THE LAPLACE TRANSFORM

LEARNING GOALS

Definition
The transform maps a function of time into a function of a complex variable

Two important singularity functions
The unit step and the unit impulse

Transform pairs
Basic table with commonly used transforms

Properties of the transform
Theorem describing properties. Many of them are useful as computational tools

Performing the inverse transformation
By restricting attention to rational functions one can simplify the inversion process

Convolution integral
Basic results in system analysis

Initial and Final value theorems
Useful result relating time and s-domain behavior

ONE-SIDED LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] = \mathbf{F}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$s = \sigma + j\omega$

$\forall s \ni$ the integral is well defined ($s \in RoC$)

It will be necessary to consider $t = 0^-$ as the lower limit

To insure uniqueness of the transform one assumes $f(t) = 0$ for $t < 0$

A SUFFICIENT CONDITION FOR EXISTENCE OF LAPLACE TRANSFORM

$$\int_0^\infty e^{-\sigma t}|f(t)|\,dt < \infty$$

Transform exists for $\mathrm{Re}\{s\} + \sigma > 0$

THE INVERSE TRANSFORM

$$\mathcal{L}^{-1}[\mathbf{F}(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \mathbf{F}(s)e^{st}\,ds$$

Contour integral in the complex plane

Evaluating the integrals can be quite time-consuming. For this reason we develop better procedures that apply only to certain useful classes of function

GEAUX

TWO SINGULARITY FUNCTIONS

This function has derivative that is zero everywhere except at the origin.
We will "define" a derivative for it

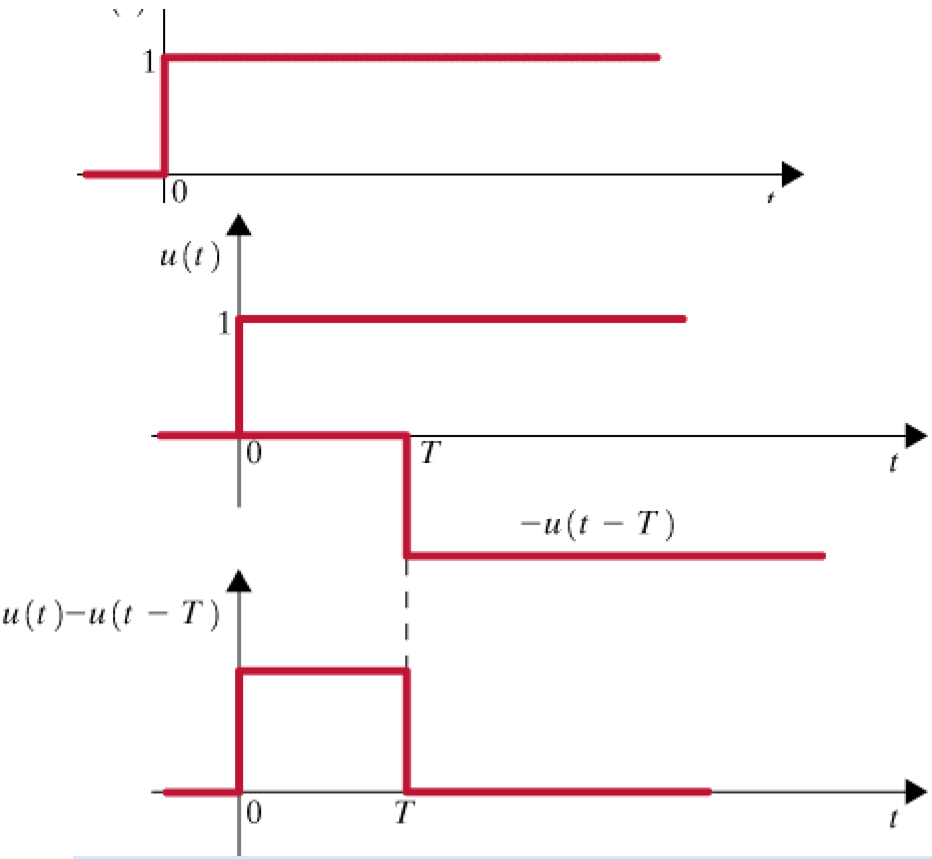
Unit step

(Important "test" function in system analysis)

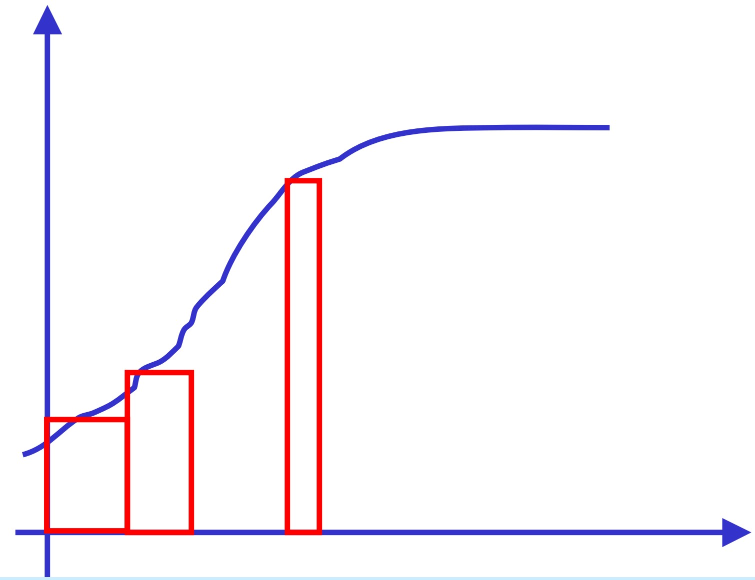
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

For positive time functions

$$f(t) = f(t)u(t)$$



Using the unit step to build functions



Using square pulses to approximate an arbitrary function

The narrower the pulse the better the approximation

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Computing the transform of the unit step

$$U(s)=\int_0^{\infty}1\times e^{-sx}dx=\lim_{T\to\infty}\int_0^T e^{-sx}dx$$

$$U(s)=\lim_{T\to\infty}\left(-\frac{1}{s}e^{-sx}\right)_0^T$$

$$U(s)=\frac{1}{s}-\lim_{T\to\infty}\frac{e^{-sT}}{s}\quad(s=\sigma+j\omega)$$

$$U(s)=\frac{1}{s}-\lim_{T\to\infty}\frac{e^{-\sigma T}e^{j\omega T}}{\sigma+j\omega}$$

$$U(s)=\frac{1}{s};\ \forall s\ni \mathrm{Re}\{s\}>0$$

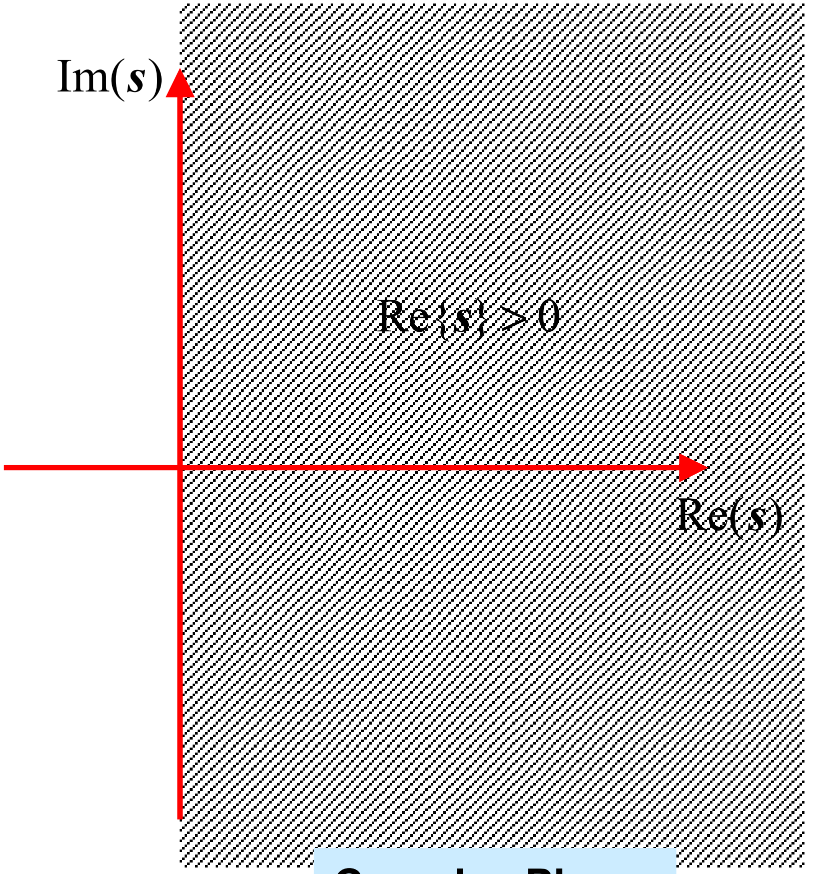
RoC

To simplify question of RoC:
A special class of functions

$$\int_0^{\infty}e^{-\sigma t}|f(t)|\,dt<\infty \Rightarrow \mathrm{RoC}\supset\{s:\mathrm{Re}\{s\}>-\sigma\}$$

In this case the RoC is at least half a plane. And any linear combination of such signals will also have a RoC that is a half plane

An example of Region of Convergence (RoC)



