Spectral Analysis of Signals

- Spectral analysis is concerned with the determination of frequency contents of a continuous-time signal $g_a(t)$ using DSP methods
- It involves the determination of either the energy spectrum or the power spectrum of the signal
- If $g_a(t)$ is sufficiently bandlimited, spectral characteristics of its discrete-time equivalent $g[n]$ should provide a good estimate of spectral characteristics of $g_a(t)$

Spectral Analysis of Signals

- In most cases, $g_a(t)$ is defined for $-\infty < t < \infty$
- Thus, $g[n]$ is of infinite extent, and defined for $-\infty < n < \infty$
- Hence, $g_a(t)$ is first passed through an analog anti-aliasing filter whose output is then sampled to generate $g[n]$
- Assumptions: (1) Effect of aliasing can be ignored, (2) A/D conversion noise can be neglected

Spectral Analysis of Signals

- Three types of spectral analysis -
- 1) Spectral analysis of stationary sinusoidal signals
- 2) Spectral analysis of of nonstationary signals with time-varying parameters
- 3) Spectral analysis of random signals

Spectral Analysis of Sinusoidal Signals

- Assumption - Parameters characterizing sinusoidal signals, such as amplitude, frequencies, and phase, do not change with time
- For such a signal $g[n]$, the Fourier analysis can be carried out by computing the DTFT

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

Spectral Analysis of Sinusoidal Signals

- In practice, the infinite-length sequence $g[n]$ is first windowed by multiplying it with a length-N window $w[n]$ to convert it into a length-N sequence $\gamma[n]$
- DTFT $\Gamma(e^{j\omega})$ of $\gamma[n]$ then is assumed to provide a reasonable estimate of $G(e^{j\omega})$
- $\Gamma(e^{j\omega})$ is evaluated at a set of R $(R \geq N)$ discrete angular frequencies equally spaced in the range $0 \leq \omega \leq 2\pi$ by computing the R-point FFT $\Gamma[k]$ of $\gamma[n]$

Spectral Analysis of Sinusoidal Signals

- We analyze the effect of windowing and the evaluation of the frequency samples of the DTFT via the DFT
- Now

$$\Gamma[k] = \Gamma(e^{j\omega})\Big|_{\omega = 2\pi k/R}, \quad 0 \le k \le R-1$$

- The normalized discrete-time angular frequency ω_k corresponding to the DFT bin number k (DFT frequency) is given by

$$\omega_k = \frac{2\pi k}{R}$$

Spectral Analysis of Sinusoidal Signals

- The continuous-time angular frequency corresponding to the DFT bin number k (DFT frequency) is given by

$$\Omega_k = \frac{2\pi k}{RT}$$

- To interpret the results of DFT-based spectral analysis correctly we first consider the frequency-domain analysis of a sinusoidal signal

Spectral Analysis of Sinusoidal Signals

- Consider $g[n] = \cos(\omega_o n + \phi), \quad -\infty < n < \infty$
- It can be expressed as

$$g[n] = \tfrac{1}{2}\left(e^{j(\omega_o n+\phi)} + e^{-j(\omega_o n+\phi)}\right)$$

- Its DTFT is given by

$$G(e^{j\omega}) = \pi \sum_{\ell=-\infty}^{\infty} e^{j\phi}\delta(\omega - \omega_o + 2\pi\ell)$$
$$+\pi \sum_{\ell=-\infty}^{\infty} e^{-j\phi}\delta(\omega - \omega_o + 2\pi\ell)$$

