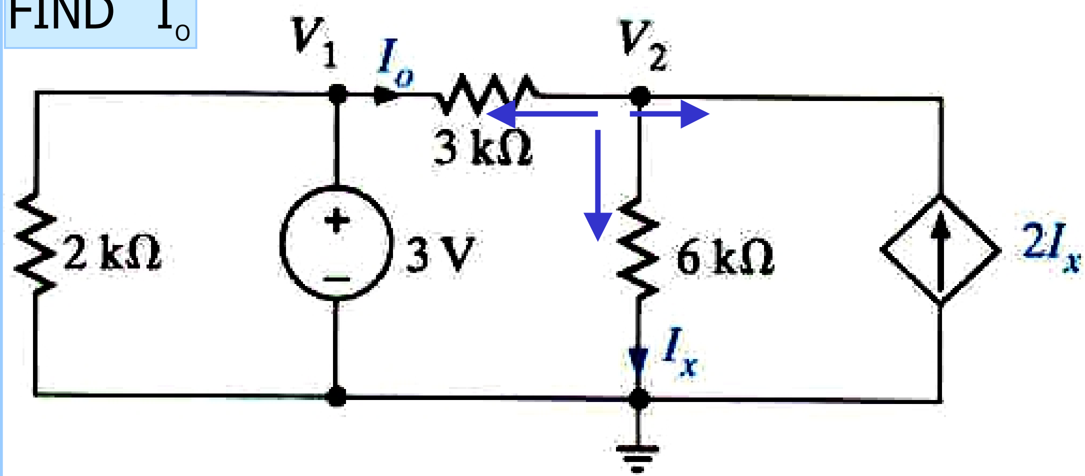
GEAUX

CIRCUITS WITH DEPENDENT SOURCES PRESENT NO SIGNIFICANT ADDITIONAL COMPLEXITY. THE DEPENDENT SOURCES ARE TREATED AS REGULAR SOURCES

WE MUST ADD ONE EQUATION FOR EACH CONTROLLING VARIABLE

LEARNING EXAMPLE

FIND I_o



VOLTAGE SOURCE CONNECTED TO REFERENCE

$V_1 = 3V$

KCL@ V_2: $\frac{V_2 - V_1}{3k} + \frac{V_2}{6k} - 2I_x = 0$

REPLACE

CONTROLLING VARIABLE IN TERMS OF NODE VOLTAGES

$I_x = \frac{V_2}{6k}$

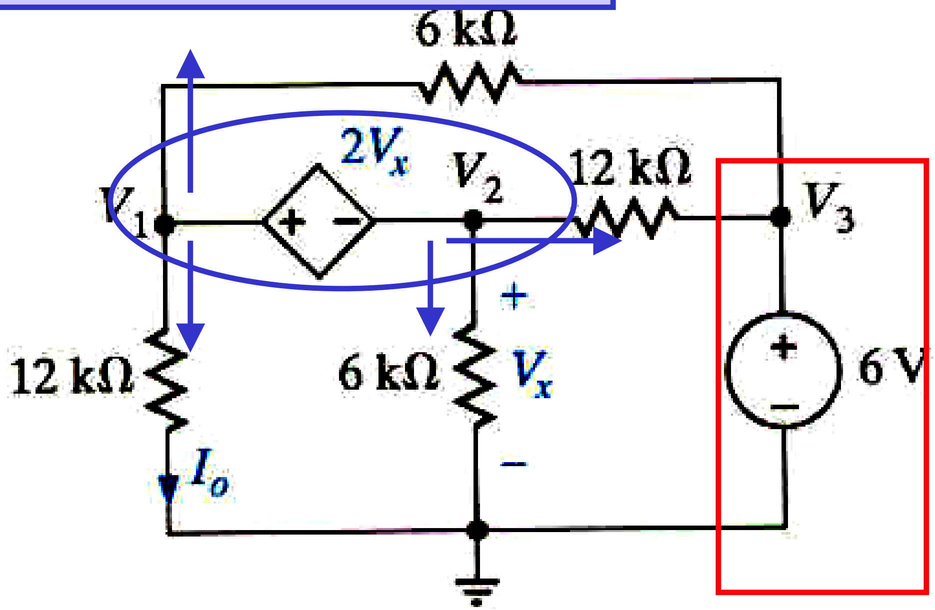
$$\frac{V_2 - V_1}{3k} + \frac{V_2}{6k} - 2\frac{V_2}{6k} = 0 \quad */6k$$

$$V_2 - 2V_1 = 0 \Rightarrow V_2 = 6V$$

$$I_o = \frac{V_1 - V_2}{3k} = -1mA$$

GEAUX

SUPER NODE WITH DEPENDENT SOURCE



VOLTAGE SOURCE CONNECTED TO REFERENCE

$V_3 = 6V$

SUPERNODE CONSTRAINT $V_1 - V_2 = 2V_x$

CONTROLLING VARIABLE IN TERMS OF NODES

$V_x = V_2 \Rightarrow V_1 = 3V_2$

KCL AT SUPERNODE

$$\frac{V_1 - V_3}{6k} + \frac{V_1}{12k} + \frac{V_2}{6k} + \frac{V_2 - V_3}{12k} = 0 \quad */12k$$