Complex Plane

THE IMPULSE FUNCTION

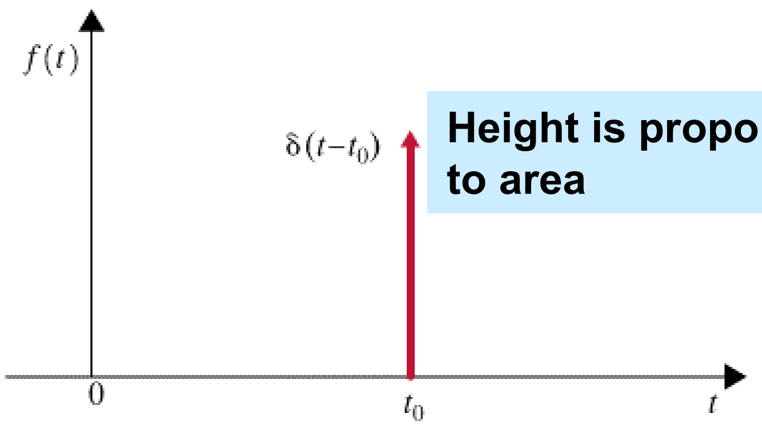
(Good model for impact, lightning, and other well known phenomena)

$$\delta(t - t_0) = 0 \qquad t \neq t_0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\, dt = 1 \qquad \varepsilon > 0$$

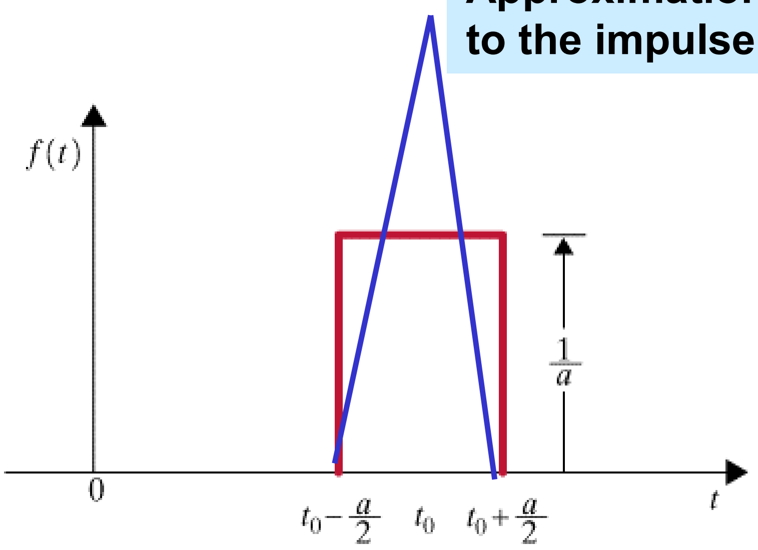
These two conditions are not feasible for "normal" functions

Approximations to the impulse



Height is proportional to area

Representation of the impulse



(a)

$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)\, dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

Sifting or sampling property of the impulse

For $t_0 = t_2$ or $t_0 = t_1$ the integral is NOT defined

$$\mathbf{F}(s) = \int_0^\infty \delta(t - t_0) e^{-st}\, dt = e^{-st_0}$$

In order to have a valid transform for $\delta(t)$ the lower limit is assumed $t = 0^-$

Laplace transform

GEAUX

LEARNING BY DOING

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)\,dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1,\, t_0 > t_2 \end{cases}$$

Evaluate the integral

$$0 < t_0 = \pi < 2\pi$$

$$\int_0^{2\pi} \cos t\,\delta(t-\pi)\,dt = \cos\pi$$

Evaluate the integral

$$t_0 = 10\pi > 2\pi$$

$$\int_0^{2\pi} e^{-t}\cos t\,\delta(t-10\pi)\,dt = 0$$

LEARNING EXAMPLE find the Laplace transform of $f(t) = t$.

$$\mathbf{F}(s) = \int_0^{\infty} te^{-st}\,dt = -t\frac{1}{s}e^{-st}\Big|_0^{\infty} - \int_0^{\infty}\left(-\frac{1}{s}e^{-st}dt\right) = -\frac{1}{s^2}e^{-st}\Big|_0^{\infty} = \frac{1}{s^2}$$

Integration by parts with $u = t, dv = e^{-st}dt$

$$du = dt, v = -\frac{1}{s}e^{-st}$$

We will develop properties that will permit the determination of a large number of transforms from a small table of transform pairs

GEAUX

Useful Properties of the Laplace Transform

$f(t)$	$F(s)$	
$Af(t)$	$A\mathbf{F}(s)$	Linearity
$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$	
$f(at)$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right), a > 0$	
$f(t - t_0)u(t - t_0), t_0 \geq 0$	$e^{-t_0 s}\mathbf{F}(s)$	Time shifting
$f(t)u(t - t_0)$	$e^{-t_0 s}\mathcal{L}[f(t + t_0)]$	Time truncation
$e^{-at}f(t)$	$\mathbf{F}(s + a)$	Multiplication by exponential
$\frac{d^n f(t)}{dt^n}$	$s^n\mathbf{F}(s) - s^{n-1}f(0) - s^{n-2}f^1(0) \cdots s^0 f^{n-1}(0)$	
$tf(t)$	$-\frac{d\mathbf{F}(s)}{ds}$	Multiplication by time
$\frac{f(t)}{t}$	$\int_s^\infty \mathbf{F}(\lambda)d\lambda$	
$\int_0^t f(\lambda)d\lambda$	$\frac{1}{s}\mathbf{F}(s)$	
$\int_0^t f_1(\lambda)f_2(t - \lambda)d\lambda$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$	

Some properties will be proved and used as efficient tools in the computation of Laplace transforms

GEAUX

LEARNING EXAMPLE

Find the transform for $f(t) = e^{-at}$

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$F(s) = -\frac{1}{s+a}\Big|_0^\infty = \frac{1}{s+a}$$

Basic Table of Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

We develop properties that expand the table and allow computation of transforms without using the definition

LINEARITY PROPERTY

$\mathcal{L}[Af(t)] = AF(s)$ **Homogeneity**

$\mathcal{L}[f_1(t) \pm f_2(t)] = \mathbf{F}_1(s) \pm \mathbf{F}_2(s)$ **Additivity**

Follow immediately from the linearity properties of the integral

APPLICATION

$$\mathbf{F}(s) = \int_0^\infty \cos\omega t \, e^{-st} \, dt$$

$$= \int_0^\infty \frac{e^{-j\omega t} + e^{-j\omega t}}{2} e^{-st} \, dt$$

$$= \frac{1}{2}\mathrm{L}[e^{-j\omega t}] + \frac{1}{2}\mathrm{L}[e^{j\omega t}]$$

$a = j\omega$ $\quad$ $a = -j\omega$

$$F(s) = \frac{1}{2}\frac{1}{s+j\omega} + \frac{1}{2}\frac{1}{s-j\omega}$$

$$= \frac{1}{2}\frac{(s-j\omega)+(s+j\omega)}{(s+j\omega)(s-j\omega)}$$

$$= \frac{s}{s^2+\omega^2}$$

GEAUX

With a similar use of linearity one shows

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

Additional entries for the table

$\sin bt$	$\dfrac{b}{s^2 + b^2}$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$

LEARNING EXAMPLE

Find the Laplace transform for

$$x(t) = \cos(3t + \pi/3)$$

$$x(t) = \cos\pi/3 \cos 3t - \sin\pi/3 \sin 3t$$

$$X(s) = \cos\pi/3 \frac{s}{s^2 + 9} - \sin\pi/3 \frac{3}{s^2 + 9}$$

LEARNING EXAMPLE

$$x_1(t) = 3 - \delta(t) + 4e^{-4t}$$

Application of Linearity

$$X(s) = 3\frac{1}{s} - 1 + 4\frac{1}{s+4}$$

Notice that the unit step is not shown explicitly. Hence

3 and $3u(t)$

are equivalent

GEAUX

MULTIPLICATION BY EXPONENTIAL

$$\mathcal{L}[y(t)] = Y(s) \Longrightarrow \mathcal{L}[e^{at}y(t)] = Y(s-a)$$

$$\mathrm{L}[e^{at}y(t)] = \int_0^\infty e^{at}y(t)e^{-st}dt = \int_0^\infty y(t)e^{-(s-a)t}dt$$

$$Y(s-a)$$

LEARNING EXAMPLE

$$f(t) = e^{-3t}\cos(10t) \qquad a = -3$$

$$y(t) = \cos 10t \Rightarrow Y(s) = \frac{s}{s^2+100} \text{ (From table)}$$

$$f(t) = e^{-3t}y(t) \Rightarrow F(s) = Y(s+3) = \frac{s+3}{(s+3)^2+100}$$

$$\mathcal{L}[\cos\beta t] = \frac{s}{s^2+\beta^2} \Longrightarrow \mathcal{L}[e^{-\sigma t}\cos\beta t] = \frac{s+\sigma}{(s+\sigma)^2+\beta^2}$$

New entries for the table of transform pairs

LEARNING EXAMPLE

$$x(t) = e^{-2t}\cos(4t+\pi/3)$$

$$x(t) = e^{-2t}(\cos\pi/3\cos 4t - \sin\pi/3\sin 4t)$$

$$a = -2,\ b = 4$$

$$X(s) = \cos\pi/3\frac{s+2}{(s+2)^2+16} - \sin\pi/3\frac{4}{(s+2)^2+16}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$

GEAUX

MULTIPLICATION BY TIME

$$\mathcal{L}[y(t)] = Y(s) \Longrightarrow \mathcal{L}[ty(t)] = -\frac{dY(s)}{ds}$$

Differentiation under an integral

$$\frac{d}{ds}\left[\int_{t_0}^{t_1} g(s,t)dt\right] = \int_{t_0}^{t_1} \frac{\partial g(s,t)}{\partial s}dt$$

$$\mathcal{L}[y(t)] = Y(s) = \int_{0^-}^{\infty} y(t)e^{-st}dt$$

$$\frac{dY}{ds}(s) = \int_{0^-}^{\infty} \frac{\partial y(t)e^{-st}}{\partial s}dt$$

$$= \int_{0^-}^{\infty} (-t)y(t)e^{-st}dt$$

$$= -\mathcal{L}[ty(t)]$$

Remember that we consider the functions to be zero for $t < 0$. Hence

$x(t) = x(t)u(t)$

LEARNING EXAMPLE

Let $u(t)$ be the unit step

Find the transform of the ramp function

$r(t) = tu(t)$

$$u(t) \leftrightarrow U(s) = \frac{1}{s}$$

$$tu(t) \leftrightarrow -\frac{d}{dt}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$t^2u(t) \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

