# MAGNETICALLY COUPLED NETWORKS

## LEARNING GOALS

**Mutual Inductance**
Behavior of inductors sharing a common magnetic field

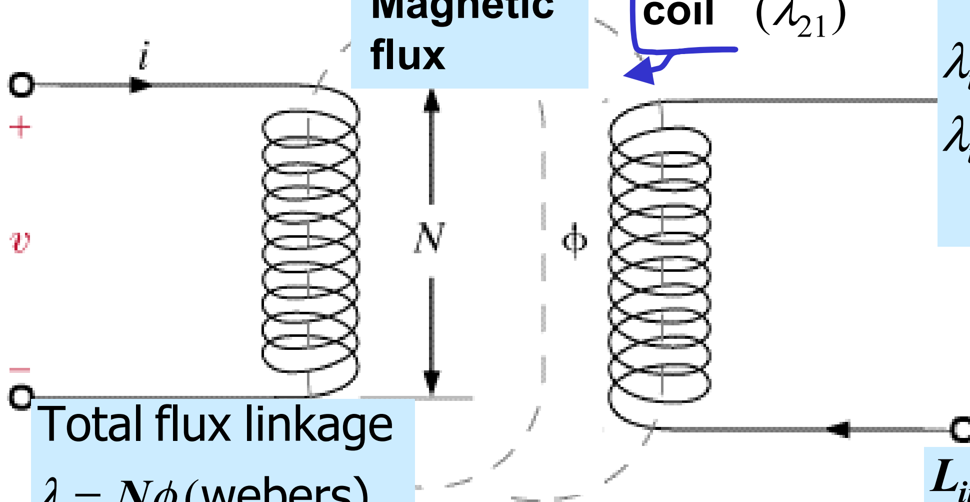
**Energy Analysis**
Used to establish relationship between mutual reluctance and self-inductance

**The ideal transformer**
Device modeling components used to change voltage and/or current levels

**Safety Considerations**
Important issues for the safe operation of circuits with transformers

# MUTUAL INDUCTANCE

## Overview of Induction Laws




Total flux linkage

$\lambda = N\phi$ (webers)

**If linkage is created by a current flowing through the coils…**

$\lambda = Li$ **(Ampere's Law)**

**The voltage created at the terminals of the components is**

$$v = L\frac{di}{dt}$$ **(Faraday's Induction Law)**

**What happens if the flux created by the current links to another coil?**

**One has the effect of mutual inductance**

**Assume *n* circuits interacting**

$$\lambda_i = \sum_{j=1}^{n} \lambda_{ij}$$

$\lambda_i$ = Total flux linking circuit $i$

$\lambda_{ij}$ = Flux linking circuit $i$ caused by a current in circuit $j$

For linear inductor models

$\lambda_{ij} = L_{ij} i_j$

$L_{ii}$ = "self inductance" of circuit $i$

$L_{ij} = L_{ji}$ = Mutual inductance between circuits $i$ and $j$

**Special case *n*=2**

$$\lambda_1 = L_{11} i_1 + L_{12} i_2$$
$$\lambda_2 = L_{21} i_1 + L_{22} i_2$$

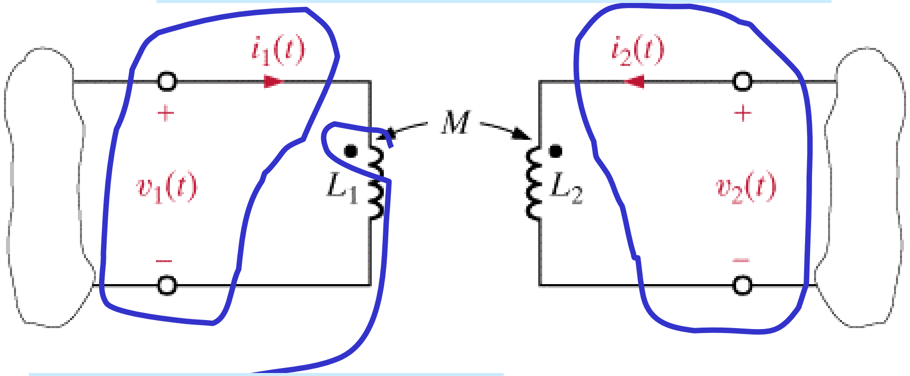
Linear Model: $L_{12} = L_{21}$

Simplify notation

$L_1 \leftarrow L_{11};\ L_2 \leftarrow L_{22};\ M \leftarrow L_{12} = L_{21}$

# THE DOT CONVENTION

Currents and voltages follow passive sign convention



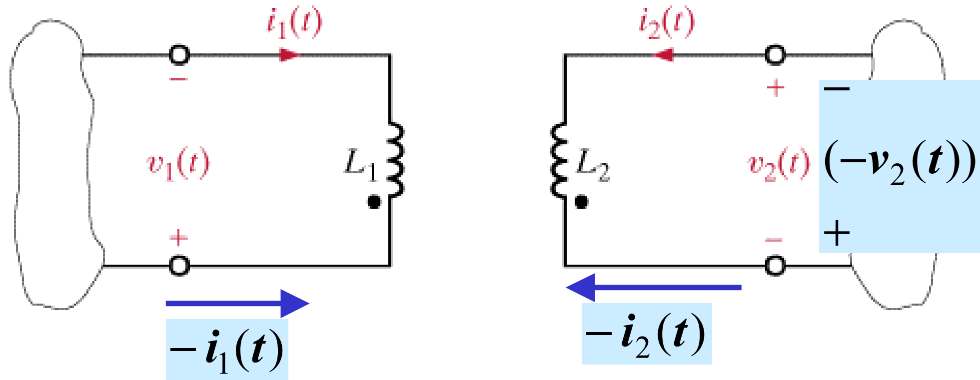

Flux 2 induced voltage has + at dot

$$v_1(t) = L_1 \frac{di_1}{dt}(t) + M \frac{di_2}{dt}(t)$$

$$v_2(t) = M \frac{di_1}{dt}(t) + L_2 \frac{di_2}{dt}(t)$$

For other cases change polarities or current directions to convert to this basic case

# LEARNING EXAMPLE




$$v_1(t) = L_1\left(-\frac{di_1}{dt}\right) + M\left(-\frac{di_2}{dt}\right)$$

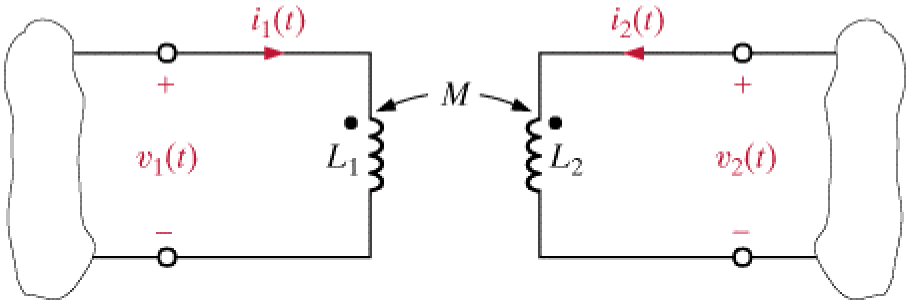
$$-v_2(t) = M\left(-\frac{di_1}{dt}\right) + L_2\left(-\frac{di_2}{dt}\right)$$

$$v_1 = -L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

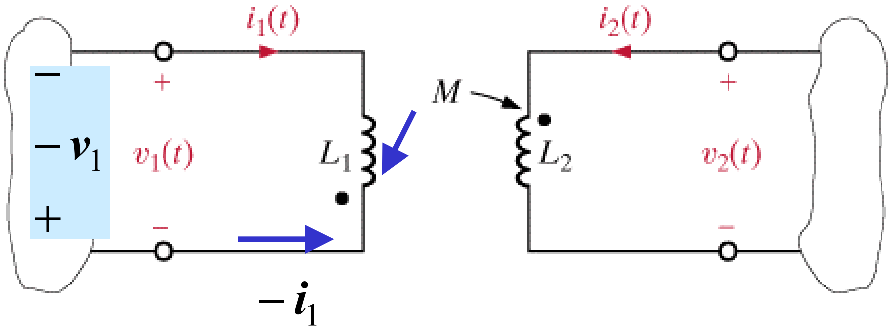
GEAUX

# More on the dot convention




$$v_1(t) = L_1 \frac{di_1}{dt}(t) + M \frac{di_2}{dt}(t)$$

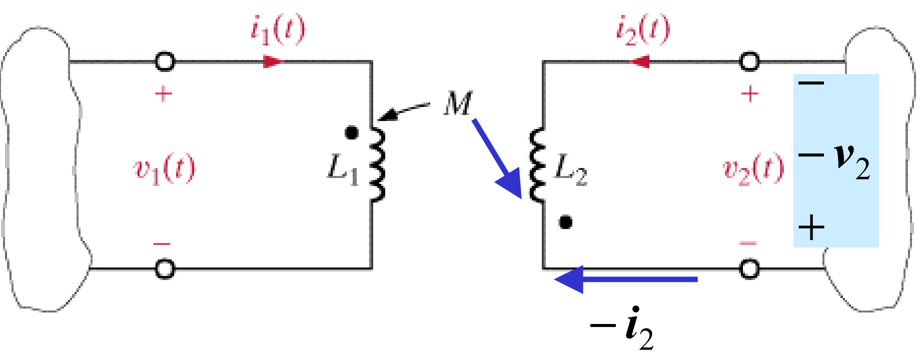
$$v_2(t) = M \frac{di_1}{dt}(t) + L_2 \frac{di_2}{dt}(t)$$




$$-v_1 = -L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

**Equivalent to a negative mutual inductance**

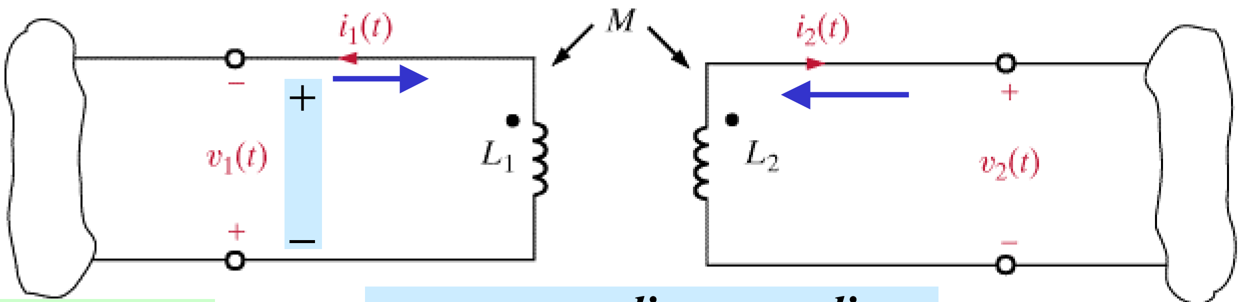



$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$-v_2 = M \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

## LEARNING EXTENSION

Write the equations for $v_1(t), v_2(t)$




**Convert to basic case**

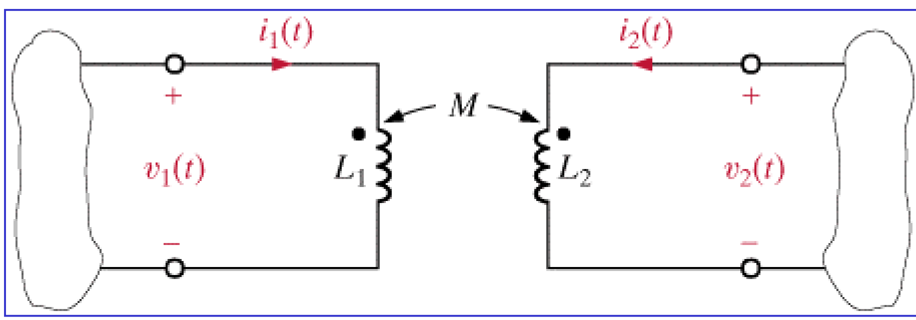
$$-v_1(t) = -L_1 \frac{di}{dt}(t) - M \frac{di_2}{dt}(t)$$

$$v_2(t) = -M \frac{di_1}{dt}(t) - L_2 \frac{di_2}{dt}(t)$$

$$v_1(t) = L_1 \frac{di}{dt}(t) + M \frac{di_2}{dt}(t)$$

$$v_2(t) = -M \frac{di_1}{dt}(t) - L_2 \frac{di_2}{dt}(t)$$

## PHASORS AND MUTUAL INDUCTANCE




$$v_1(t) = L_1 \frac{di_1}{dt}(t) + M \frac{di_2}{dt}(t)$$

$$v_2(t) = M \frac{di_1}{dt}(t) + L_2 \frac{di_2}{dt}(t)$$

**Assuming complex exponential sources**

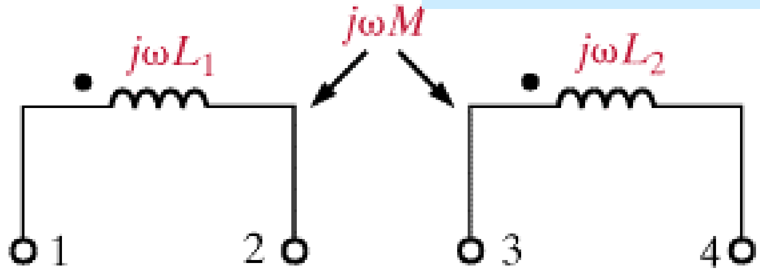
$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$V_2 = j\omega M I_1 + j\omega L_2 I_2$$