Spectral Analysis of Sinusoidal Signals

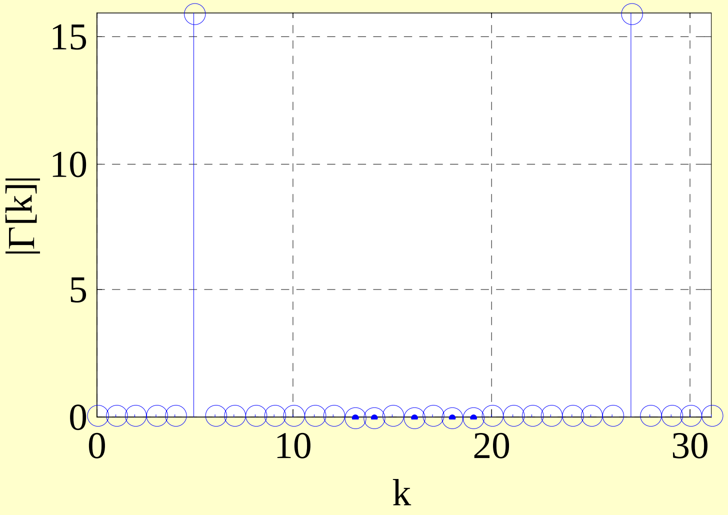
- $G(e^{j\omega})$ is a periodic function of ω with a period 2π containing two impulses in each period
- In the range $-\pi \leq \omega \leq \pi$, there is an impulse at $\omega = \omega_o$ of complex amplitude $\pi e^{j\phi}$ and an impulse at $\omega = -\omega_o$ of complex amplitude $\pi e^{-j\phi}$
- To analyze $g[n]$ using DFT, we employ a finite-length version of the sequence given by $\gamma[n] = \cos(\omega_o n + \phi),\ 0 \leq n \leq N-1$

Spectral Analysis of Sinusoidal Signals

- Example - Determine the 32-point DFT of a length-32 sequence $g[n]$ obtained by sampling at a rate of 64 Hz a sinusoidal signal $g_a(t)$ of frequency 10 Hz
- Since $F_T = 64$ Hz is much larger than the Nyquist frequency of 20 Hz, there is no aliasing due to sampling

Spectral Analysis of Sinusoidal Signals

- DFT magnitude plot



- Since $\gamma[n]$ is a pure sinusoid, its DTFT $\Gamma(e^{j2\pi f})$ contains two impulses at $f = \pm 10$ Hz and is zero everywhere else

Spectral Analysis of Sinusoidal Signals

- Its 32-point DFT is obtained by sampling $\Gamma(e^{j2\pi f})$ at $f = 64 \times 2/32 = 2k\ Hz, 0 \leq k \leq 31$
- The impulse at $f = 10$ Hz appears as $\Gamma[5]$ at the DFT frequency bin location

$$k = \frac{fR}{F_T} = \frac{10 \times 32}{64} = 5$$

and the impulse at $f = -10$ Hz appears as $\Gamma[27]$ at bin location $k = 32 - 5 = 27$

Spectral Analysis of Sinusoidal Signals

- Note: For an N-point DFT, first half DFT samples for $k = 0$ to $k = (N/2) - 1$ corresponds to the positive frequency axis from $f = 0$ to $f = F_T/2$ excluding the point $f = F_T/2$ and the second half for $k = N/2$ to $k = N - 1$ corresponds to the negative frequency axis from $f = -F_T/2$ to $f = 0$ excluding the point $f = 0$

Spectral Analysis of Sinusoidal Signals

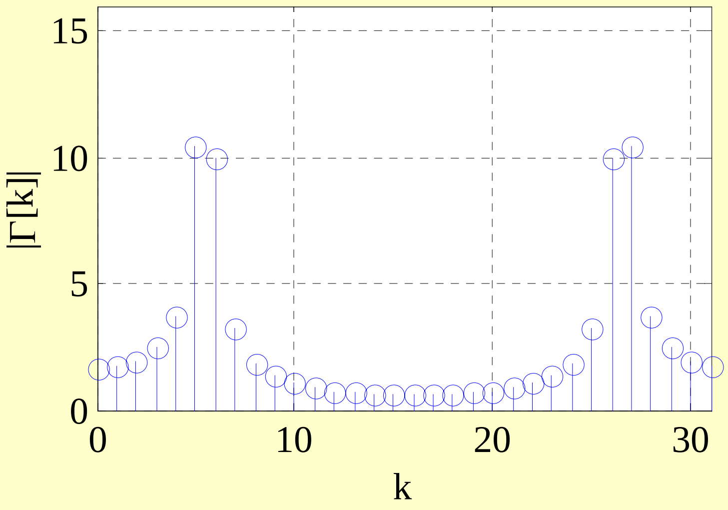
- Example - Determine the 32-point DFT of a length-32 sequence $\gamma[n]$ obtained by sampling at a rate of 64 Hz a sinusoidal signal $x_a(t)$ of frequency 11 Hz
- Since $\gamma[n]$ is a pure sinusoid, its DTFT $\Gamma(e^{j2\pi f})$ contains two impulses at $f = \pm 11$ Hz and is zero everywhere else