By succesive application of the property one shows

$$t^n(u(t)) \leftrightarrow \frac{n!}{s^{n+1}}$$

This result, plus linearity, allows computation of the transform of any polynomial

LEARNING BY DOING

$x(t) = 1 + 2t + 6t^3$

$$X(s) = \frac{1}{s} + 2\frac{1}{s^2} + 3\frac{3!}{s^4}$$

GEAUX

TIME SHIFTING PROPERTY

$$f(t)u(t) \leftrightarrow F(s) \Rightarrow f(t-t_0)u(t-t_0) \leftrightarrow e^{-st_0}F(s)$$

$$\mathcal{L}[f(t-t_0)u(t-t_0)] = \int_0^\infty f(t-t_0)u(t-t_0)e^{-st}\,dt$$

$$= \int_{t_0}^\infty f(t-t_0)e^{-st}\,dt$$

let $\lambda = t - t_0$ *and* $d\lambda = dt$, *then*

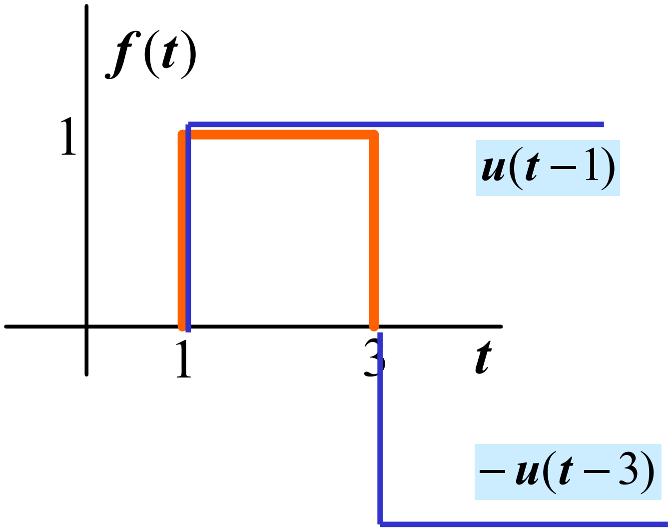
$$\mathcal{L}[f(t-t_0)u(t-t_0)] = \int_0^\infty f(\lambda)e^{-s(\lambda+t_0)}\,d\lambda$$

$$= e^{-t_0 s}\int_0^\infty f(\lambda)e^{-s\lambda}\,d\lambda$$

$$= e^{-t_0 s}\mathbf{F}(s) \quad t_0 \geq 0$$

LEARNING EXAMPLE

$$f(t) = \begin{cases} 1 & 1 \leq t \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$



$$f(t) = u(t-1) - u(t-3)$$

$$F(s) = e^{-s}\frac{1}{s} - e^{-3s}\frac{1}{s} = \frac{1}{s}\left(e^{-s} - e^{-3s}\right)$$

GEAUX

LEARNING EXTENSION

FIND THE TRANSFORM FOR

$$f(t) = te^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

One can apply the time shifting property if the time variable always appears as it appears in the argument of the step. In this case as *t-1*

$$f(t) = (t-1+1)e^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

$$f(t) = (t-1)e^{-(t-1)}u(t-1) + e^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1) = (t-1)e^{-(t-1)}u(t-1)$$

$$tu(t) \leftrightarrow \frac{1}{s^2}$$

$$te^{-t}u(t) \leftrightarrow \frac{1}{(s+1)^2}$$

$$\therefore (t-1)e^{-(t-1)}u(t-1) \leftrightarrow \frac{e^{-s}}{(s+1)}$$

One could also write

$$f(t) = e(t-1)e^{-t}u(t-1) \quad g(t)$$

And apply the time truncation property

$$f(t) = g(t)u(t-1) \Rightarrow F(s) = e^{-s}\mathcal{L}[g(t+1)]$$

$$g(t+1) = ete^{-(t+1)} = te^{-t}$$

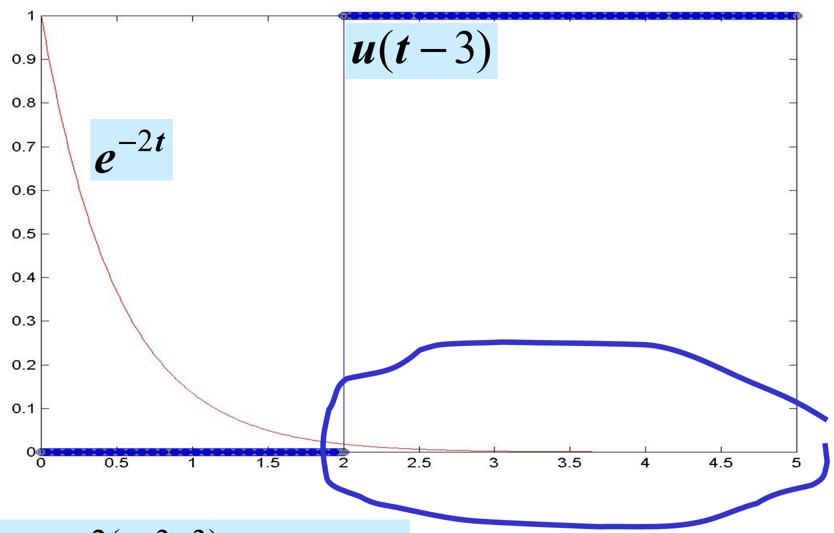
$$\mathcal{L}[g(t+1)] = \frac{1}{(s+1)^2}$$

The two properties are only different representations of the same result

GEAUX

LEARNING EXAMPLE

$f(t) = e^{-2t}u(t-3)$



$$f(t) = e^{-2(t-3+3)}u(t-3)$$
$$= e^{-6}e^{-2(t-3)}u(t-3)$$

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2} \Rightarrow e^{-2(t-3)}u(t-3) \leftrightarrow e^{-3s}\frac{1}{s+2}$$

$$F(s) = e^{-6}\frac{e^{-3s}}{s+2}$$

Using time truncation

$$f(t) = g(t)u(t-3) \Rightarrow F(s) = e^{-3s}\mathcal{L}[g(t+3)]$$

$$g(t) = e^{-2t} \Rightarrow g(t+3) = e^{-2(t+3)} = e^{-6}e^{-2t}$$

$$\mathcal{L}[g(t+3)] = e^{-6}\frac{1}{s+2}$$

LEARNING EXAMPLE

$g(t)$

$$x(t) = \sin(2t - \pi/6)u(t-2)$$

$$x(t) = \sin(2(t-2+2) - \pi/6)u(t-2)$$

$$\theta = 4 - \pi/6$$

$$x(t) = \sin(2(t-2) + \theta)u(t-2)$$

$$x(t) = \cos\theta \sin(2(t-2))u(t-2) + \sin\theta\cos(2(t-2))u(t-2)$$

$$\sin 2t u(t) \leftrightarrow \frac{2}{s^2+4}$$

$$\Rightarrow \sin(2(t-2))u(t-2) \leftrightarrow e^{-2s}\frac{2}{s^2+4}$$

$$\cos 2t u(t) \leftrightarrow \frac{s}{s^2+4}$$

$$\Rightarrow \cos(2(t-2))u(t-2) \leftrightarrow e^{-2s}\frac{s}{s^2+4}$$

$$X(s) = e^{-2s}\left(\cos\theta\frac{2}{s^2+4} + \sin\theta\frac{s}{s^2+4}\right)$$

$$X(s) = e^{-2s}\mathcal{L}[g(t+2)]$$

GEAUX

LEARNING EXTENSION

Compute the Laplace transform of the following functions

A) $f(t) = e^{-4t}(t - e^{-t})$ $= te^{-4t} - e^{-5t}$ $= \frac{1}{(s+4)^2} - \frac{1}{s+5}$

B) $g(t) = \frac{te^{-4x}}{a^2 + 4}$ $G(s) = \frac{e^{-4x}}{a^2+4} L[t]$ $G(s) = \frac{e^{-4x}}{s^2(a^2+4)}$

C) $x(t) = \cos(bt)u(t-1)$ $X(s) = e^{-s} L[\cos(b(t+1))]$

$$\cos(b(t+1)) = \cos b \cos bt - \sin b \sin bt$$

$$L[\cos(b(t+1))] = \cos b \frac{s}{s^2+b^2} - \sin b \frac{b}{s^2+b^2}$$

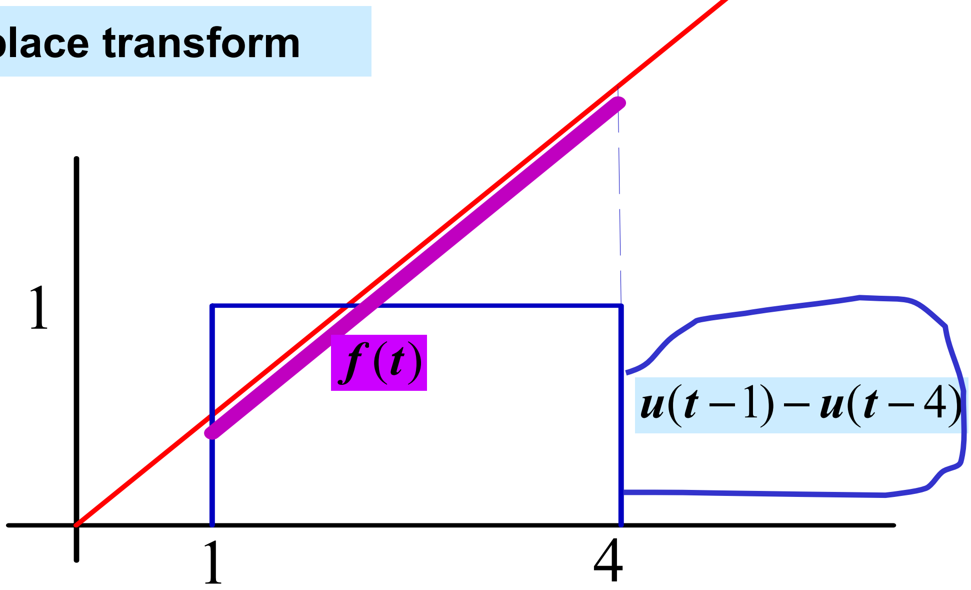
$$X(s) = e^{-s}\left(\cos b \frac{s}{s^2+b^2} - \sin b \frac{b}{s^2+b^2}\right)$$