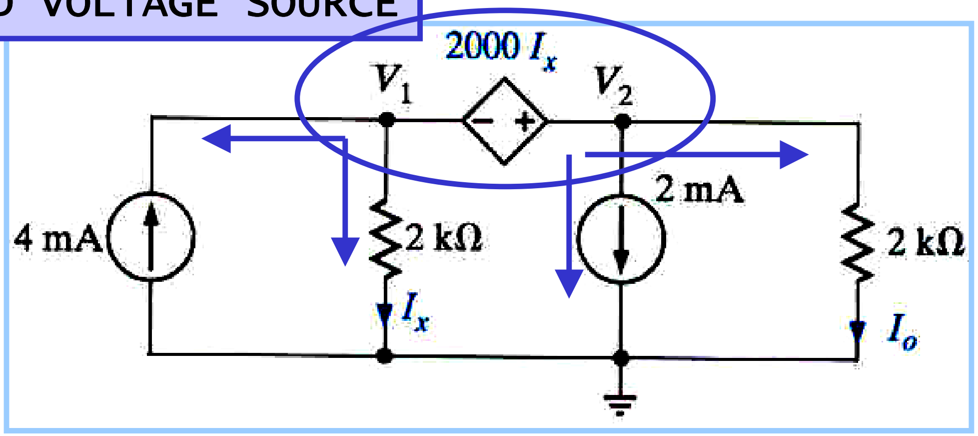
$$2(V_1 - 6) + V_1 + 2V_2 + V_2 - 6 = 0$$

$$3V_1 + 3V_2 = 18 \Rightarrow 4V_1 = 18$$

$$I_o = \frac{V_1}{12k} = \frac{3}{8}\text{mA}$$

GEAUX

CURRENT CONTROLLED VOLTAGE SOURCE



CONSTRAINT DUE TO SOURCE $V_2 - V_1 = 2kI_x$

CONTROLLING VARIABLE IN TERMS OF NODES

$$\Rightarrow V_1 = 2kI_x \Rightarrow V_2 = 2V_1 \qquad I_x = \frac{V_1}{2k}$$

KCL AT SUPERNODE $-4mA + \frac{V_1}{2k} + 2mA + \frac{V_2}{2k} = 0$

$V_1 + V_2 = 4[V]$ $*/2$ and add

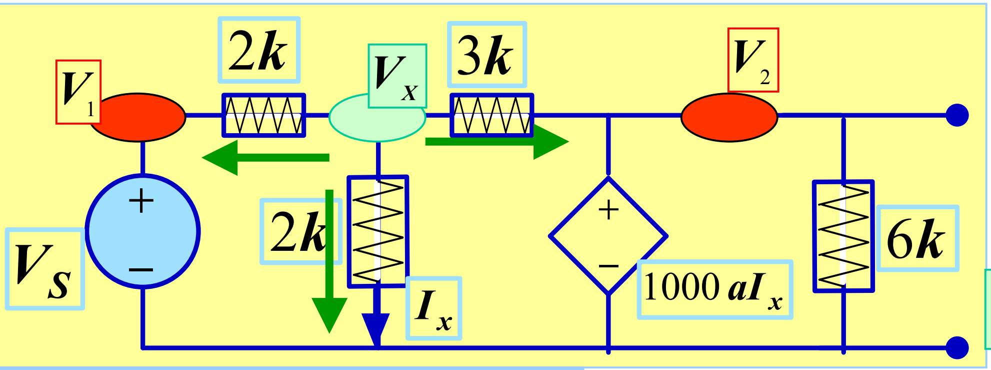
$-2V_1 + V_2 = 0$

$3V_2 = 8[V]$

$$I_o = \frac{V_2}{2k} = \frac{4}{3}mA$$

GEAUX

An example with dependent sources



'a' has units of [Volt/Amp]

IDENTIFY AND LABEL NODES

2 nodes are connected to the reference through voltage sources

$$V_1 = V_S$$

$$V_2 = 1000 a I_X$$

KCL @ VX

$$\frac{V_X - V_S}{2k} + \frac{V_X}{2k} + \frac{V_X - v_2}{3k} = 0$$

EXPRESS CONTROLLING VARIABLE IN TERMS OF NODE VOLTAGES

$$I_X = \frac{V_X}{2k}$$

REPLACE Ix IN V2

$$V_2 = \frac{1k * aV_X}{2k}$$

$$V_2 = \frac{aV_X}{2}$$

REPLACE V2 IN KCL

$$3(V_X - V_S) + 3V_X + 2(V_X - aV_X / 2) = 0$$

$$(8 - a)V_X = 3V_S$$

What happens when a=8?