Spectral Analysis of Sinusoidal Signals

- Since
$$\frac{fR}{F_T} = \frac{11 \times 32}{64} = 5.5$$
the impulse at $f = 11$ Hz of the DTFT appear between the DFT bin locations $k = 5$ and $k = 6$
- Likewise, the impulse at $f = -11$ Hz of the DTFT appear between the DFT bin locations $k = 26$ and $k = 27$

Spectral Analysis of Sinusoidal Signals

- DFT magnitude plot



- Note: Spectrum contains frequency components at all bins, with two strong components at $k = 5$ and $k = 6$, and two strong components at $k = 26$ and $k = 27$

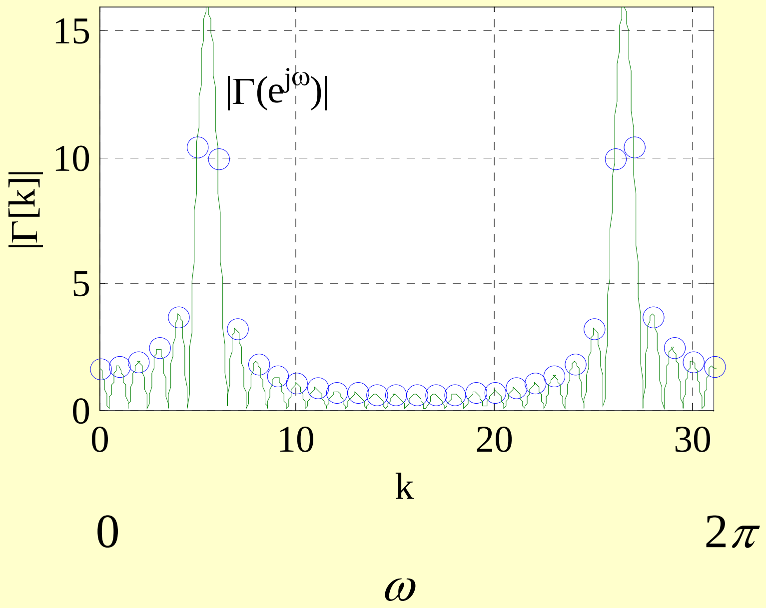
Spectral Analysis of Sinusoidal Signals

- The phenomenon of the spread of energy from a single frequency to many DFT frequency locations is called **leakage**
- To understand the cause of leakage, recall that the N-point DFT $\Gamma[k]$ of a length-N sequence $\gamma[n]$ is given by the samples of its DTFT $\Gamma(e^{j\omega})$:

$$\Gamma[k] = \Gamma(e^{j\omega_k})\Big|_{\omega_k = 2\pi k/N}, \; 0 \le k \le N-1$$

Spectral Analysis of Sinusoidal Signals

- Plot of the DTFT of the length-32 sinusoidal sequence of frequency 11 Hz sampled at 64 Hz is shown below along with its 32-point DFT



Spectral Analysis of Sinusoidal Signals

- The DFT samples are indeed obtained by the frequency samples of the DTFT
- Now the sequence

$$\gamma[n] = \cos(\omega_o n + \phi),\ \ 0 \le n \le N-1$$

has been obtained by windowing the infinite-length sinusoidal sequence $g[n]$ with a rectangular window $w[n]$:

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Spectral Analysis of Sinusoidal Signals

- The DTFT $\Gamma(e^{j\omega})$ of $\gamma[n]$ is given by

$$\Gamma(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\varphi})\Psi(e^{j(\omega-\varphi)})\,d\varphi$$

where $G(e^{j\omega})$ is the DTFT of $g[n]$:

$$G(e^{j\omega}) = \pi \sum_{\ell=-\infty}^{\infty} e^{j\phi}\delta(\omega-\omega_o+2\pi\ell) + \pi \sum_{\ell=-\infty}^{\infty} e^{-j\phi}\delta(\omega+\omega_o+2\pi\ell)$$

and $\Psi(e^{j\omega})$ is the DTFT of $w[n]$:

$$\Psi(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Spectral Analysis of Sinusoidal Signals

- Hence

$$\Gamma(e^{j\omega}) = \tfrac{1}{2} e^{j\phi} \Psi(e^{j(\omega-\omega_o)}) + \tfrac{1}{2} e^{-j\phi} \Psi(e^{j(\omega+\omega_o)})$$