LEARNING EXTENSION

Compute the Laplace transform

$$f(t) = \begin{cases} t & 1 \le t \le 4 \\ 0 & elsewhere \end{cases}$$

$$f(t) = t(u(t-1) - u(t-4))$$



$$f(t) = (t-1+1)u(t-1) - (t-4+4)u(t-4)$$

$$f(t) = (t-1)u(t-1) + u(t-1) - (t-4)u(t-4) - u(t-4)$$

$$F(s) = e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s} - e^{-4s}\frac{1}{s^2} - e^{-4s}\frac{1}{s}$$

Using the definition

$$F(s) = \int_1^4 te^{-st}dt$$

GEAUX

PERFORMING THE INVERSE TRANSFORM

FACT: Most of the Laplace transforms that we encounter are *proper* rational functions of the form

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

$m \le n$

Zeros = roots of numerator
Poles = roots of denominator

KNOWN: PARTIAL FRACTION EXPANSION

If $Q(s) = Q_1(s)Q_2(s)$ is a COPRIME factorization of the denominator with

$\deg(Q_i) = n_i \ (\therefore \sum n_i = n)$, then

$$F(s) = K_0 + \frac{P_1(s)}{Q_1(s)} + \frac{P_2(s)}{Q_2(s)};\ \deg(P_i) < n_i$$

If m<n and the poles are simple

$$\frac{P_1(s)}{Q(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n}$$

Simple, complex conjugate poles

$$\frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

$$= \frac{K_1}{s+\alpha-j\beta} + \frac{K_1^*}{s+\alpha+j\beta} + \cdots$$

$$= \frac{C_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{C_2\beta}{(s+\alpha)^2+\beta^2} + \cdots$$

Pole with multiplicity *r*

$$\frac{P_1(s)}{Q_1(s)(s+p_1)^r}$$

$$= \frac{K_{11}}{(s+p_1)} + \frac{K_{12}}{(s+p_1)^2} + \cdots + \frac{K_{1r}}{(s+p_1)^r} + \cdots$$

THE INVERSE TRANSFORM OF EACH PARTIAL FRACTION IS IMMEDIATE. WE ONLY NEED TO COMPUTE THE VARIOUS CONSTANTS

SIMPLE POLES

$$\mathbf{F}(s) = \frac{\mathbf{P}(s)}{\mathbf{Q}(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n} \quad \times/(s+p_i)$$

$$\left.\frac{(s+p_i)\mathbf{P}(s)}{\mathbf{Q}(s)}\right|_{s=-p_i} = 0 + \cdots + 0 + K_i + 0 + \cdots + 0 \qquad i = 1,2,\ldots,n$$

LEARNING EXAMPLE

$$F(s) = \frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)}$$

Write the partial fraction expansion

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}$$

Determine the coefficients (residues)

$$K_1 = sF(s)\big|_{s=0} = \frac{12\times1\times3}{2\times4\times5} = \frac{9}{10}$$

$$K_2 = (s+2)F(s)\big|_{s=-2} = \frac{12(-1)(1)}{(-2)(2)(3)} = 1$$

$$K_3 = (s+4)F(s)\big|_{s=-4} = \frac{12(-3)(-1)}{(-4)(-2)(1)} = \frac{36}{8}$$

$$K_4 = (s+5)F(s)\big|_{s=-5} = \frac{12(-4)(-2)}{(-5)(-3)(-1)} = -\frac{32}{5}$$

"FORM" of the inverse transform

$$f(t) = \left(K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}\right)u(t)$$

Get the inverse of each term and write the final answer

$$f(t) = \left(\frac{9}{10} + e^{-2t} + \frac{36}{8}e^{-4t} - \frac{32}{5}e^{-5t}\right)u(t)$$

The step function is necessary to make the function zero for t<0

LEARNING EXTENSIONS

Find the inverse transform

1. Partial fraction
2. Residues
3. Inverse of each term

A) $F(s) = \frac{10(s+6)}{(s+1)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$

Partial fraction

$K_1 = (s+1)F(s)\big|_{s=-1} = \frac{10(-1+6)}{(-1+3)}$

Inverse of each term

residues

$f(t) = \left(25e^{-t} - 15e^{-3t}\right)u(t)$

$K_2 = (s+3)F(s)\big|_{s=-3} = \frac{10(-3+6)}{(-3+1)}$

Makes the function zero for t<0

Form of solution: $f(t) = (K_1 e^{-t} + K_2 e^{-3t})u(t)$

B) $F(s) = \frac{12(s+2)}{s(s+1)} = \frac{K_1}{s} + \frac{K_2}{s+1}$

$f(t) = (K_1 + K_2 e^{-t})u(t)$

$K_1 = sF(s)\big|_{s=0} = \frac{12(2)}{1}$

$f(t) = (24 - 12e^{-t})u(t)$

$K_2 = (s+1)F(s)\,\big|_{s=-1} = \frac{12(-1+2)}{-1}$

COMPLEX CONJUGATE POLES

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{K_1}{s+\alpha-j\beta} + \frac{K_1^*}{s+\alpha+j\beta} + \cdots$$

$$(s+\alpha-j\beta)\mathbf{F}(s)\Big|_{s=-\alpha+j\beta} = K_1 = |K_1|\angle\theta \quad \Rightarrow$$

$$\mathbf{F}(s) = \frac{|K_1|\angle\theta}{s+\alpha-j\beta} + \frac{|K_1|\angle-\theta}{s+\alpha-j\beta} + \cdots$$

$$= \frac{|K_1|e^{j\theta}}{s+\alpha-j\beta} + \frac{|K_1|e^{-j\theta}}{s+\alpha+j\beta} + \cdots$$

$$f(t) = \mathcal{L}^{-1}[\mathbf{F}(s)] = |K_1|e^{j\theta}e^{-(\alpha-j\beta)t} + |K_1|e^{-j\theta}e^{-(\alpha+j\beta)t}$$

$$= |K_1|e^{-\alpha t}[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}] + \cdots$$

Euler's Identity

$$\cos\phi = \frac{e^{j\phi}+e^{-j\phi}}{2}$$

$$f(t) = 2|K_1|e^{-\alpha t}\cos(\beta t+\theta) + \ldots$$

USING QUADRATIC FACTORS

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{P_1(s)}{Q_1(s)\left[(s+\alpha)^2+\beta^2\right]} = \frac{C_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{C_2\beta}{(s+\alpha)^2+\beta^2} + \ldots$$

$$f(t) = C_1e^{-\alpha t}\cos\beta t + C_2e^{-\alpha t}\sin\beta t + \ldots$$

Avoids using complex algebra.
Must determine the coefficients in different way

The two forms are equivalent !

LEARNING EXAMPLE

$$Y(s)=\frac{10(s+2)}{s(s^2+4s+5)}$$

$$f(t)=2|K_1|e^{-\alpha t}\cos(\beta t+\theta)+\dots$$

$$s^2+4s+5=(s+2-j1)(s+2+j1)$$
$$=(s+2)^2+1$$

$$Y(s)=\frac{10(s+2)}{s(s+2-j1)(s+2+j1)}=\frac{K_0}{s}+\frac{K_1}{s+2-j1}+\frac{K_1^*}{s+2+j1}$$

MUST use radians in exponent

$$K_0=sY(s)|_{s=0}=\frac{10(2)}{(2-j1)(2+j1)}=\frac{20}{5}=4$$

$$K_1=(s+2-j1)Y(s)|_{s=-2+j1}=\frac{10(j1)}{(-2+j1)(j2)}=\frac{5}{\sqrt{5}\angle 153.43^\circ}=2.236\angle -153.43^\circ=2.236e^{-j2.678}$$

$$y(t)=\left(4+2\times 2.236\cos(t-2.678)\right)u(t)$$

Using quadratic factors

$$Y(s)=\frac{10(s+2)}{s(s^2+4s+5)}=\frac{C_0}{s}+\frac{C_1(s+2)}{(s+2)^2+1}+\frac{C_2}{(s+2)^2+1}=\frac{C_0((s+2)^2+1)+C_1(s+2)s+C_2s}{s(s^2+4s+5)}$$

$$\therefore 10(s+2)=C_0((s+2)^2+1)+C_1(s+2)s+C_2s$$

$s^2: 0=C_0+C_1 \quad \Rightarrow C_1=-C_0=-4$

$s: 10=4C_0+2C_1+C_2 \quad \Rightarrow C_2=2$

$s^0: 20=5C_0 \quad \Rightarrow C_0=4$

Alternative way to determine coefficients

For $s=0: 20=5C_0$

For $s=-2: 0=C_0-2C_2$

For $s=-1: 10=2C_0-C_1-C_2$

$$y(t)=(C_0+C_1e^{-2t}\cos t+C_2e^{-t}\sin t)u(t)$$

GEAUX

MULTIPLE POLES

$$\mathcal{L}^{-1}\left[\frac{1}{(s+p)^n}\right] = \frac{1}{(n-1)!}t^{n-1}e^{-pt}$$