**Phasor model for mutually coupled linear inductors**

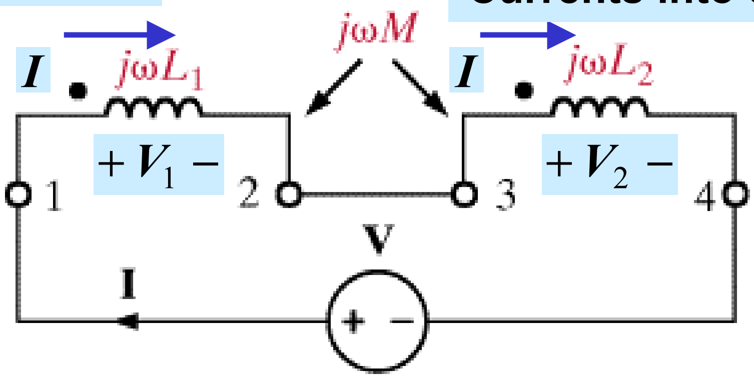
## LEARNING EXAMPLE

**The coupled inductors can be connected in four different ways. Find the model for each case**




### CASE I

**Currents into dots**




$$V = V_1 + V_2$$

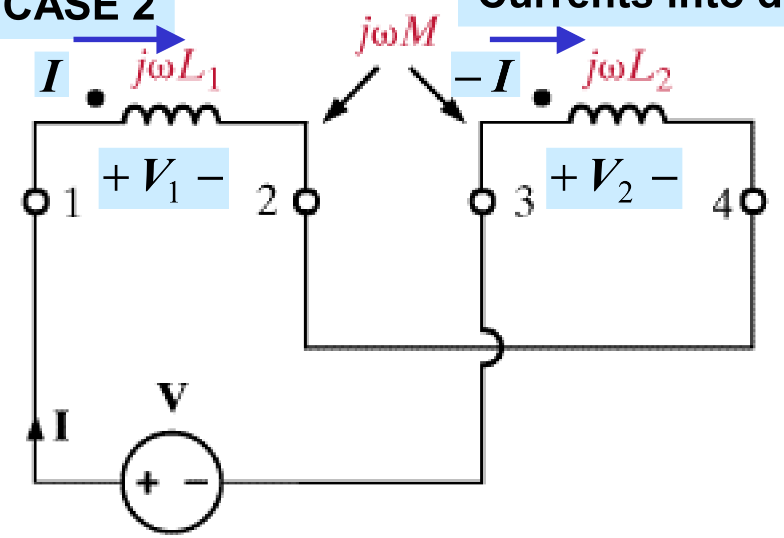
$$V_1 = j\omega L_1 I + j\omega M I$$

$$V_2 = j\omega M I + j\omega L_2 I$$

$$V = j\omega(L_1 + L_2 + 2M) = j\omega L_{eq} I$$

### CASE 2

**Currents into dots**




$$V = V_1 - V_2$$

$$V_1 = j\omega L_1 - j\omega M I$$

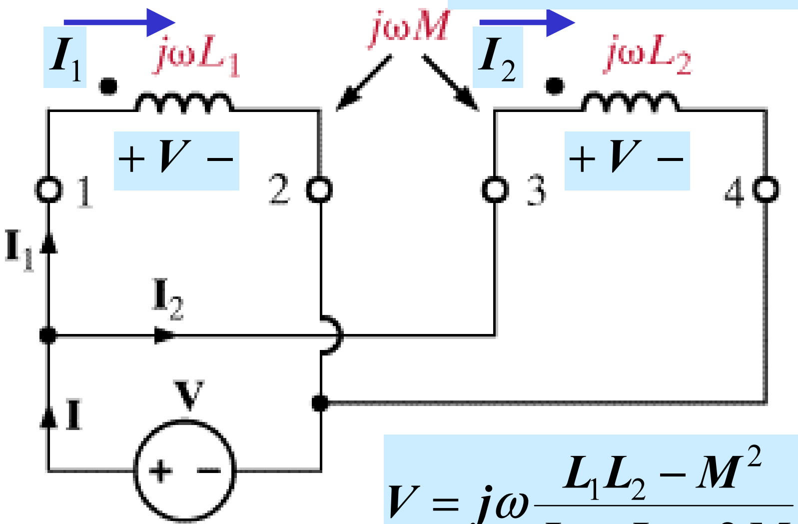
$$V_2 = j\omega M I - j\omega L_2 I$$

$$V = j\omega\underbrace{(L_1 - 2M + L_2)}_{L_{eq}}I$$

$L_{eq} \geq 0$ imposes a physical constraint on the value of $M$

## CASE 3

**Currents into dots**




$$I = I_1 + I_2 \qquad \Rightarrow I_2 = I - I_1$$

$$V = j\omega L_1 I_1 + j\omega M I_2$$

$$V = j\omega M I_1 + j\omega L_2 I_2$$

$$V = j\omega L_1 I_1 + j\omega M (I - I_1)$$

$$V = j\omega M I_1 + j\omega L_2 (I - I_1)$$

$$V = j\omega (L_1 - M) I_1 + j\omega M I \qquad \times / (L_2 - M)$$

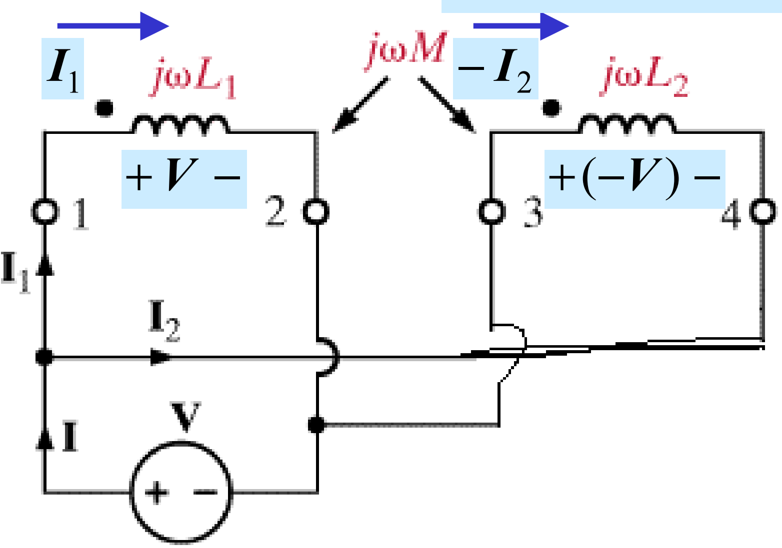
$$V = -j\omega (L_2 - M) I_1 + j\omega L_2 I \qquad \times / (L_1 - M)$$

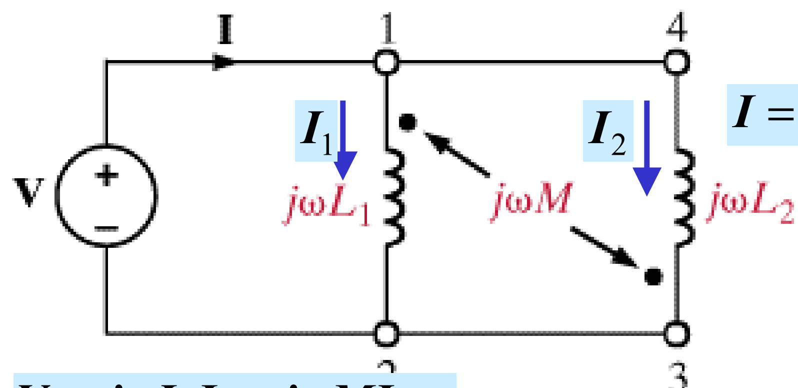
$$(L_1 + L_2 - 2M) V = j\omega \left( M (L_2 - M) + L_2 (L_1 - M) \right) I$$

$$V = j\omega \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} I$$

## CASE 4

**Currents into dots**




$$I = I_1 + I_2$$

$$V = j\omega L_1 I_1 - j\omega M I_2$$

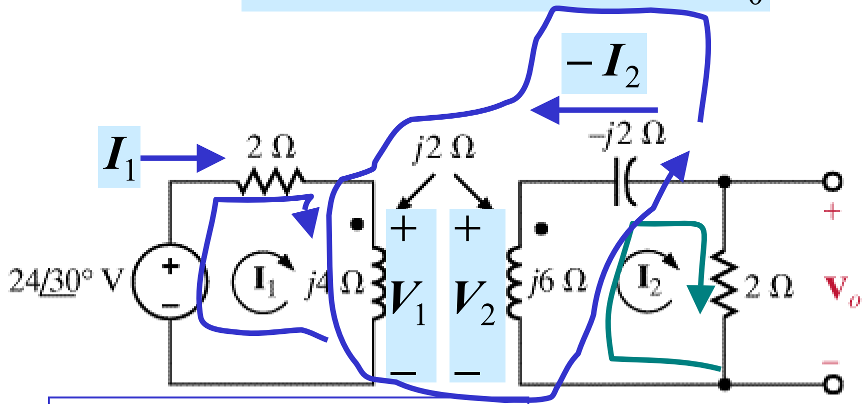
$$-V = j\omega M I_1 - j\omega L_2 I_2$$

$$V = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} I$$

GEAUX

**LEARNING EXAMPLE**

FIND THE VOLTAGE $V_0$




**1. Coupled inductors. Define their voltages and currents**

KVL: $24\angle 30° = 2I_1 + V_1$

$V_S$

KVL: $-V_2 - j2I_2 + 2I_2 = 0$

**2. Write loop equations in terms of coupled inductor voltages**

MUTUAL INDUCTANCE CIRCUIT

$$V_1 = j4I_1 + j2(-I_2)$$

$$V_2 = j2I_1 + j6(-I_2)$$

$V_0 = 2I_2$

**3. Write equations for coupled inductors**

$$V_S = (2 + j4)I_1 - j2I_2 \quad \times / j2$$

$$0 = -j2I_1 + (2 - j2 + j6)I_2 \quad \times / 2 + j4$$

**4. Replace into loop equations and do the algebra**

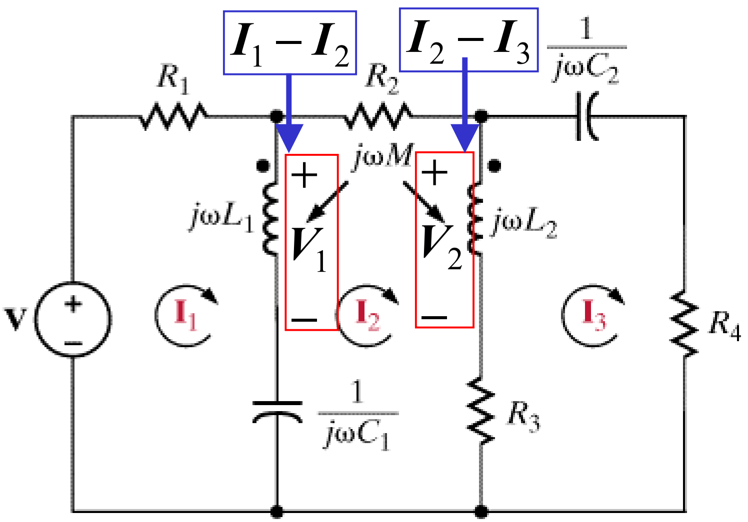
$$j2V_S = \left(4 + (2 + j4)^2\right)I_2$$

$$I_2 = \frac{j2V_S}{-8 + j16} \times \frac{-j}{-j} = \frac{2V_S}{16 + 8j}$$

$$V_0 = 2I_2 = \frac{V_S}{4 + 2j} = \frac{24\angle 30°}{4.47\angle 26.57°} = 5.37\angle 3.42°$$

GEAUX

## LEARNING EXAMPLE — Write the mesh equations




**1. Define variables for coupled inductors**

**2. Write loop equations in terms of coupled inductor voltages**

$$V = R_1 I_1 + V_1 + \frac{I_1 - I_2}{j\omega C_1}$$

$$-V_1 + R_2 I_2 + V_2 + R_3(I_2 - I_3) + \frac{I_2 - I_1}{j\omega C_1} = 0$$

$$-V_2 + \frac{I_3}{j\omega C_2} + R_4 I_3 + R_3(I_3 - I_2) = 0$$

**3. Write equations for coupled inductors**

$$V_1 = j\omega L_1(I_1 - I_2) + j\omega M(I_2 - I_3)$$

$$V_2 = j\omega M(I_1 - I_2) + j\omega L_2(I_2 - I_3)$$

**4. Replace into loop equations and rearrange terms**

$$V = \left(R_1 + j\omega L_1 + \frac{1}{j\omega C_1}\right) I_1 - \left(j\omega L_1 - j\omega M + \frac{1}{j\omega C_1}\right) I_2 - j\omega M I_3$$