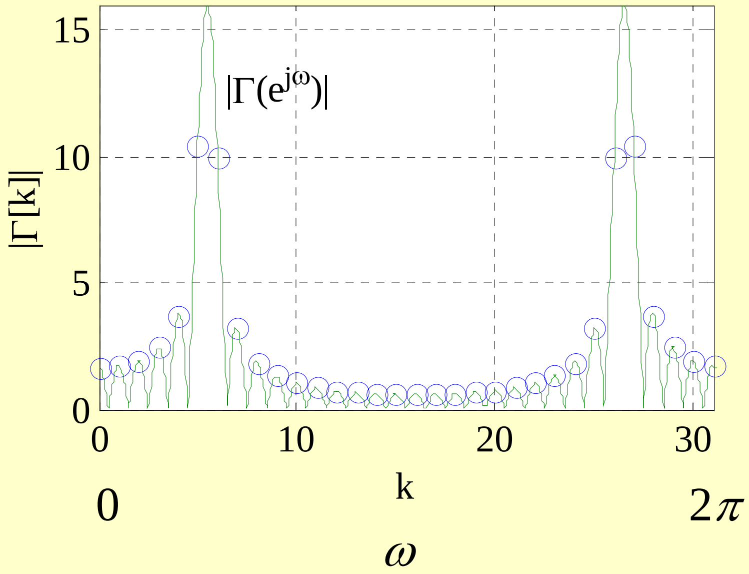
- Thus, $\Gamma(e^{j\omega})$ is a sum of frequency shifted and amplitude scaled DTFT $\Psi(e^{j\omega})$ with the amount of shifts given by $\pm\omega_o$
- For the length-32 sinusoid of frequency 11 Hz sampled at 64 Hz, normalized angular frequency of the sinusoid is 11/64 = 0.172

Spectral Analysis of Sinusoidal Signals

- Its DTFT $\Gamma(e^{j\omega})$ is obtained by frequency shifting the DTFT $\Psi(e^{j\omega})$ to the right and to the left by the amount $0.172\times 2\pi = 0.344\pi$, adding both shifted versions and scaling the sum by a factor of 1/2
- In the frequency range $0 \le \omega \le 2\pi$, which is one period of the DTFT, there are two peaks, one at 0.344π and the other at $2\pi(1-0.172)$ $= 1.656\pi$

Spectral Analysis of Sinusoidal Signals

- Plot of $|\Gamma(e^{j\omega})|$ and the 32-point DFT $|\Gamma[k]|$



Spectral Analysis of Sinusoidal Signals

- The two peaks of $|\Gamma[k]|$ located at bin locations $k = 5$ and $k = 6$ are frequency samples on both sides of the main lobe located at 0.172
- The two peaks of $|\Gamma[k]|$ located at bin locations $k = 26$ and $k = 27$ are frequency samples on both sides of the main lobe located at 0.828

Spectral Analysis of Sinusoidal Signals

- All other DFT samples are given by the samples of the sidelobes of $\Psi(e^{j\omega})$ causing the leakage of the frequency components at to other bin locations with the amount of leakage determined by the relative amplitudes of the main lobe and the sidelobes
- Since the relative sidelobe level $A_{s\ell}$ of the rectangular window is very high, there is a considerable amount of leakage to the bin locations adjacent to the main lobes

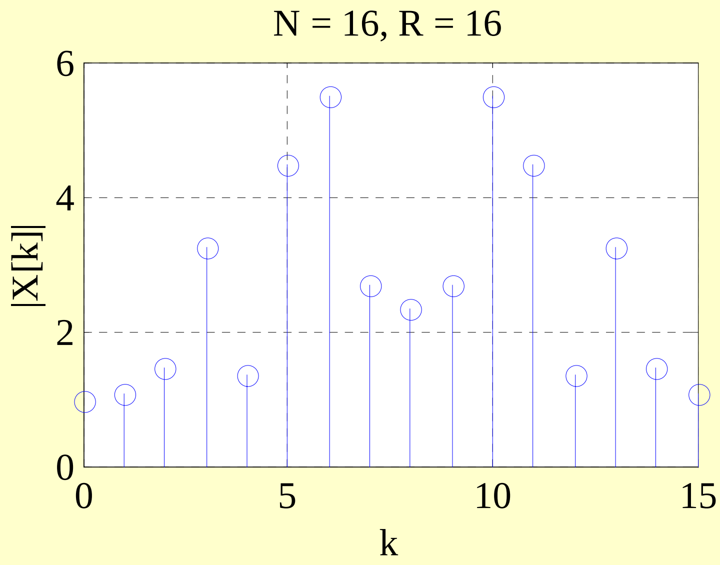
Spectral Analysis of Sinusoidal Signals

- Problem gets more complicated if the signal being analyzed has more than one sinusoid
- We now examine the effects of the length R of the DFT, the type of window being used, and its length N on the results of spectral analysis
- Consider