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+p_1)^r} = \frac{K_{11}}{s+p_1} + \frac{K_{12}}{(s+p_1)^2} + \cdots + \frac{K_{1r}}{(s+p_1)^r} + \cdots \times/(s+p_1)^r$$

$$(s+p_1)^r\mathbf{F}(s)\Big|_{s=-p_1} = K_{1r}$$

$$\frac{d}{ds}\left[(s+p_1)^r\mathbf{F}(s)\right]\Big|_{s=-p_1} = K_{1r-1}$$

$$\frac{d^2}{ds^2}\left[(s+p_1)^r\mathbf{F}(s)\right]\Big|_{s=-p_1} = (2!)K_{1r-2}$$

$$K_{1j} = \frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}\left[(s+p_1)^r\mathbf{F}(s)\right]\Big|_{s=-p_1}$$

The method of identification of coefficients, or even the method of selecting values of *s*, may provide a convenient alternative for the determination of the residues

LEARNING EXAMPLE

$$F(s) = \frac{(s+2)^2}{s^3(s+5)} = \frac{K_{11}}{s} + \frac{K_{12}}{s^2} + \frac{K_{13}}{s^3} + \frac{K_2}{s+5}$$

$$K_2 = (s+5)F(s)\Big|_{s=-5} = \frac{(-3)^2}{(-5)^3}$$

$$s^3F(s) = \frac{(s+2)^2}{s+5} = K_{11}s^2 + K_{12}s + K_{13} + K_2\frac{s^3}{s+5}$$

$$= \frac{K_{11}s^2(s+5) + K_{12}s(s+5) + K_{13}(s+5) + K_2s^3}{s+5}$$

$s^3 : 0 = K_{11} + K_2$

$s^2 : 1 = 5K_{11} + K_{12}$

$s^1 : 2 = 5K_{12} + K_{13}$

$s^0 : 4 = 5K_{13}$

For K_{11}

$r = 3, j = 1$

$$K_{13} = s^3F(s)\Big|_{s=0} = \frac{4}{5}$$

$$K_{12} = \frac{d}{ds}\left(\frac{(s+2)^2}{s+5}\right)\Bigg|_{s=0}$$

$$= \frac{2(s+2)(s+5) - (s+2)^2}{(s+5)^2}\Bigg|_{s=0}$$

For K_{11} must differentiate one more time

GEAUX

LEARNING EXAMPLE

$$\mathbf{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)}$$

$$\mathbf{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2} + \frac{K_{13}}{(s+1)^3} + \frac{K_2}{s+2}$$

$$f(t) = \left(K_{11}e^{-t} + K_{12}te^{-t} + K_{13}\left(\frac{1}{2}t^2e^{-t}\right) + K_2e^{-2t} \right)u(t)$$

$$K_2 = (s+2)F(s)\big|_{s=-2} = \frac{10(1)}{(-1)^3} = -10$$

$$K_{13} = (s+1)^3F(s)\big|_{s=-1} = \frac{10(2)}{(1)} = 20$$

$$K_{12} = \frac{d}{ds}\left((s+1)^3F(s)\right)\Big|_{s=-1} = \frac{d}{ds}\left(\frac{10(s+3)}{s+2}\right)\Bigg|_{s=-1}$$

$$= \frac{10(s+2) - 10(s+3)}{(s+2)^2}\Bigg|_{s=-1} = \frac{-10}{(s+2)^2}\Bigg|_{s=-10} = -10$$

$$K_{11} = \frac{1}{2!}\frac{d^2}{ds^2}\left((s+1)^3F(s)\right)\Big|_{s=-1} = \frac{1}{2!}\frac{d}{ds}\frac{-10}{(s+2)^2}$$

$$K_{11} = \frac{1}{2}\frac{10(2(s+2))}{(s+2)^4}\Bigg|_{s=-1} = \frac{10}{(s+2)^3}\Bigg|_{s=-1} = 10$$

Using identification of coefficients

$$(s+1)^3F(s) = \frac{10(s+3)}{(s+2)}$$

$$= \frac{K_{11}(s+1)^2(s+2) + K_{12}(s+1)(s+2) + K_{13}(s+2) + K_2(s+1)^3}{(s+2)}$$

$$s^3 : 0 = K_{11} + K_2$$

$$s^2 : 0 = 4K_{11} + K_{12} + 3K_2$$

$$s^1 : 10 = 5K_{11} + 3K_{12} + K_{13} + 3K_2$$

$$s^0 : 30 = 2K_{11} + 2K_{12} + 2K_{13} + K_2$$

$$K_{1j} = \frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}\left[(s+p_1)^r\mathbf{F}(s)\right]\Big|_{s=-p_1}$$

GEAUX

LEARNING EXTENSION

Find the inverse transform $F(s) = \dfrac{s}{(s+1)^2}$

Partial fraction

$$F(s) = \frac{s}{(s+1)^2} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2}$$

Form of the inverse

$$f(t) = (K_{11}e^{-t} + K_{12}te^{-t})u(t)$$

Residues

$$K_{12} = (s+1)^2 F(s)|_{s=-1} = -1$$

$$K_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{ds^{r-j}} [(s+p_1)^r \mathbf{F}(s)] \Big|_{s=-p_1}$$

alternatively

$$(s+1)^2 F(s) = s = K_{11}(s+1) + K_{12}$$

$$\therefore \frac{d}{ds}(s+1)^2 F(s) = 1 = K_{11}$$

$$f(t) = (e^{-t} - te^{-t})u(t)$$

LEARNING EXTENSION

Find the inverse transform

$$F(s)=\frac{(s+2)}{s^2(s+1)}$$

Partial fraction expansion

$$F(s)=\frac{(s+2)}{s^2(s+1)}=\frac{K_{11}}{s}+\frac{K_{12}}{s^2}+\frac{K_2}{s+1}$$

Form of the inverse

$$f(t)=(K_{11}+K_{12}t+K_2e^{-t})u(t)$$

Residues

$$K_2=(s+1)F(s)|_{s=-1}=\frac{(-1+2)}{(-1)^2}$$

$$K_{12}=s^2F(s)|_{s=0}=\frac{2}{1}$$

$$s^2F(s)=\frac{s+2}{s+1}=sK_{11}+K_{12}+K_2\frac{s^2}{(s+1)}$$

$$\frac{d}{ds}(s^2F(s))|_{s=0}=K_{11}$$

$$\frac{d}{ds}\left(\frac{s+2}{s+1}\right)\bigg|_{s=0}=\frac{(s+1)-(s+2)}{(s+1)^2}\bigg|_{s=0}=-1$$

inverse

$$f(t)=(-1+2t+e^{-t})u(t)$$

GEAUX

CONVOLUTION INTEGRAL

CLAIM: Given an ODE

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots a_0 y = b_m \frac{d^m u}{dt^m} + \ldots + b_0 u$$

there exists a function, $h(t), t \geq 0$, such that

$$y(t) = \int_0^t h(t-x)u(x)dx = h(t) \otimes u(t)$$

is a particular solution of the equation for $t \geq 0$

(Actually, the zero state response)

RESULT : If f_1, f_2, are positive time functions

$$f(t) = \int_0^t f_1(t-\lambda)f_2(\lambda)d\lambda = \int_0^t f_1(\lambda)f_2(t-\lambda)\,d\lambda$$

$$F(s) = F_1(s)F_2(s)$$

PROOF

$$\mathcal{L}[f(t)] = \int_0^\infty \left[\int_0^t f_1(t-\lambda)f_2(\lambda)\,d\lambda\right]e^{-st}\,dt$$

$$\mathcal{L}[f(t)] = \int_0^\infty \left[\int_0^\infty f_1(t-\lambda)u(t-\lambda)f_2(\lambda)\,d\lambda\right]e^{-st}\,dt$$

$$\mathcal{L}[f(t)] = \int_0^\infty f_2(\lambda)\left[\int_0^\infty f_1(t-\lambda)u(t-\lambda)e^{-st}\,dt\right]d\lambda$$

Shifting

$$\mathcal{L}[f(t)] = \int_0^\infty f_2(\lambda)\mathbf{F}_1(s)e^{-s\lambda}\,d\lambda$$

$$= \mathbf{F}_1(s)\int_0^\infty f_2(\lambda)e^{-s\lambda}\,d\lambda$$

$$= \mathbf{F}_1(s)\mathbf{F}_2(s)$$

EXAMPLE

FIND $Y(s)$

$$y(t) + \int_0^t e^{-(t-x)}y(x)dx = t; \quad t > 0$$

$$y(t) + e^{-t} \otimes y(t) = t \Rightarrow$$

$$Y(s) + \frac{1}{s+2}Y(s) = \frac{1}{s^2}$$

$$\left(1 + \frac{1}{s+2}\right)Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{s+2}{s^2(s+3)}$$

GEAUX

LEARNING EXAMPLE

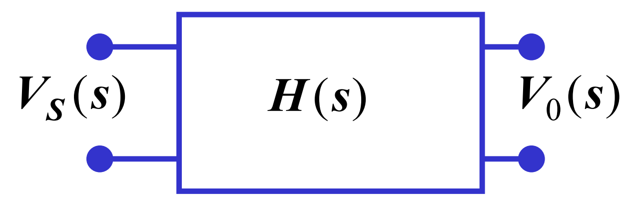
Using convolution to determine a network response

Network function

$$H(s)=\frac{V_0(s)}{V_S(s)}=\frac{10}{s+5}$$

Input

$$V_S(s)=\frac{1}{s}$$



$$V_0(s)=H(s)V_S(s)$$

$$V_0(s)=\frac{10}{s+5}\times\frac{1}{s}$$

$$\frac{10}{s+5}\leftrightarrow 10e^{-5t}u(t)$$

$$\frac{1}{s}\leftrightarrow u(t)$$

$$v_0(t)=e^{-5t}u(t)\otimes u(t)$$

RESULT : If $f_1, f_2,$ are positive time functions

$$f(t)=\int_0^t f_1(t-\lambda)f_2(\lambda)d\lambda=\int_0^t f_1(\lambda)f_2(t-\lambda)d\lambda$$

$$F(s)=F_1(s)F_2(s)$$

For $t\geq 0$

$$v_0(t)=10\int_0^t e^{-5(t-\lambda)}d\lambda=10e^{-5t}\int_0^t e^{5\lambda}d\lambda=10e^{-5t}\left[\frac{1}{5}e^{\lambda}\right]_0^t$$

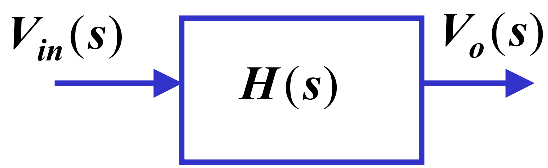
$$v_0(t)=2e^{-5t}\left[e^{t}-1\right]=2\left(1-e^{-5t}\right);\ t\geq 0$$

In general convolution is not an efficient approach to determine the output of a system. But it can be a very useful tool in special cases

LEARNING EXAMPLE

This example illustrates an idealized modeling approach and the use of convolution as a system simulation tool.