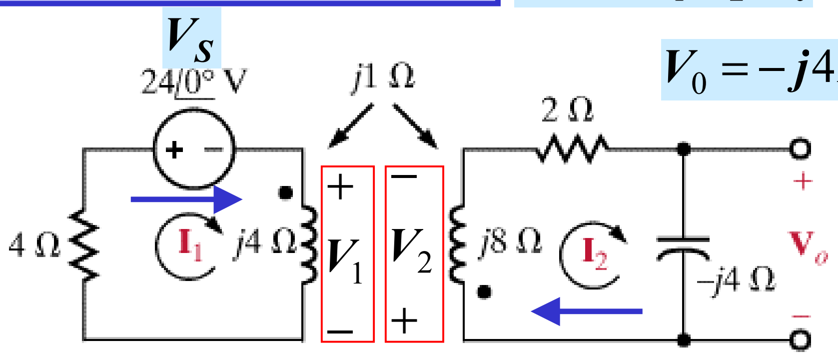
$$0 = -\left(j\omega L_1 - j\omega M + \frac{1}{j\omega C_1}\right) I_1 + \left(j\omega L_2 - j\omega M + R_2 - j\omega M + j\omega L_2 + R_3 + \frac{1}{j\omega C_1}\right) I_2 - (-j\omega M + j\omega L_2 + R_3) I_3$$

$$0 = -j\omega M I_1 - (j\omega L_2 - j\omega M + R_3) I_2 + \left(j\omega L_2 + \frac{1}{j\omega C_2} + R_4 + R_3\right) I_3$$

GEAUX

**LEARNING EXTENSION**

FIND $I_1, I_2, V_0$




$V_0 = -j4I_2$

**1. Define variables for coupled inductors**

**2. Loop equations**

**Voltages in Volts**
**Impedances in Ohms**
**Currents in ____**

$$V_S + V_1 + 4I_1 = 0$$
$$V_2 + (2 - j4)I_2 = 0$$

**3. Coupled inductors equations**

$$V_1 = j4I_1 + jI_2$$
$$V_2 = jI_1 + j8I_2$$

**4. Replace and rearrange**

$$(4 + j4)I_1 + jI_2 = -V_S \quad \times / -j$$
$$jI_1 + (2 + j4)I_2 = 0 \quad \times/(4 + j4)$$

$$(1 + 8(1 + j)(1 + 2j))I_2 = jV_S$$

$$I_2 = \frac{jV_S}{-7 + 24j} \times \frac{-j}{-j} = \frac{24\angle 0°}{24 + 7j} = \frac{24\angle 0°}{25\angle 16.26°}$$

$$I_2 = 0.96\angle -16.26°(A)$$

$$jI_1 + (2 + j4)I_2 = 0 / \times j \Rightarrow I_1 = j(2 + j4)I_2$$

$$I_1 = 1\angle 90° \times 4.47\angle 63.43° \times 0.96\angle -16.26°$$

$$I_1 = 4.29\angle 137.17°(A)$$

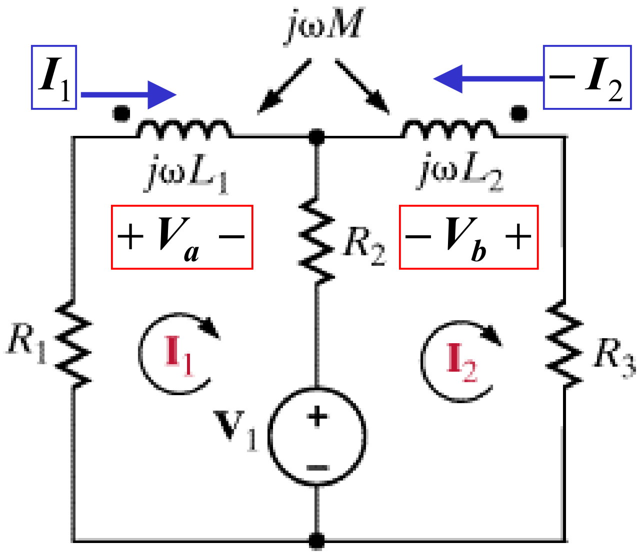
$$V_0 = -j4I_2 = 1\angle -90° \times 4 \times 0.96\angle -16.26°$$

$$V_0 = 3.84\angle -106.26°(V)$$

GEAUX

## LEARNING EXTENSION

## WRITE THE KVL EQUATIONS




$$(R_1 + R_2 + j\omega L_1)I_1 - (R_2 + j\omega M)I_2 = -V_1$$

$$-(R_2 + j\omega M)I_1 + (R_2 + R_3 + j\omega L_2)I_2 = V_1$$

1. Define variables for coupled inductors

2. Loop equations in terms of inductor voltages

$$V_a + R_2(I_1 - I_2) + V_1 + R_1 I_1 = 0$$

$$-V_b + R_3 I_2 - V_1 + R_2(I_2 - I_1) = 0$$

3. Equations for coupled inductors

$$V_a = j\omega L_1 I_1 + j\omega M(-I_2)$$

$$V_b = j\omega M I_1 + j\omega L_2(-I_2)$$

4. Replace into loop equations and rearrange

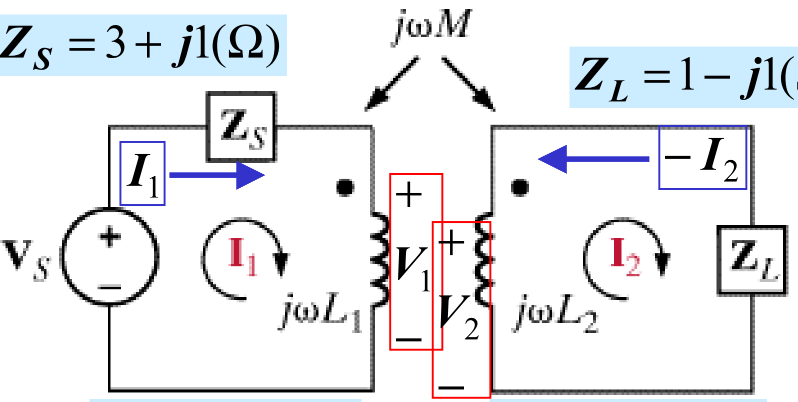
# LEARNING EXAMPLE

## DETERMINE IMPEDANCE SEEN BY THE SOURCE

$$Z_i = \frac{V_S}{I_1}$$

$Z_S = 3 + j1(\Omega)$

$Z_L = 1 - j1(\Omega)$




$j\omega L_1 = j2(\Omega)$ $j\omega L_2 = j2(\Omega)$

$j\omega M = j1(\Omega)$

1. Variables for coupled inductors

2. Loop equations in terms of coupled inductors voltages

$$Z_S I_1 + V_1 = V_S$$
$$-V_2 + Z_L I_2 = 0$$

3. Equations for coupled inductors

$$V_1 = j\omega L_1 I_1 + j\omega M(-I_2)$$
$$V_2 = j\omega M I_1 + j\omega L_2(-I_2)$$

4. Replace and do the algebra

$$(Z_S + j\omega L_1)I_1 - (j\omega M)I_2 = V_S \quad \times/(Z_L + j\omega L_2)$$
$$-(j\omega M)I_1 + (Z_L + j\omega L_2)I_2 = 0 \quad \times/\, j\omega M$$

$$\left((Z_S + j\omega L_1)(Z_L + j\omega L_2) - (j\omega M)^2\right)I_1 = (Z_L + j\omega L_2)V_S$$

$$Z_i = \frac{V_S}{I_1} = (Z_S + j\omega L_1) - \frac{(j\omega M)^2}{Z_L + j\omega L_2}$$

$$Z_i = 3 + j3 - \frac{(j1)^2}{1 + j1} = 3 + j3 + \frac{1}{1 + j} \times \frac{1 - j}{1 - j}$$

$$Z_i = 3 + j3 + \frac{1 - j}{2} = 3.5 + j2.5(\Omega)$$

$$Z_i = 4.30\angle 35.54°(\Omega)$$

**WARNING: This is NOT a phasor**

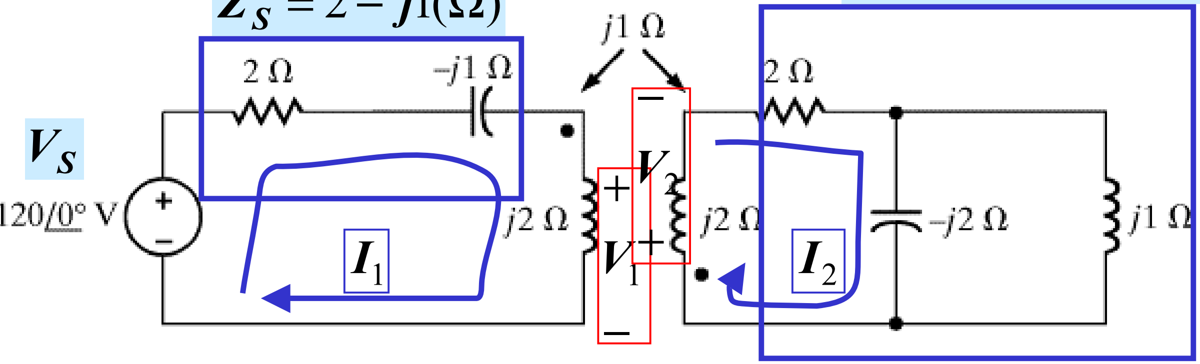
GEAUX

## LEARNING EXTENSION

## DETERMINE IMPEDANCE SEEN BY THE SOURCE

$Z_S = 2 - j1(\Omega)$

$Z_L = 2 + (-j2 \| j1)$




$$Z_L = 2 + \frac{2}{-j} = 2 + 2j(\Omega)$$

1. Variables for coupled inductors

2. Loop equations

$$V_1 + Z_S I_1 = V_S$$
$$V_2 + Z_L I_2 = 0$$

3. Equations for coupled inductors

$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$
$$V_2 = j\omega M I_1 + j\omega L_2 I_2$$

4. Replace and do the algebra

$$Z_i = \frac{V_S}{I_1} = (Z_S + j\omega L_1) - \frac{(j\omega M)^2}{Z_L + j\omega L_2}$$

$$Z_i = [(2 - j1) + j2] - \frac{(j1)^2}{(2+2j)+2j} = (2+j) + \frac{1}{2(1+2j)} \times \frac{(1-2j)}{(1-2j)}$$

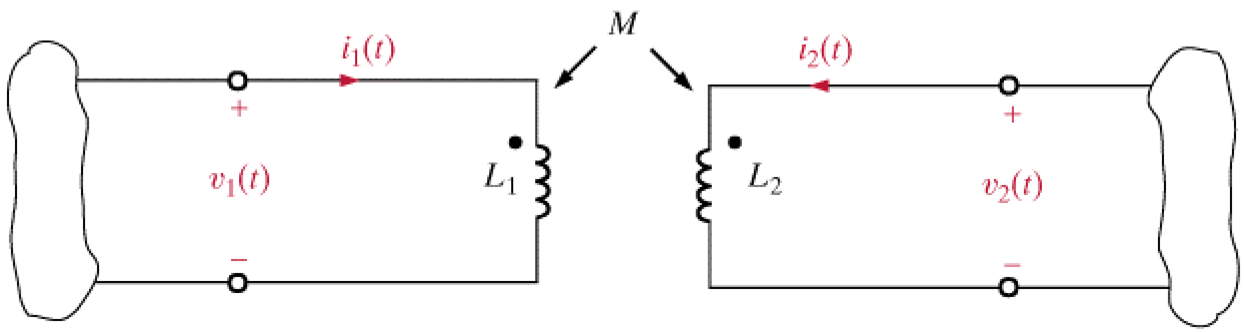
$$Z_i = 2 + j + \frac{1-2j}{2(1+2^2)} = 2.1 + 0.8j(\Omega)$$

$$Z_i = 2.25\angle 20.85°(\Omega)$$

One can choose directions for currents.
If I2 is reversed one gets the same equations than in previous example.
Solution for I1 must be the same and expression for impedance must be the same

# ENERGY ANALYSIS

**We determine the total energy stored in a coupled network**




**This development is different from the one in the book. But the final result is obviously the same**

EQUATIONS FOR COUPLED INDUCTORS

$$v_1(t) = L_1 \frac{di_1}{dt}(t) \pm M \frac{di_2}{dt}(t) \qquad \times / i_1(t)$$

$$v_2(t) = \pm M \frac{di_1}{dt}(t) + L_2 \frac{di_2}{dt}(t) \qquad \times / i_2(t)$$