$$x[n] = \frac{1}{2}\sin(2\pi f_1 n) + \sin(2\pi f_2 n), \quad 0 \le n \le N-1$$

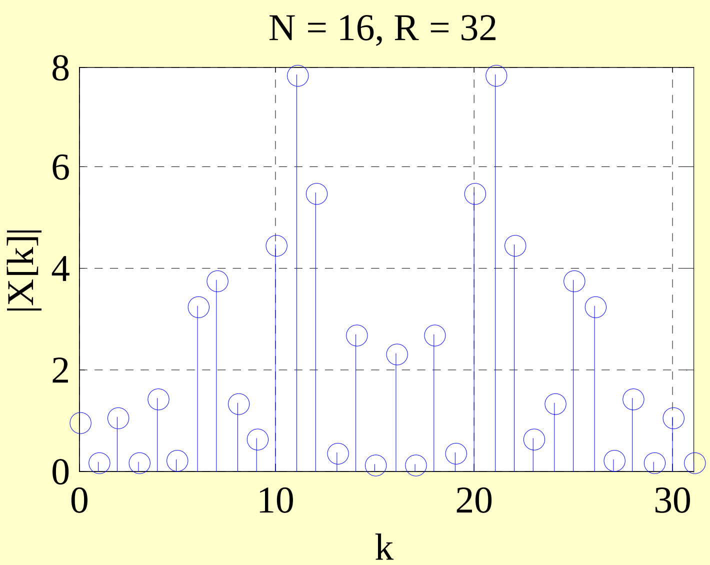
Spectral Analysis of Sinusoidal Signals

- <u>Example</u> - $N = 16, \quad f_1 = 0.22, \quad f_2 = 0.34$



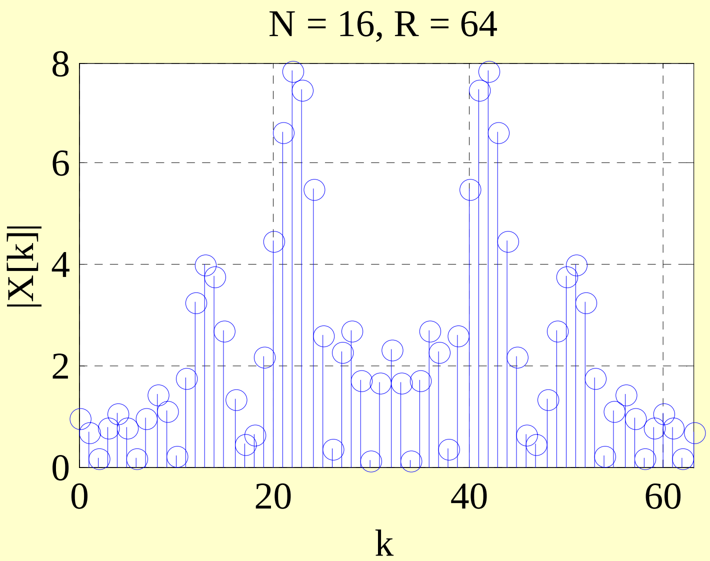
- From the above plot it is difficult to determine whether there is one or more sinusoids in $x[n]$ and the exact locations of the sinusoids