This slide shows how one can obtain a “black box” model for a system



$$= H(s)V_{in}(s)$$

The black box model is a description of the system based only on input/output data. There is no information on what is “inside the box”

Unknown linear system represented in the Laplace domain

Ideal approach to modeling

Measure the impulse response

$$v_{in}(t) = \delta(t) \Rightarrow V_{in}(s) = 1,$$

$$\therefore V_o(s) = H(s), v_o(t) = h(t)$$

For any other input one has

$$v_o(t) = \int_0^t h(t-x) v_{in}(x)dx$$

In practice, a good approximation to an impulse may be difficult, or impossible to apply. Hence we try to use “more sensible inputs.”

Using the step response

$$v_{in}(t) = u(t) \Rightarrow V_{in}(s) = \frac{1}{s}, V_{os}(s) = \frac{H(s)}{s}$$

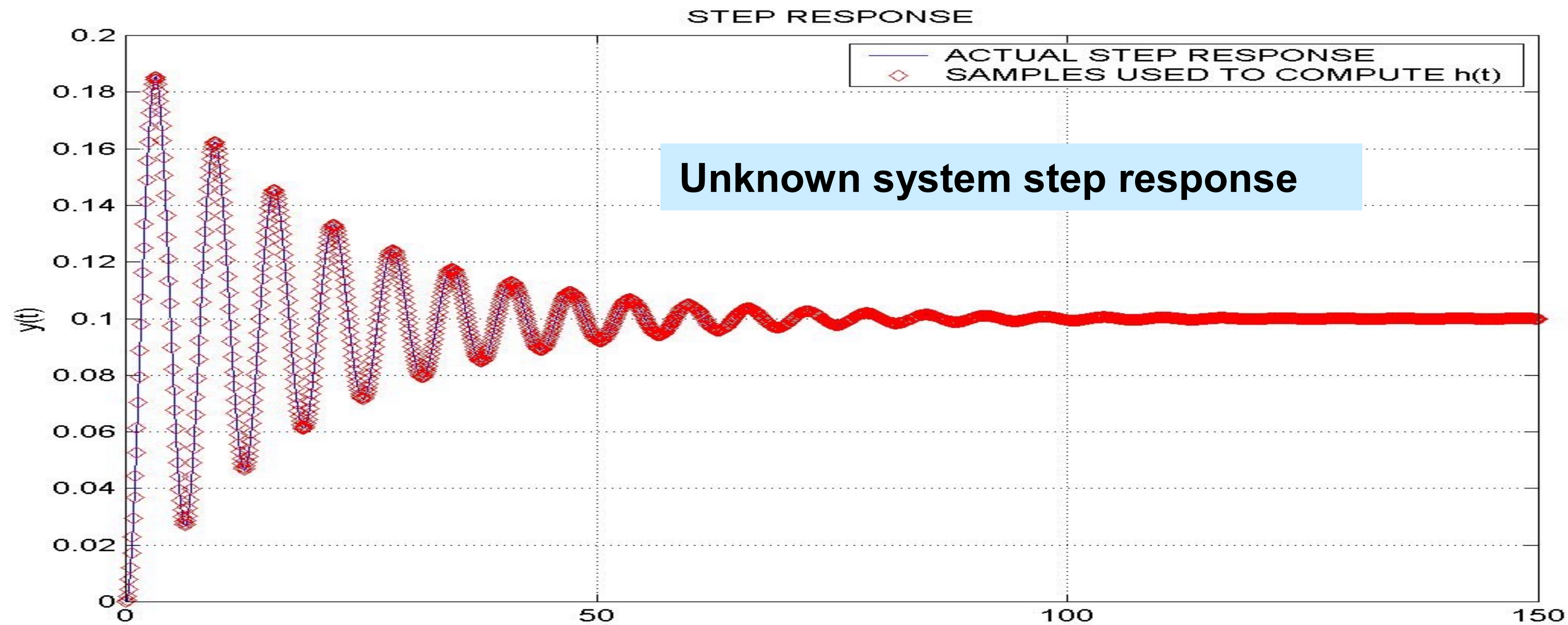
$$\therefore H(s) = sV_{os}(s) \Rightarrow h(t) = \frac{d}{dt} v_{os}(t)$$

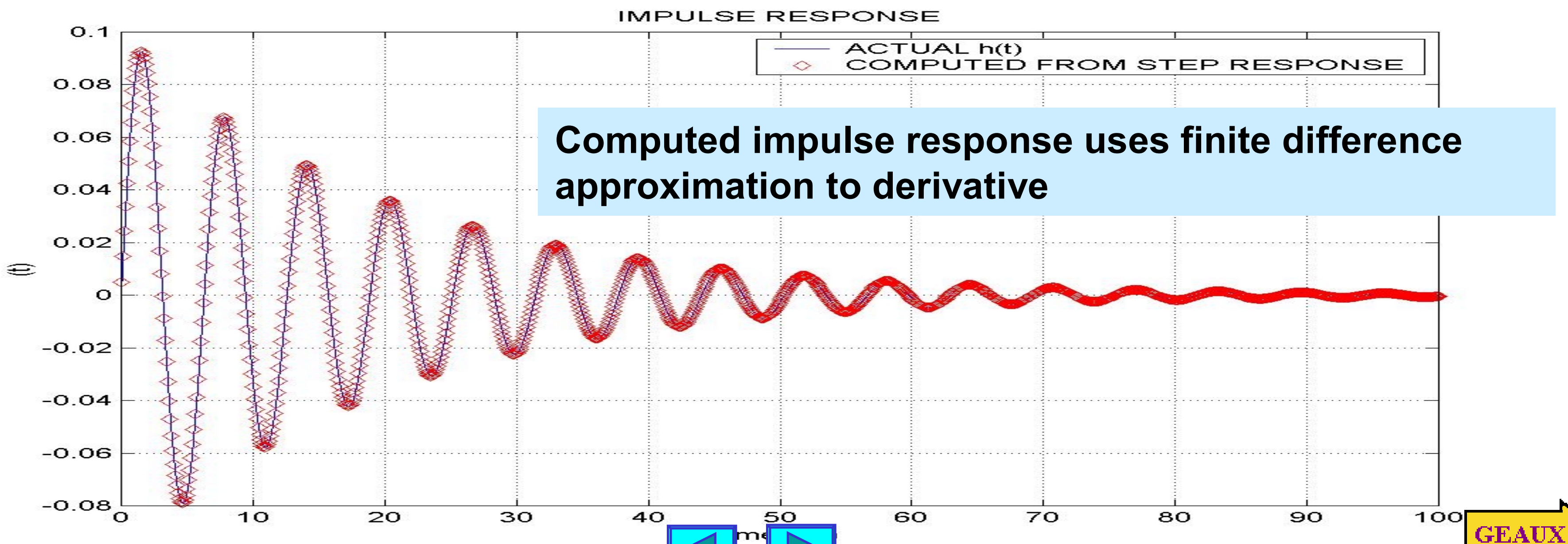
The impulse response is the derivative of the step response of a system

Once the impulse response is obtained, the convolution can be evaluated numerically