TOTAL POWER SUPPLIED TO NETWORK

$$p_T(t) = v_1(t) i_1(t) + v_2(t) i_2(t)$$

$$p_T(t) = L_1 i_1(t) \frac{di_1}{dt}(t) \pm M i_1(t) \frac{di_2}{dt}(t) \pm M \frac{di_1}{dt}(t) i_2(t) + L_2 i_2(t) \frac{di_2}{dt}(t)$$

$$\frac{1}{2}\frac{di_1^2}{dt}(t) \qquad M \frac{di_1 i_2}{dt}(t) \qquad \frac{1}{2}\frac{di_2^2}{dt}(t)$$

$$p_T(t) = \frac{d}{dt}\left(\frac{1}{2} L_1 i_1^2(t) \pm M i_1(t) i_2(t) + \frac{1}{2} L_2 i_2^2(t)\right) \qquad \int_{-\infty}^{t}$$

$$w(t) = \frac{1}{2} L_1 i_1^2(t) \pm M i_1(t) i_2(t) + \frac{1}{2} L_2 i_2^2(t)$$

$$+\frac{1}{2}\frac{M^2}{L_2} i_1^2(t) - \frac{1}{2}\frac{M^2}{L_2} i_1^2(t)$$

$$w(t) = \frac{1}{2}\left(L_1 - \frac{M^2}{L_2}\right) i_1^2(t) + \frac{1}{2}\left(L_2 i_2(t) \pm \frac{M}{\sqrt{L_2}} i_1(t)\right)^2$$

$$w(t) \geq \Leftrightarrow L_1 - \frac{M^2}{L_2} \geq 0 \Leftrightarrow M \leq \sqrt{L_1 L_2}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

**Coefficient of coupling**

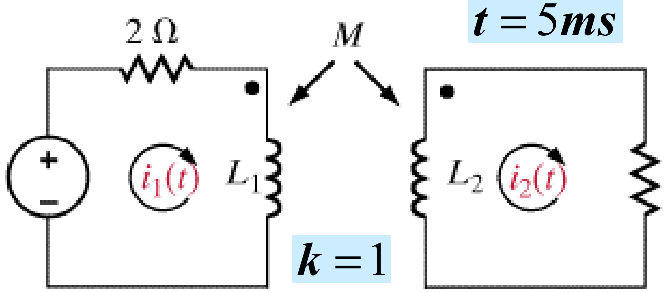
GEAUX

**LEARNING EXAMPLE**

**Compute the energy stored in the mutually coupled inductors**

$t = 5ms$

**Assume steady state operation**




**We can use frequency domain techniques**

$L_1 = 2.653mH, \; L_2 = 10.61mH$

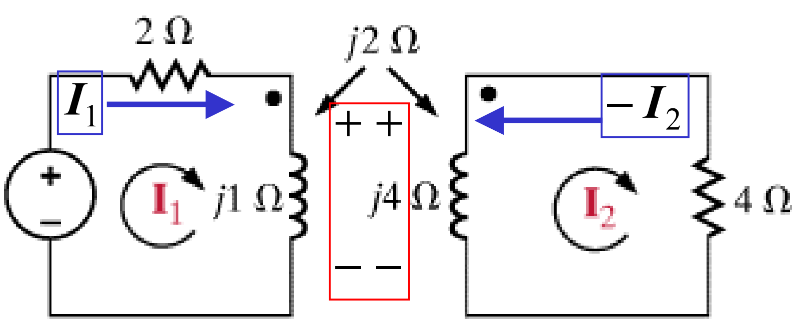
$$w(t) = \frac{1}{2}L_1 i_1^2(t) + M i_1(t) i_2(t) + \frac{1}{2} L_2 i_2^2(t)$$

MUST COMPUTE $M, i_1(t), i_2(t)$

$L_1, L_2, k \Rightarrow M = k\sqrt{L_1 L_2}$ $\quad M = 5.31mH$

$\omega L_1 = 377 \times 2.653 \times 10^{-3} = 1\Omega$

$\omega L_2 = 4\Omega, \; \omega M = 2\Omega$




**Circuit in frequency domain**

**Merge the writing of the loop and coupled inductor equations in one step**

$$2I_1 + (j1I_1 - j2I_2) = 24\angle 0°$$

$$4I_2 - (j2I_1 - j4I_2) = 0$$

SOLVE TO GET

$I_1 = 9.41\angle -11.31°(A), \quad I_2 = 3.33\angle 33.69°(A)$

$$\therefore i_1(t) = 9.41\cos(377t - 11.31°)(A)$$

$$i_2(t) = 3.33\cos(377t + 33.69°)(A)$$

WARNING: The term $377t$ is in radians!

$t = 0.005s \Rightarrow 377t = 1.885(rad) = 108°$

$i_1(0.005) = -1.10(A), \; i_2(0.005) = -2.61(A)$

$$w(0.005) = 0.5 \times 2.653 \times 10^{-3}(-1.10)^2 + 5.31 \times 10^{-3}(-1.10) \times (-2.61) + 0.5 \times 101.61 \times 10^{-3} \times (-2.61)^2 (J)$$

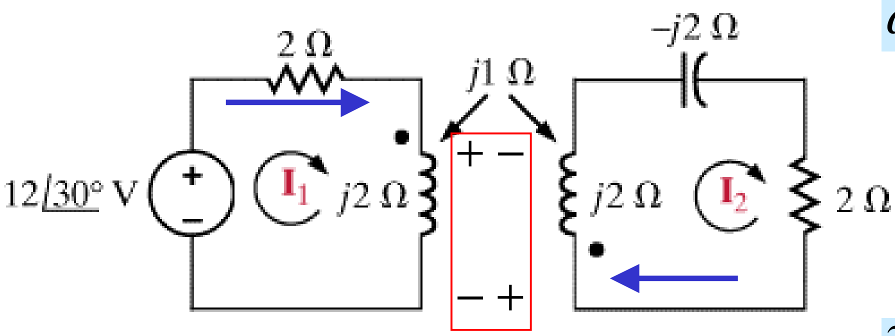
$w(0.005) = 22.5mJ$

GEAUX

## LEARNING EXTENSION

DETERMINE ENERGY STORED AT $t = 10ms$

$f = 60Hz$




$$f = 60Hz \Rightarrow \omega = 378.9(s^{-1})$$

$$\omega L_1 = 2 \Rightarrow L_1 = 0.00528(H) = L_2$$

$$M = 0.00264(H)$$

$$w(t) = \frac{1}{2}L_1 i_1^2(t) - M i_1(t) i_2(t) + \frac{1}{2}L_2 i_2^2(t)$$

$$2I_1 + (j2I_1 + j1I_2) = 12\angle 30°$$

$$(j1I_1 + j2I_2) + (2 - j2)I_2 = 0$$

$$(2 + j2)I_1 + jI_2 = 12\angle 30°$$

$$jI_1 + 2I_2 = 0 \qquad \Rightarrow I_2 = -0.5jI_1$$

$$(2 + j2 + 0.5)I_1 = 12\angle 30°$$

$$I_1 = \frac{12\angle 30°}{2.5 + j2} = \frac{12\angle 30°}{3.20\angle 38.66°} = 3.75\angle -8.66°(A)$$

$$I_2 = -0.5jI_1 = 0.5\angle -90° \times 3.75\angle -8.66°$$
$$= 1.875\angle -98.66°$$

**Go back to time domain**

$$i_1(t) = 3.75\cos(378.9t - 8.66°)(A)$$

$$i_2(t) = 1.875\cos(378.9t - 98.66°)(A)$$

$$378.9(rad/\sec) \times 0.010(\sec) = 3.789(rad) = 217.1°$$

$$i_1(0.010) = -3.3(A)$$

$$i_2(0.010) = -0.91(A)$$

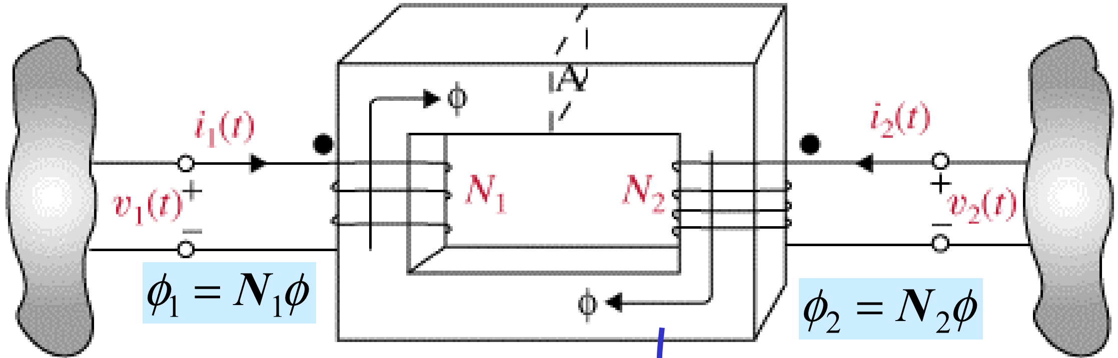
$$w(0.010) = 0.5 * 0.00528 * (-3.3)^2 - 0.00264 * (-3.3)(-0.91) + 0.5 * 0.00528 * (0.91)^2 (J)$$

$$w(0.010) = 0.00264 * (3.3^2 - (3.3)(0.91) + 0.91^2)$$

$$w(0.010) = 0.030J = 30mJ$$

GEAUX

# THE IDEAL TRANSFORMER




Insures that 'no magnetic flux goes astray'

$$\left.\begin{aligned} v_1(t) &= N_1\frac{d\phi}{dt}(t) \\ v_2(t) &= N_2\frac{d\phi}{dt}(t) \end{aligned}\right\} \Rightarrow \frac{v_1}{v_2} = \frac{N_1}{N_2}$$

First ideal transformer equation

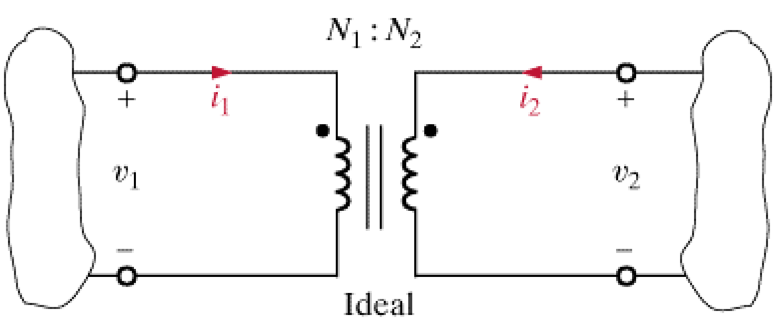
$$v_1(t)i_1(t) + v_2(t)i_2(t) = 0$$

Ideal transformer is lossless

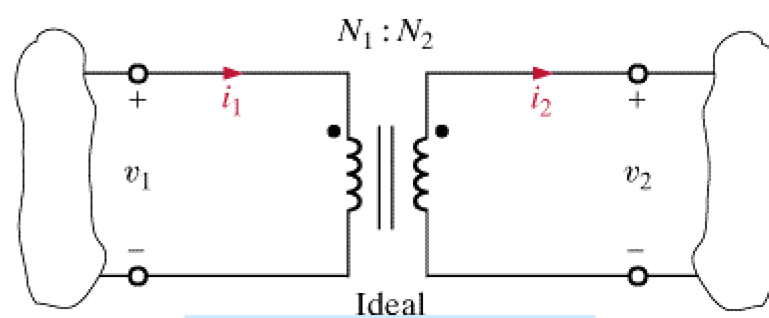
$$\frac{i_1}{i_2} = -\frac{N_2}{N_1}$$

Second ideal transformer equations

Since the equations are algebraic, they are unchanged for Phasors. Just be careful with signs



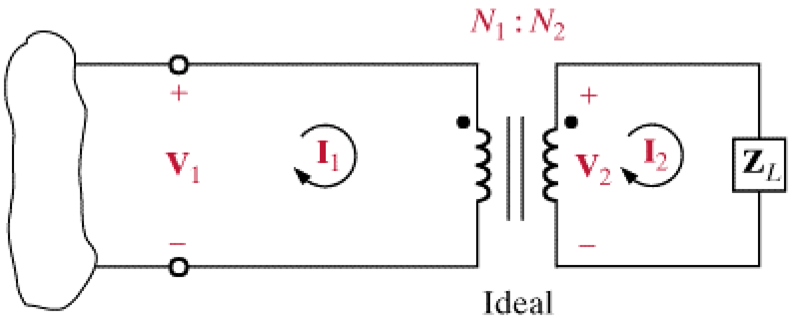

Circuit Representations




$$\frac{v_1}{v_2} = \frac{N_1}{N_2}; \qquad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

GEAUX

# REFLECTING IMPEDANCES




$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \text{ (both + signs at dots)}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} \text{ (Current } I_2 \text{ leaving transformer)}$$

$$V_2 = Z_L I_2 \text{ (Ohm's Law)}$$

$$V_1 \frac{N_2}{N_1} = Z_L I_1 \frac{N_1}{N_2}$$

$$V_1 = \left(\frac{N_1}{N_2}\right)^2 Z_L I_1$$

$$\frac{V_1}{I_1} = Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

$Z_1$ = impedance, $Z_L$, reflected into the primary side

## For future reference

$$S_1 = V_1 I_1^* = \left(V_2 \frac{N_1}{N_2}\right)\left(I_2 \frac{N_2}{N_1}\right)^* = V_2 I_2^* = S_2$$

$$n = \frac{N_2}{N_1} = \text{turns ratio}$$

## Phasor equations for ideal transformer

$$V_1 = \frac{V_2}{n}$$

$$I_1 = n I_2$$

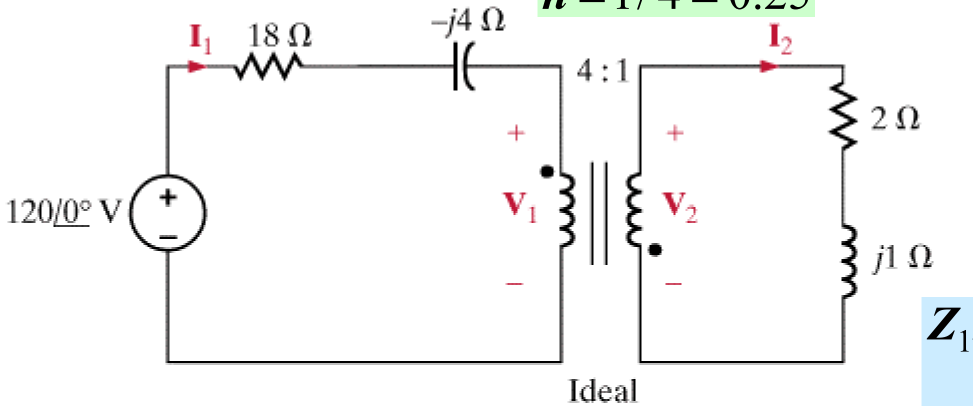
$$Z_1 = \frac{Z_L}{n^2}$$

$$S_1 = S_2$$

**LEARNING EXAMPLE**

**Determine all indicated voltages and currents**

$\boldsymbol{n} = 1/4 = 0.25$




$$I_1 = \frac{120\angle 0°}{50 + j12} = \frac{120\angle 0°}{51.42\angle 13.5°} = 2.33\angle -13.5°$$

$$V_1 = Z_1 I_1 = \frac{Z_1}{Z_1 + Z_2} 120\angle 0°$$

$$Z_1 I_1 = (32 + j16) \times 2.33\angle -13.5°$$

**SAME COMPLEXITY**

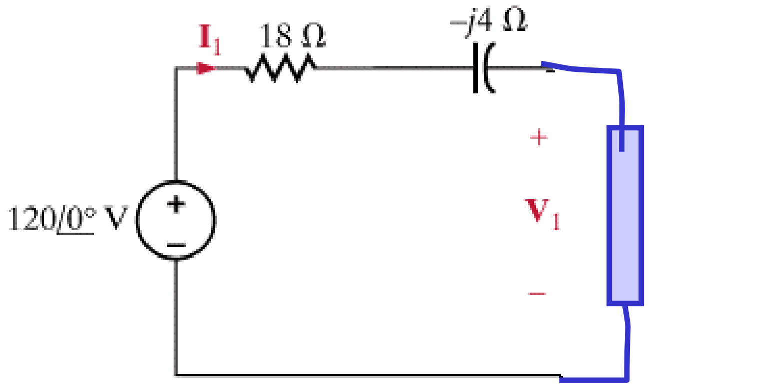
$$\frac{Z_1}{Z_1 + Z_2} 120\angle 0° = \frac{32 + j16}{51.42\angle 13.5°} \times 120$$

**Strategy: reflect impedance into the primary side and make transformer "transparent to user."**

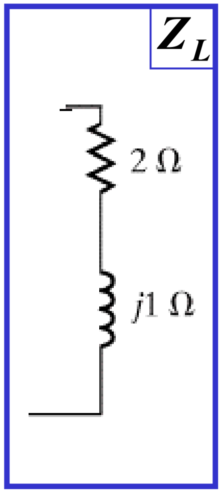
$$Z_1 = \frac{Z_L}{n^2}$$

$$V_1 = 35.78\angle 26.57° \times 2.33\angle -13.5° = 83.36\angle 13.07°$$

**CAREFUL WITH POLARITIES AND CURRENT DIRECTIONS!**




$Z_1 = 32 + j16$



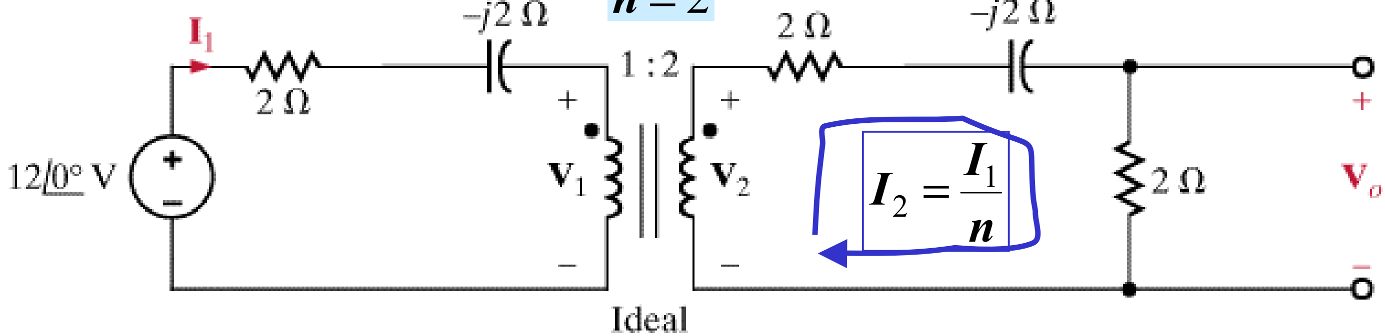

$$I_2 = \frac{I_1}{n} = -4 I_1 \text{ (current into dot)}$$

$$V_2 = -n V_1 = -0.25 V_1 \text{ (+ is opposite to dot)}$$

GEAUX

**LEARNING EXTENSION** Find the current $I_1$

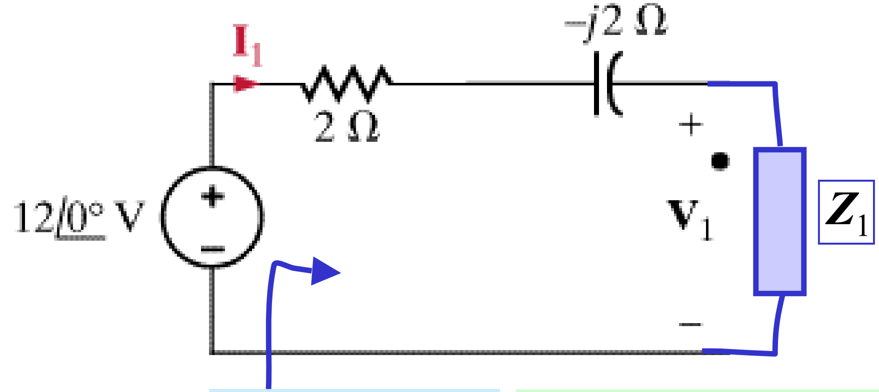
$n = 2$




**Strategy: reflect impedance into the primary side and make transformer "transparent to user."**

$$Z_1 = \frac{Z_L}{n^2}$$

$$Z_1 = \frac{4 - j2}{4} = 1 - j0.5\Omega$$




$$Z_i = 3 - j2.5\Omega$$

**Voltage in Volts**
**Impedance in Ohms**
**...Current in Amps**

$$I_1 = \frac{12\angle 0°}{3 - j2.5} = \frac{12\angle 0°}{3.91\angle -39.81°} = 3.07\angle 39.81°(A)$$

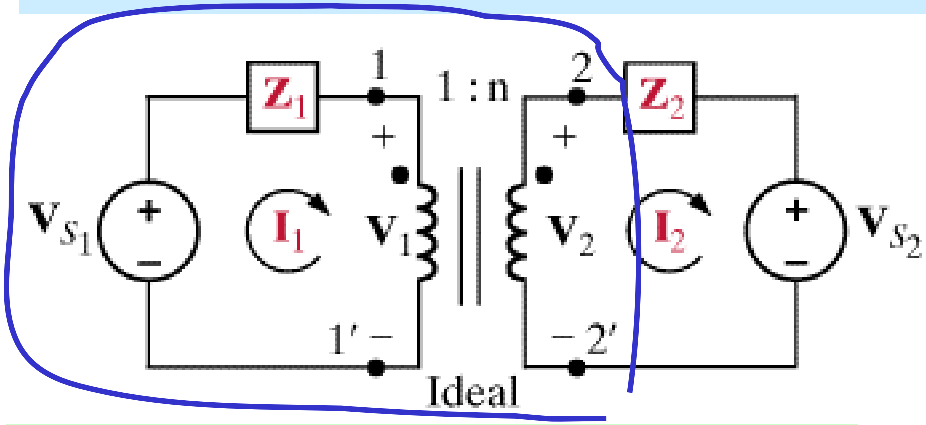
**LEARNING EXTENSION** Find $V_o$

**Strategy: Find current in secondary and then use Ohm's Law**

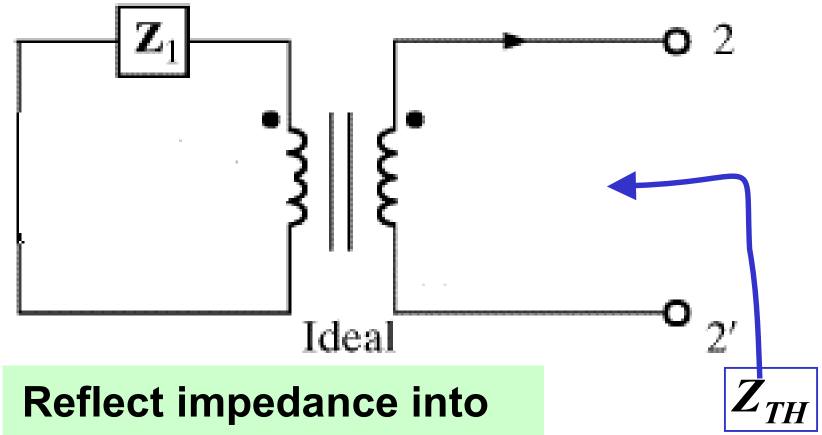
$$I_2 = \frac{I_1}{2} \Rightarrow V_0 = 2\Omega \times \frac{I_1}{2} = 3.07\angle 39.81°(V)$$

GEAUX

# USING THEVENIN'S THEOREM TO SIMPLIFY CIRCUITS WITH IDEAL TRANSFORMERS

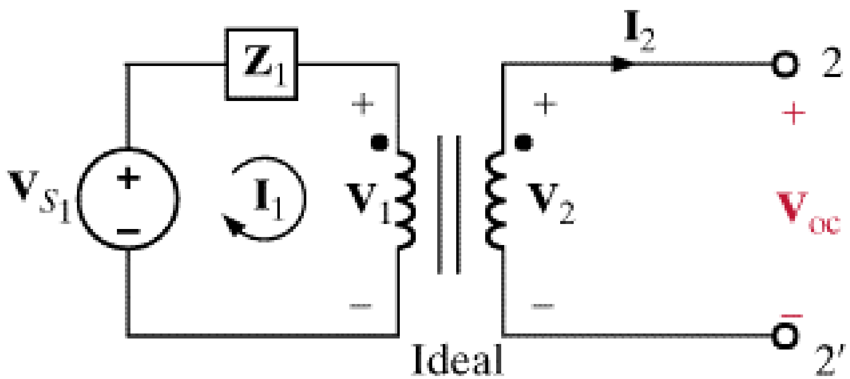



Replace this circuit with its Thevenin equivalent




Reflect impedance into secondary

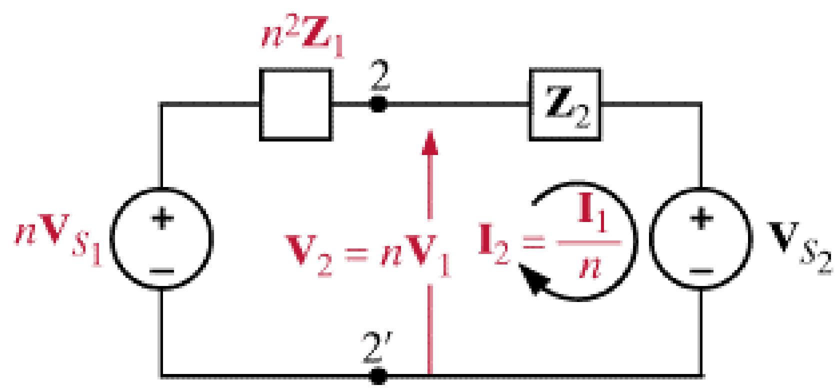
$$Z_{TH} = n^2 Z_1$$




$$\left.\begin{array}{l} I_2 = 0 \\ I_1 = nI_2 \end{array}\right\} \Rightarrow I_1 = 0 \Rightarrow V_1 = V_{S1}$$

$$\left.\begin{array}{l} V_1 = V_{S1} \\ V_2 = nV_1 \end{array}\right\} \Rightarrow V_{OC} = nV_{S1}$$

To determine the Thevenin impedance...