GEAUX

A CASE STUDY IN MODELING





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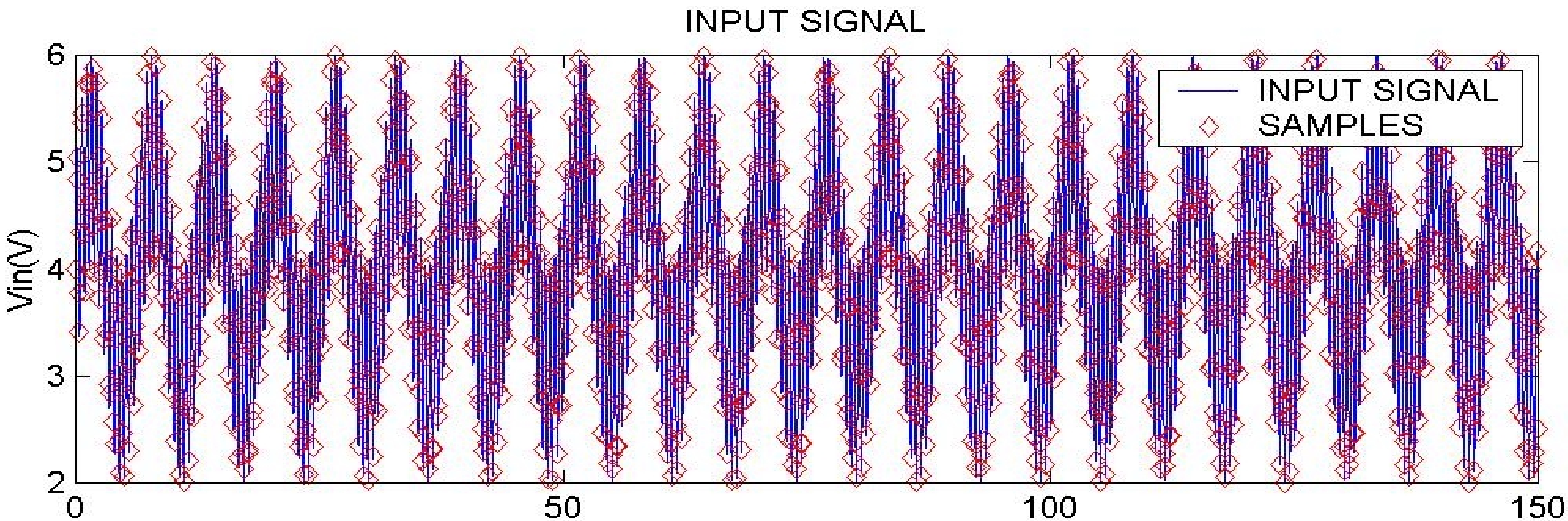
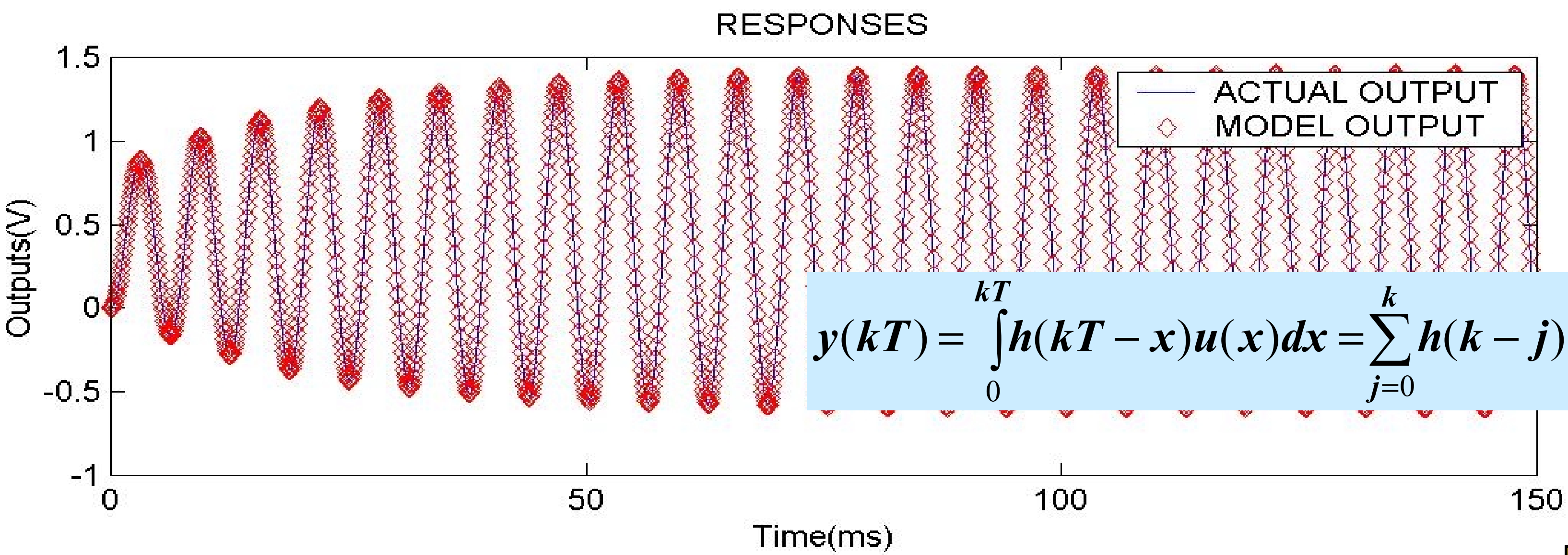
Test of the model

The model output uses the computed impulse response and samples of the input signal. Convolution integral is evaluated numerically

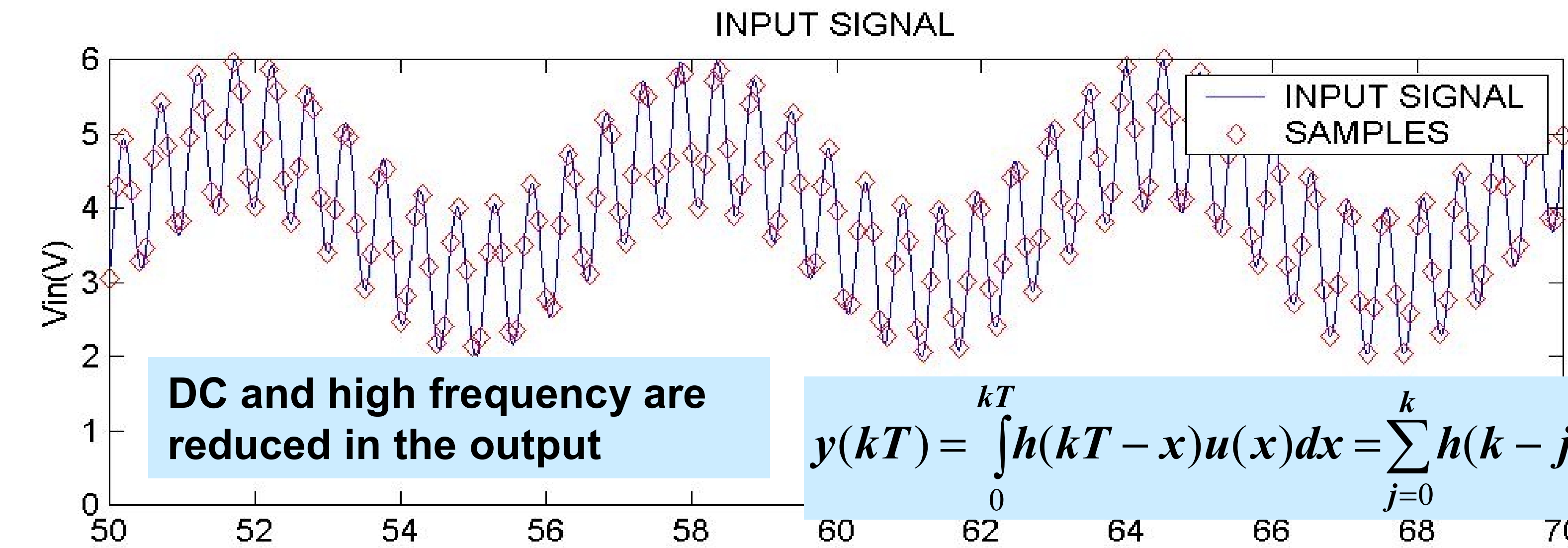




$$y(kT) = \int_0^{kT} h(kT - x)u(x)dx = \sum_{j=0}^{k} h(k-j)u(jT)T$$

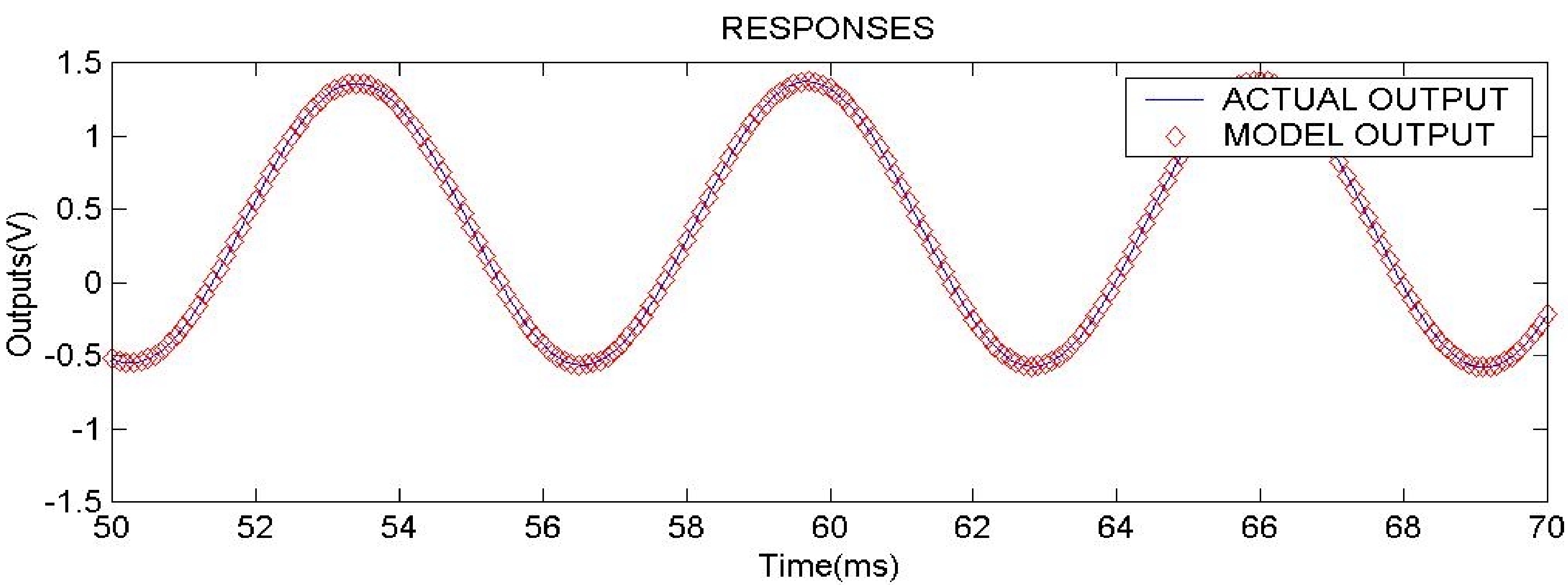
GEAUX

Detailed view of a segment of the signals showing bandpass action



DC and high frequency are reduced in the output

$$y(kT) = \int_0^{kT} h(kT-x)u(x)dx = \sum_{j=0}^{k} h(k-j)u(jT)T$$



GEAUX

INITIAL AND FINAL VALUE THEOREMS

These results relate behavior of a function in the time domain with the behavior of the Laplace transform in the s-domain

INITIAL VALUE THEOREM

Assume that both $f(t), {}^{df}\!/_{dt}$, have Laplace transform. Then

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

$L[\frac{df}{dt}] = sF(s) - f(0)$

And if the derivative is transformable then

$\lim_{s \to \infty} L[\frac{df}{dt}] = 0$

FINAL VALUE THEOREM

Assume that both $f(t), {}^{df}\!/_{dt}$, have Laplace transform and that $\lim_{t \to \infty} f(t)$ exists. Then

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

$\int_0^\infty \frac{df}{dt}(t) e^{-st} dt = sF(s) - f(0)$

Taking limits as $s \to 0$

$\int_0^\infty \frac{df}{dt}(t) dt = \lim_{s \to 0} sF(s) - f(0)$

NOTE: $\lim_{t \to \infty} f(t)$ will exist if $F(s)$ has poles with negative real part and at most a single pole at $s = 0$

GEAUX

LEARNING EXAMPLE

Given $F(s)=\dfrac{10(s+1)}{s(s^2+2s+2)}$.

Determine the initial and final values for $f(t)$

Clearly, *f(t)* has Laplace transform. And *sF(s) -f(0)* is also defined.

$$f(0)=\lim_{s\to\infty} sF(s)$$

$$f(0)=\lim_{s\to\infty}\frac{10(s+1)}{s^2+2s+2}=0$$

***F(s)* has one pole at *s=0* and the others have negative real part. The final value theorem can be applied.**

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0}\frac{10(s+1)}{s^2+2s+2}=5$$

NOTE: Computing the inverse one gets

$$f(t)=5+5\sqrt{2}e^{-t}\cos(t-\frac{3\pi}{4})$$

LEARNING EXTENSION

Given $F(s)=\dfrac{(s+1)^2}{s(s+2)(s^2+2s+2)}$.

Determine the initial and final values for $f(t)$

$$f(0)=\lim_{s\to\infty}\frac{(s+1)^2}{(s+2)(s^2+2s+2)}=0$$

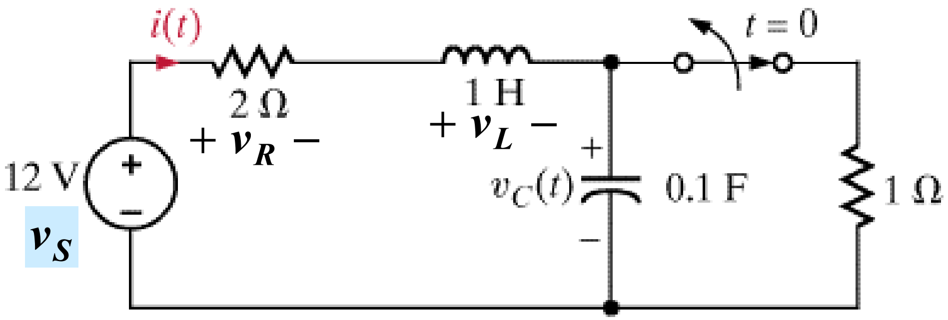
$$\lim_{t\to\infty} f(t)=\lim_{s\to 0}\frac{(s+1)^2}{(s+2)(s^2+2s+2)}=\frac{1}{4}$$

GEAUX

One way of using Laplace transform techniques in circuit analysis uses the following steps:

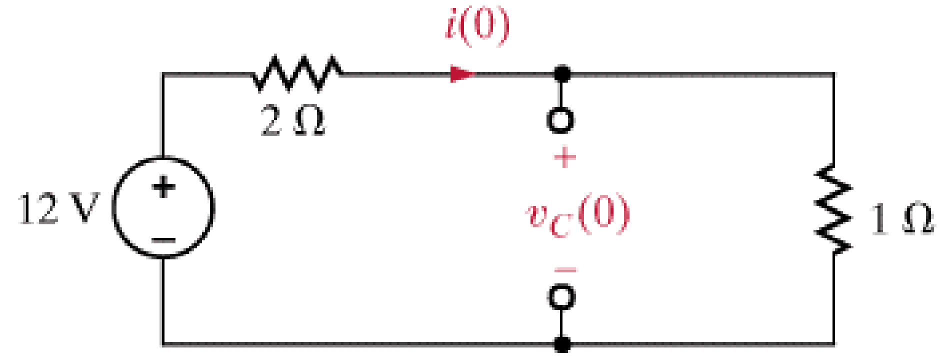
1. Derive the differential equation that describes the network
2. Apply the transform as a tool to solve the differential equation

LEARNING BY APPLICATION FIND $i(t), t > 0$



We will write the equation for *i(t)* and solve it using Laplace Transform

To find the initial conditions we use the steady state assumption for t<0



Circuit in steady state for t<0

For *t>0*

$$v_S = v_R + v_L + v_C$$

$$v_S = 12u(t),\ t > 0$$

$$v_R = Ri(t)$$

$$v_L = L\frac{di}{dt}(t)$$

$$v_C = v_C(0) + \frac{1}{C}\int_0^t i(x)dx$$

One could write KVL in the Laplace domain and skip the time domain

$$12u(t) = 2i(t) + \frac{di}{dt}(t) + v_C(0) + \frac{1}{C}\int_0^t i(x)dx$$

L

$$\frac{12}{s} = 2I(s) + sI(s) - i(0) + \frac{v_C(0)}{s} + \frac{I(s)}{0.1s}$$

$$i(0) = \frac{12(V)}{3\Omega} = 4(A);\quad v_C(0) = \frac{1}{1+2}\times 12(V) = 4(V)$$

$$\frac{12}{s} + 4 - \frac{4}{s} = \left(2 + s + \frac{10}{s}\right)I(s)$$

Replace and rearrange

$$I(s) = \frac{4(s+2)}{s^2 + 2s + 10} = \frac{4(s+2)}{(s+1-j3)(s+1+j3)}$$

$$I(s) = \frac{K_1}{s+1-j3} + \frac{K^*}{s+1+j3} \qquad K_1 = \left.\frac{4(s+2)}{s+1+j3}\right|_{s=-1+j3}$$

$$K_1 = \frac{4(1+j3)}{j6} = \frac{4\times 3.16\angle 71.57^\circ}{6\angle 90^\circ} = 2.11\angle -18.43^\circ$$

$$i(t) = 2|K_1|e^{-\sigma t}\cos(\omega t + \theta)$$

$$\sigma = 1$$

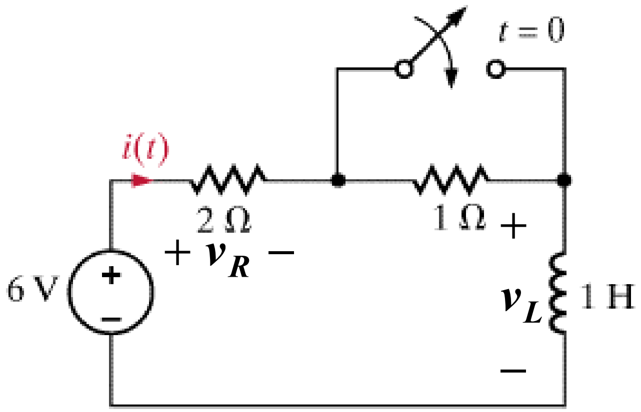
$$\omega = 3$$

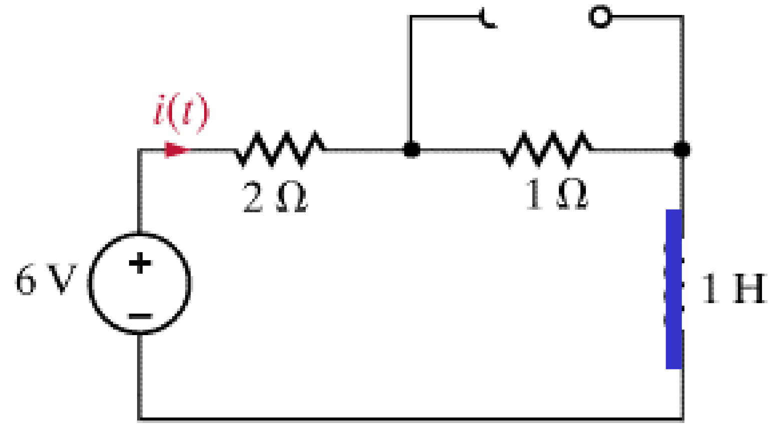
$$\theta = -18.43^\circ$$

GEAUX

LEARNING EXTENSION

Assuming the circuit in steady state for $t < 0$, determine $i(t), t > 0$.





Circuit in steady state for *t<0*

$$i(0) = \frac{6V}{3\Omega} = 2(A)$$

Equation for *t>0*

$$6u(t) = 2i(t) + \frac{di}{dt}(t)$$

Transforming to the Laplace domain

$$\frac{6}{s} = 2I(s) + sI(s) - i(0)$$

Next we must determine *i(0)*

$$\frac{6}{s} + 2 = (s+2)I(s)$$

$$I(s) = \frac{2(s+3)}{s(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+2}$$

$$K_1 = sI(s)\big|_{s=0} = 3 \qquad K_2 = (s+2)I(s)\big|_{s=-2} = -1$$

$$i(t) = 3 - e^{-2t}(A);\ t > 0$$

Laplace