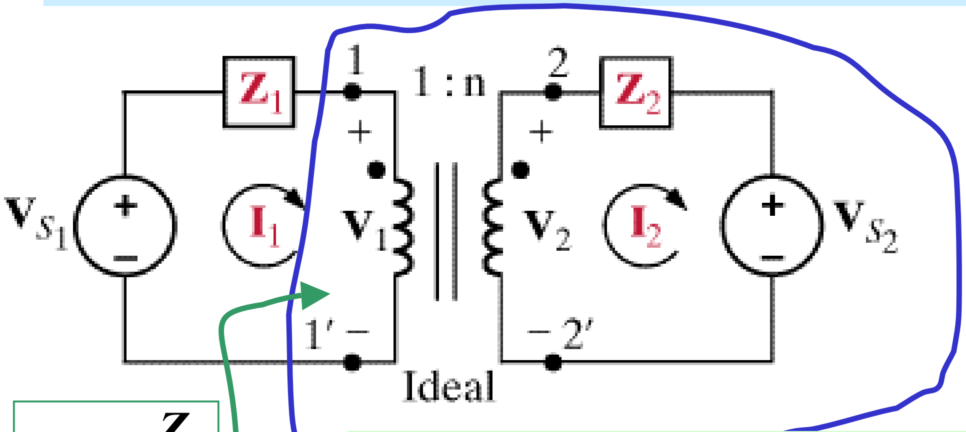



Equivalent circuit with transformer "made transparent."

One can also determine the Thevenin equivalent at 1 - 1'

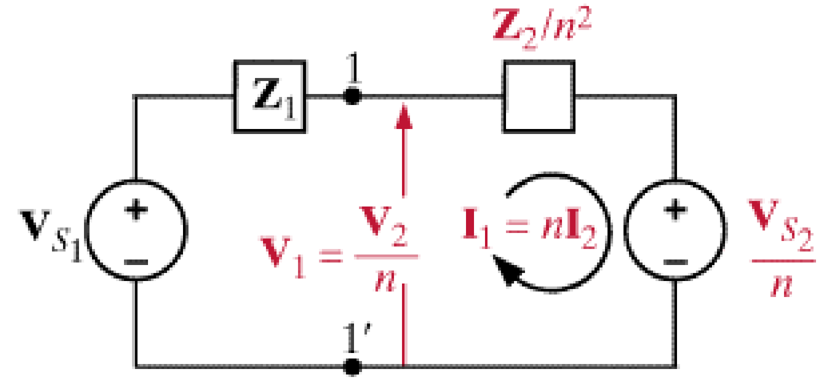
GEAUX

# USING THEVENIN'S THEOREM: REFLECTING INTO THE PRIMARY




$$Z_{TH} = \frac{Z_2}{n^2}$$

Find the Thevenin equivalent of this part

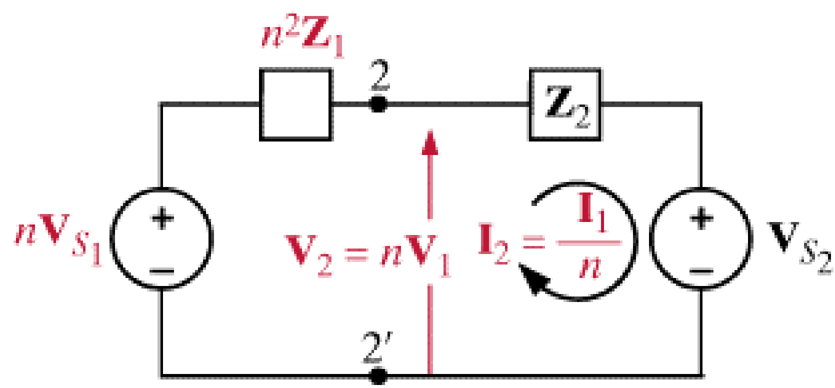



Equivalent circuit reflecting into primary

In open circuit $I_1 = 0$ and $I_2 = 0$

$$V_{OC} = \frac{V_{S2}}{n}$$

Thevenin impedance will be the secondary impedance reflected into the primary circuit

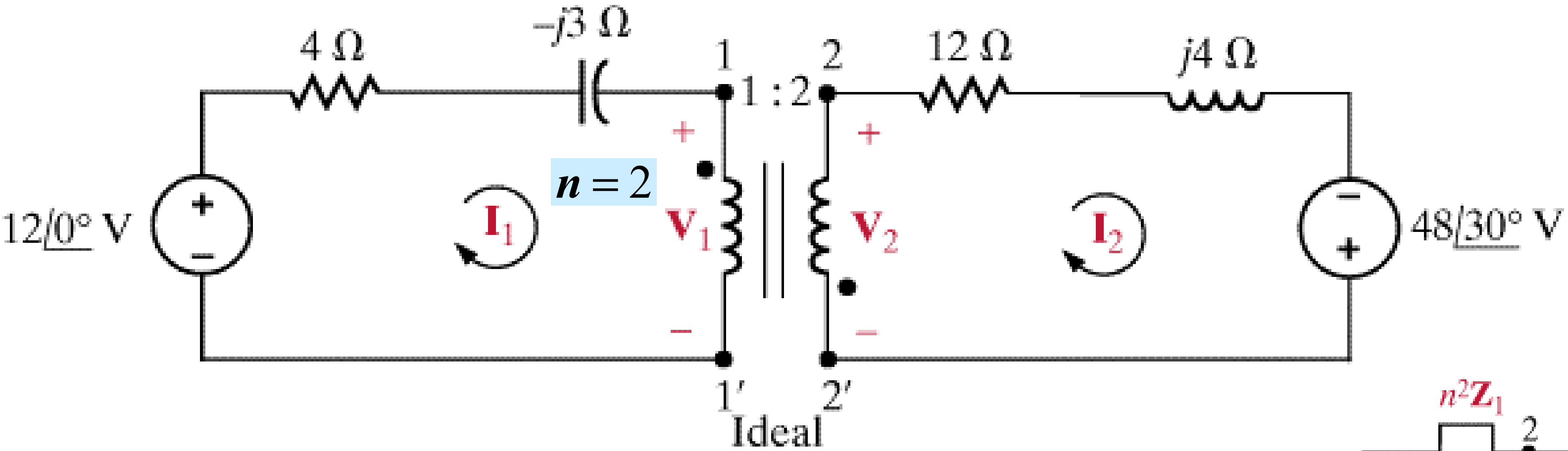


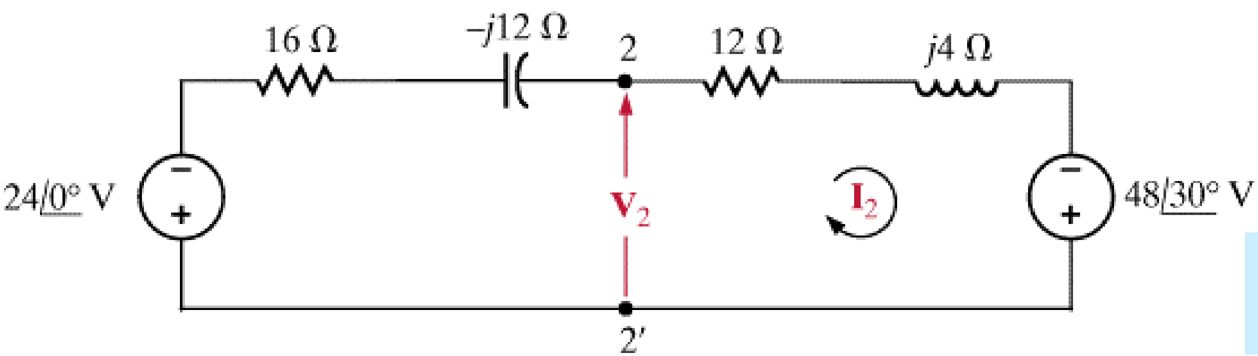

Equivalent circuit reflecting into secondary

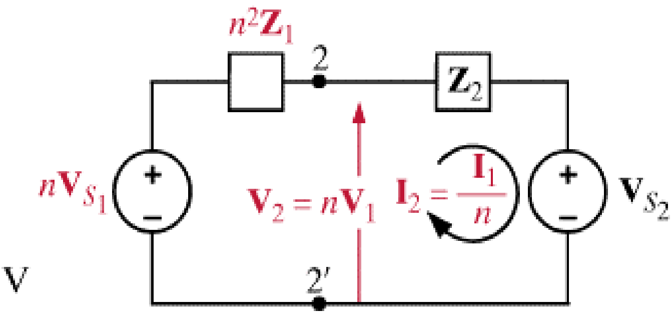
GEAUX

**LEARNING EXAMPLE**

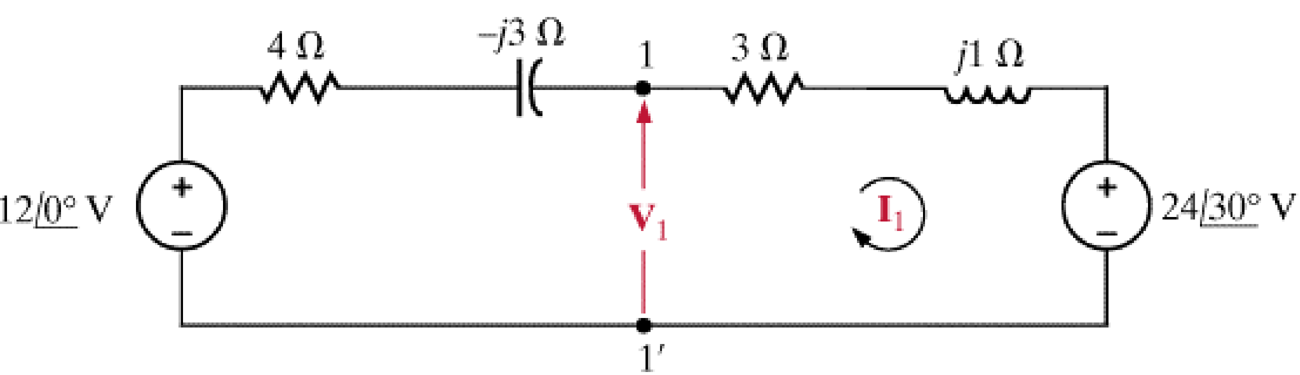
**Draw the two equivalent circuits**

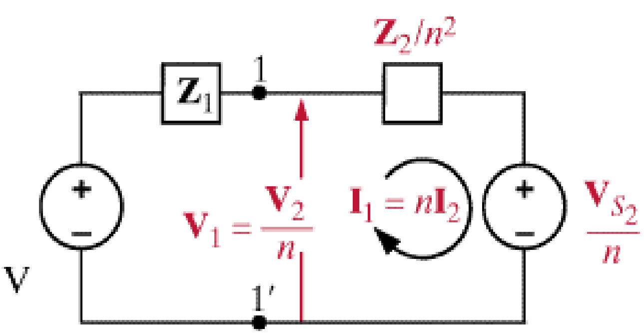



**Equivalent circuit reflecting into secondary**



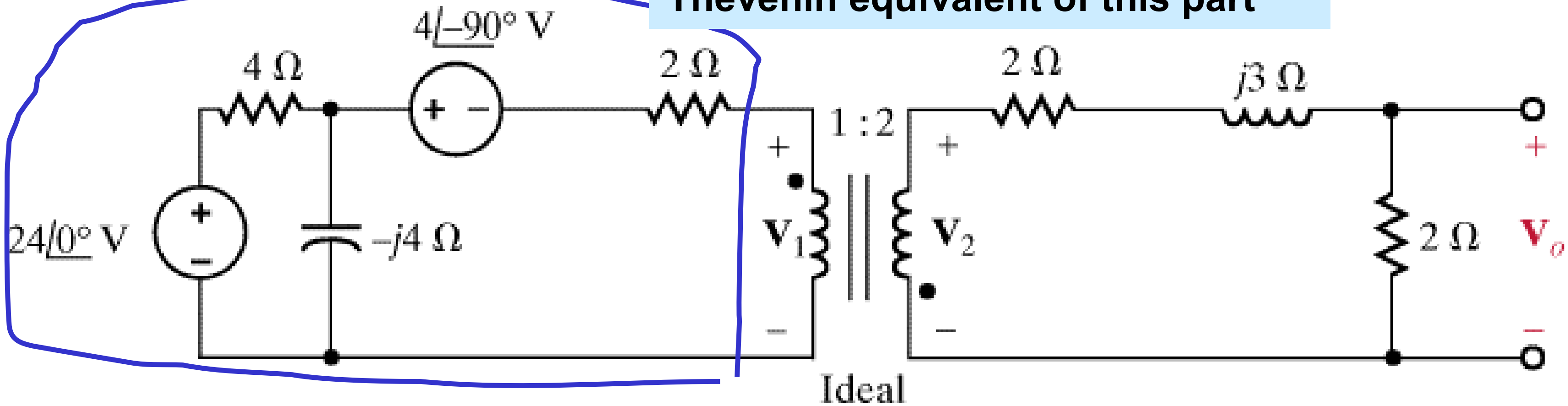

**Equivalent circuit reflecting into primary**

GEAUX

**LEARNING EXAMPLE**

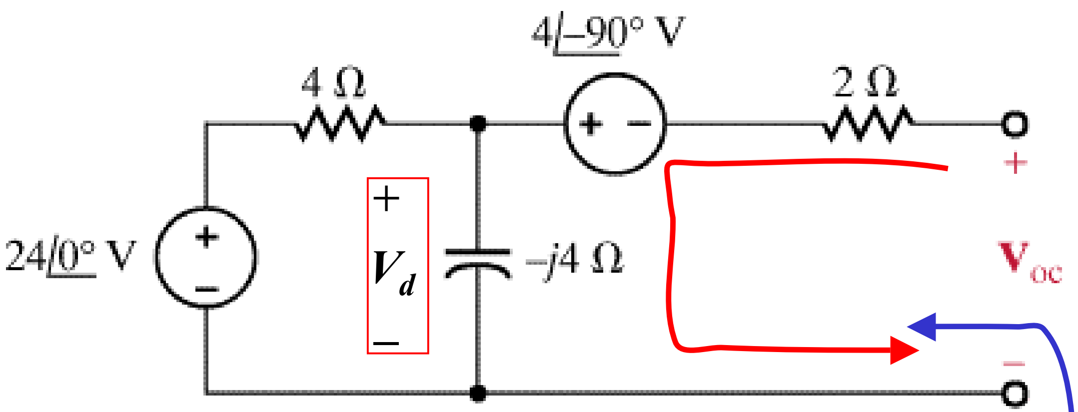
Find $V_o$

Thevenin equivalent of this part




To compute $V_o$ is better to reflect into secondary

**But before doing that it is better to simplify the primary using Thevenin's Theorem**




$$V_{OC} = V_d - 4\angle -90^\circ$$

$$V_d = \frac{-j4}{4-j4} 24\angle 0^\circ = \frac{24\angle -90^\circ}{1-j}$$

$$V_{OC} = 14.47\angle -33.69^\circ (V)$$

$$Z_{TH} = 2 + \frac{-j16}{4-j4} = \frac{8-j8-j16}{4-j4}$$

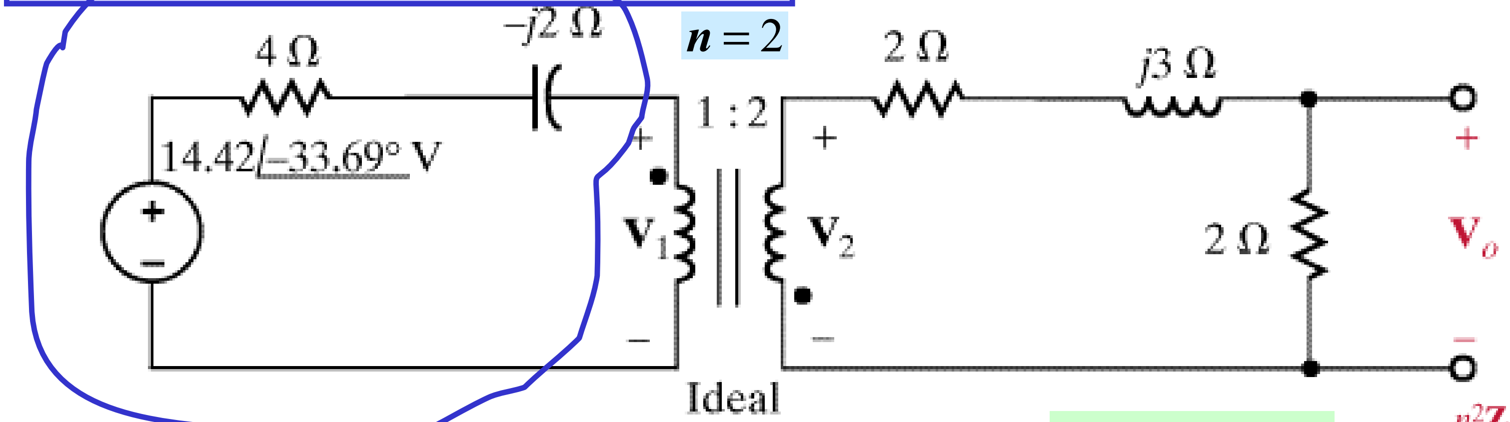
$$Z_{TH} = \frac{2-j6}{1-j} \times \frac{1+j}{1+j} = \frac{8-j4}{2}$$

$$Z_{TH} = 2 + (4 \| -j4) \qquad Z_{TH} = 4 - j2(\Omega)$$

**This equivalent circuit is now transferred to the secondary**

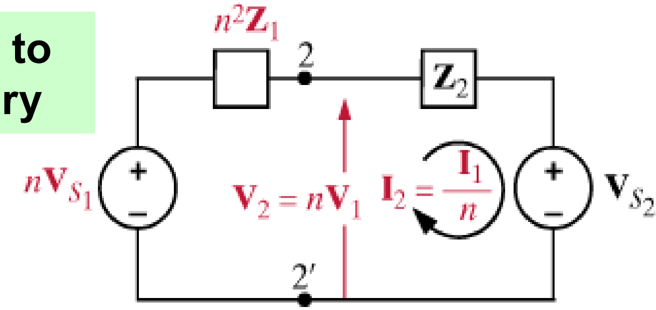
GEAUX

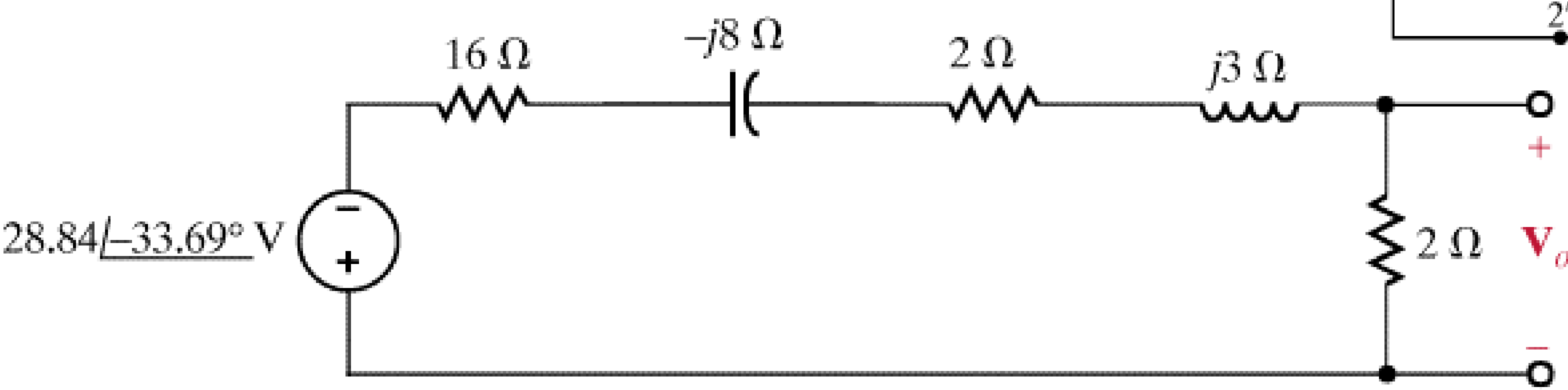
## LEARNING EXAMPLE (continued…)




Thevenin equivalent of primary side

Transfer to secondary



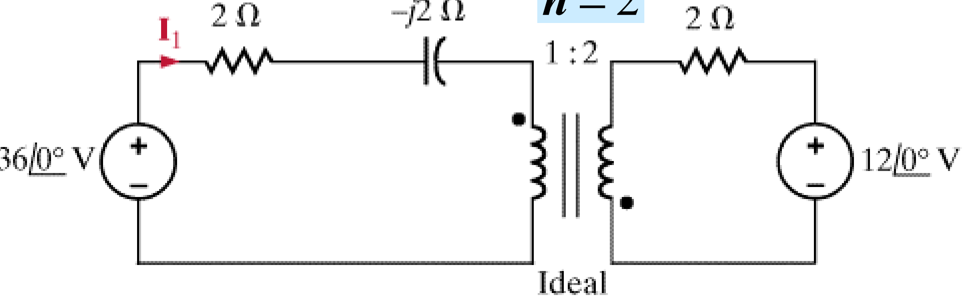

Circuit with primary transferred to secondary

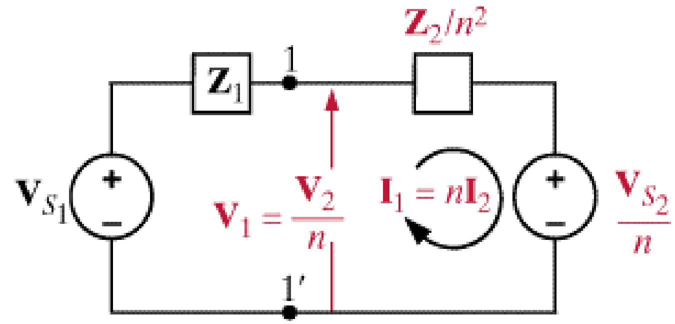
$$V_o = \frac{2}{20 - j5} 28.84\angle -33.69° = \frac{2 \times 28.84\angle -33.69°}{20.62\angle -14.04°}$$

## LEARNING EXTENSION

Find $I_1$

$n = 2$

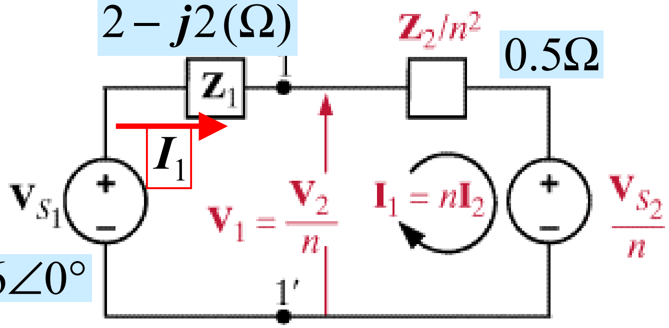



Equivalent circuit reflecting into primary

$2 - j2(\Omega)$

$0.5\Omega$




$36\angle 0°$

$$= -\frac{12\angle 0°}{2}$$

**Notice the position of the dot marks**

$$I_1 = \frac{36\angle 0° + 6\angle 0°}{2 - j1.5}$$

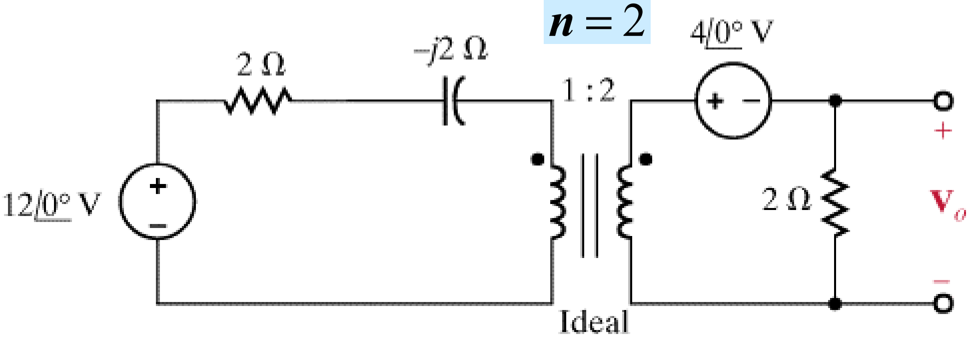
$$I_1 = \frac{36\angle 0°}{2.5\angle -36.86°}$$

GEAUX

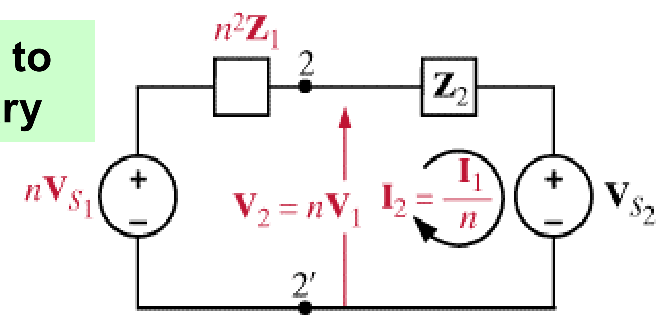
**LEARNING EXTENSION**

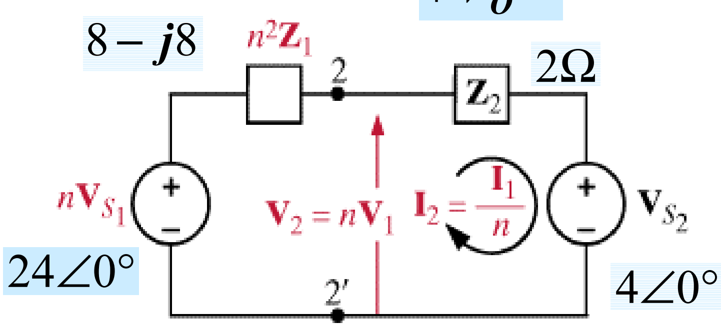
Find $V_o$

$n = 2$




**Transfer to secondary**



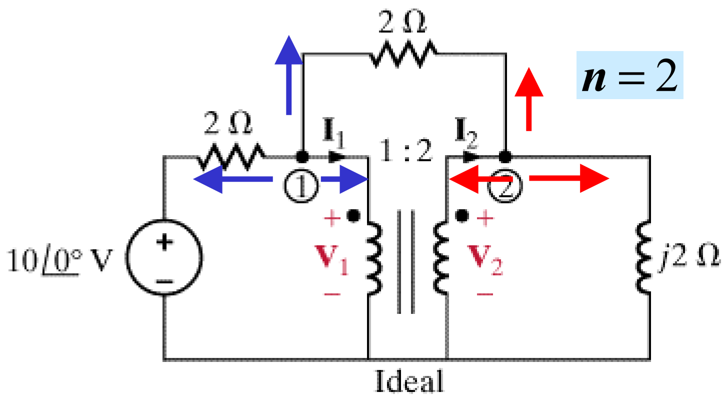




$$V_o = \frac{2}{(8 - j8) + 2} 20\angle 0^\circ$$

$$V_o = \frac{40\angle 0^\circ}{12.81\angle -38.66^\circ}$$

**LEARNING EXAMPLE**

Find $I_1, I_2, V_1, V_2$




Nothing can be transferred. Use transformer equations and circuit analysis tools

Phasor equations for ideal transformer

$$V_1 = \frac{V_2}{n}$$

$$I_1 = nI_2$$

$$\text{@ Node 1: } \frac{V_1 - 10\angle 0^\circ}{2} + \frac{V_1 - V_2}{2} + I_1 = 0$$

$$\text{@Node 2: } \frac{V_2 - V_1}{2} + \frac{V_2}{j2} - I_2 = 0$$

4 equations in 4 unknowns!

$$2V_1 - V_2 + 2I_1 = 10\angle 0^\circ \quad \Rightarrow I_1 = 5\angle 0^\circ$$

$$-V_1 + (1-j)V_2 - 2I_2 = 0$$

$$V_2 = 2V_1$$

$$I_1 = 2I_2 \quad \rightarrow \quad I_2 = 2.5\angle 0^\circ$$

$$-V_1 + (1-j)(2V_1) = 5\angle 0^\circ$$

$$V_1 = \frac{5\angle 0^\circ}{1-j2} = \frac{5\angle 0^\circ}{2.24\angle -63.43^\circ} = \sqrt{5}\angle 63.43^\circ$$

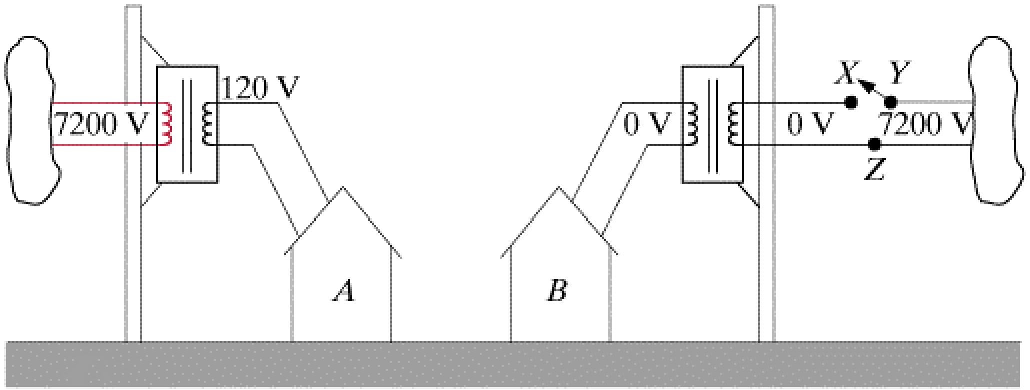
$$V_2 = 2\sqrt{5}\angle 63.43^\circ$$

GEAUX

# SAFETY CONSIDERATIONS: AN EXAMPLE

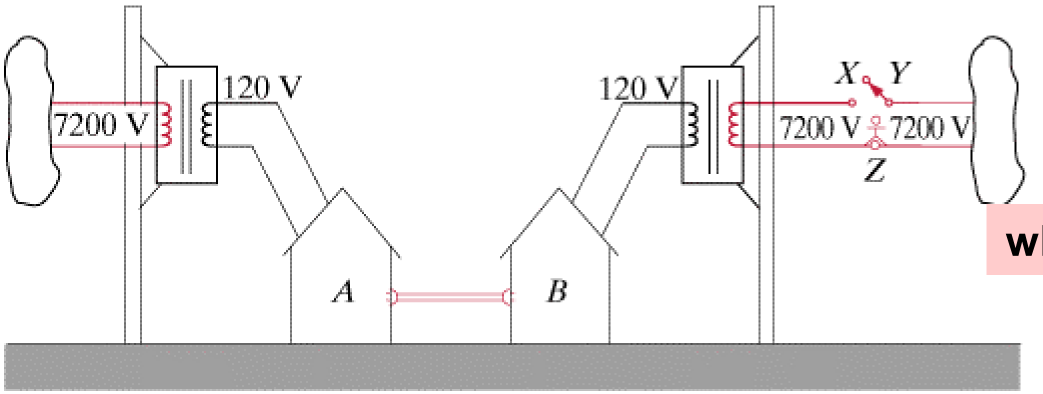
Houses fed from different distribution transformers

Braker X-Y opens, house B is powered down




(a)

When technician resets the braker he finds 7200V between points X-Z




when he did not expect to find any

(b)

Good neighbor runs an extension and powers house B

GEAUX

**LEARNING BY APPLICATION**

## Why high voltage transmission lines?

**CASE STUDY: Transmit 24MW over 100miles with 95% efficiency**

**A. AT 240V**
**B. AT 240kV**

Given : Conductor resistance, $R = \frac{\rho l}{A}$

$\rho =$ resistivity of material (e.g., copper $= 8\times10^{-8}\,\Omega$/m)

$l =$ length of conductor $= 160.9\,Km = 1.609\times10^{5}\,m$

$A =$ cross section $= \pi r^2$

Required : Maximum losses, $P_{loss} = 24MW \times 0.05 = 1.2MW$

Known : Line losses : $P_{loss} = RI^2$

A. At 240V one needs a current $I_l = \frac{24\times10^6 W}{240V} = 10^5 A$

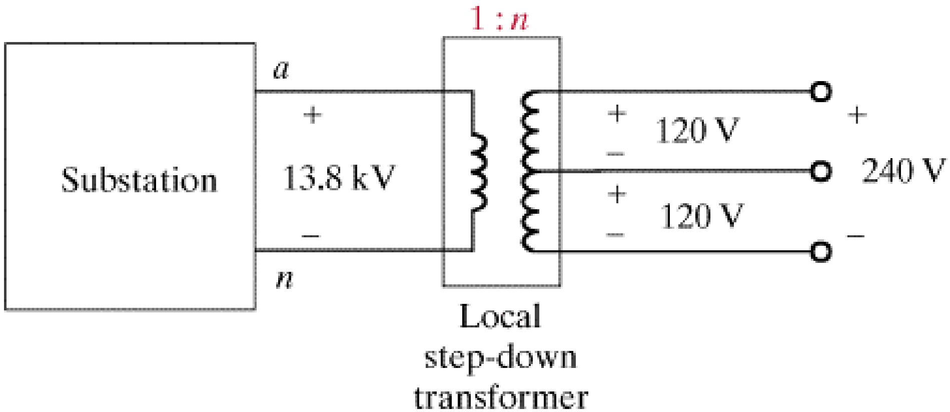
$$1.2\times10^6 W = 2\times\frac{8\times10^{-8}\times1.609\times10^5}{\pi r^2}\times10^{10} \Rightarrow r = 8.624m$$

B. At 240kV one needs a current $I_l = \frac{24\times10^6 W}{240\times10^3 V} = 10^2 A$

$$1.2\times10^6 W = 2\times\frac{8\times10^{-8}\times1.609\times10^5}{\pi r^2}\times10^{4} \Rightarrow r = 0.8624cm$$

GEAUX

**LEARNING EXAMPLE** Rating a distribution transformer




Households per transformer $= 10$

Maximum current per household $= 200A$

**Determining ratio**

$$\frac{V_2}{V_1} = n \Rightarrow \quad n = \frac{240}{13800} = \frac{1}{57.5}$$

**Determining power rating**

Max secondary current $= 2000A$

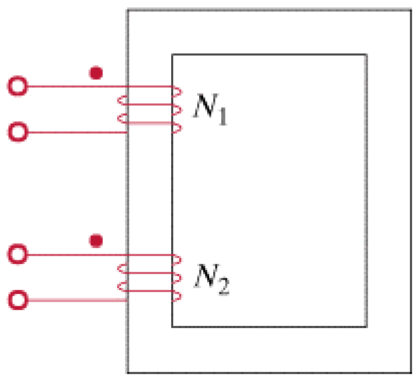
$$n = \frac{I_1}{I_2} \Rightarrow \quad I_1 = \frac{2000}{57.6} = 34.78A$$

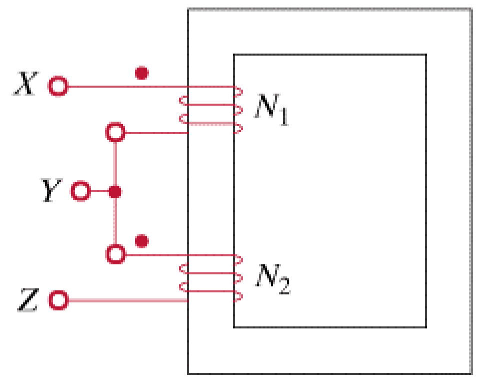
$$\therefore P = 13{,}800 \times 34.78 = 480kVA$$

Also: $P = 240(V) \times 2000(A)$

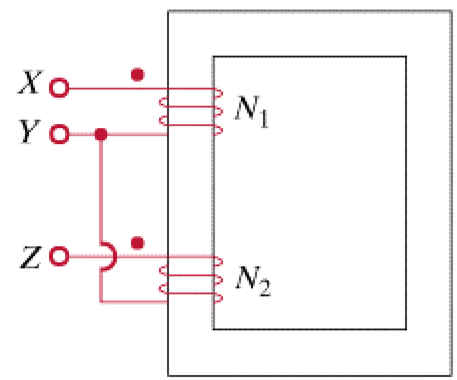
**LEARNING BY DESIGN**

**Use a 120V - 12V transformer to build a 108V autotransformer**




Additive connection

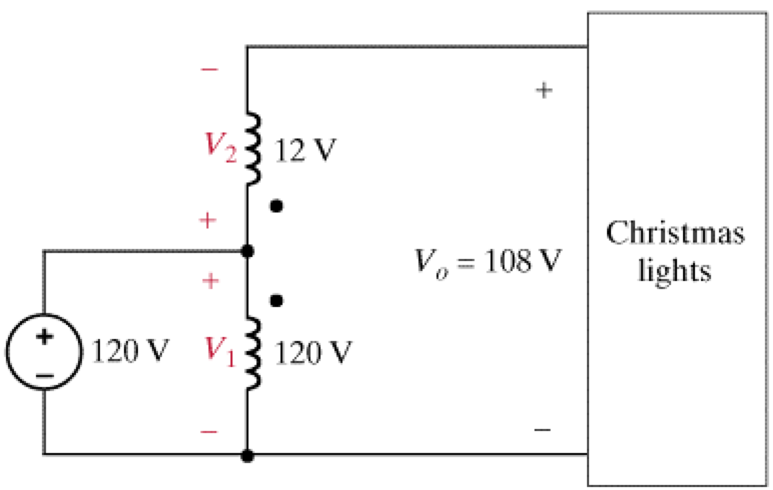



Subtractive connection

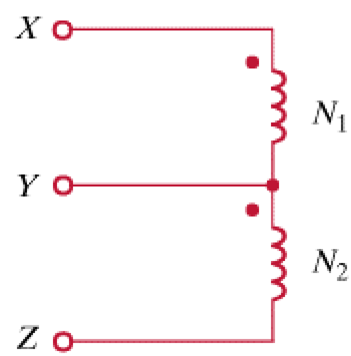
**Conventional transformer**

**Auto transformer connections**

**Use the subtractive connection on the 120V - 12V transformer**

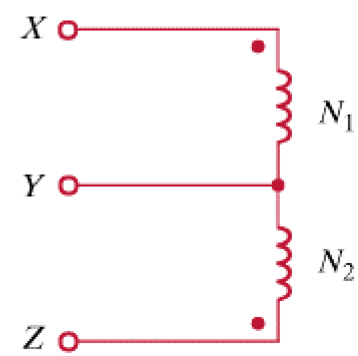



$$V_{xz} = V_{xy} + V_{yz}$$




Additive connection

$$V_{xz} = V_{xy} - V_{xz}$$




Subtractive connection

**Circuit representations**

Transformers