Spectral Analysis of Sinusoidal Signals



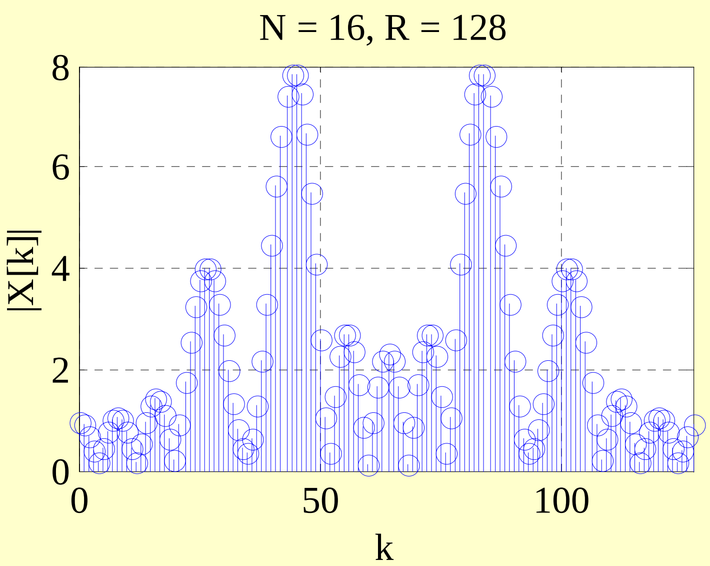
- An increase in the DFT length to $R = 32$ leads to some concentrations around $k = 7$ and $k = 11$ in the normalized frequency range from 0 to 0.5

Spectral Analysis of Sinusoidal Signals



- There are two clear peaks when $R = 64$

Spectral Analysis of Sinusoidal Signals



- An increase in the accuracy of the peak locations is obtained by increasing DFT length to R = 128 with peaks occurring at k = 27 and k =45

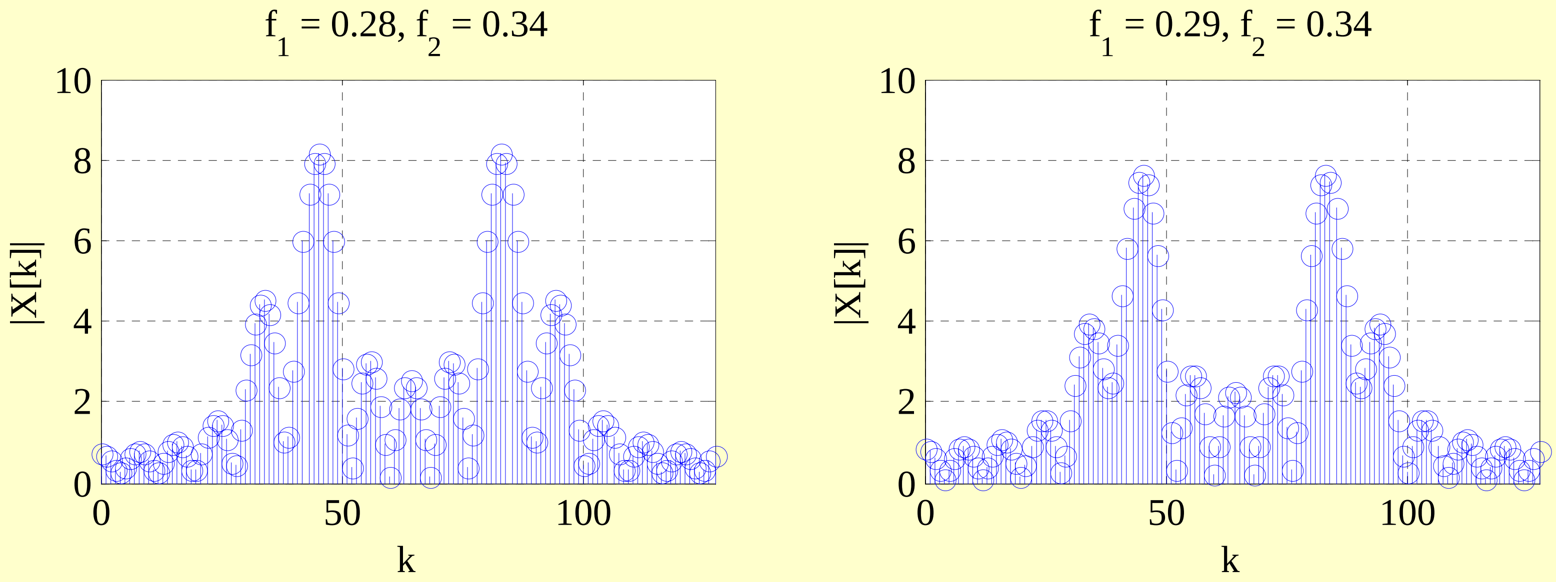
Spectral Analysis of Sinusoidal Signals

- However, there are a number of minor peaks and it is not clear whether additional sinusoids of lesser strengths are present
- General conclusion - An increase in the DFT length improves the sampling accuracy of the DTFT by reducing the spectral separation of adjacent DFT samples

Spectral Analysis of Sinusoidal Signals

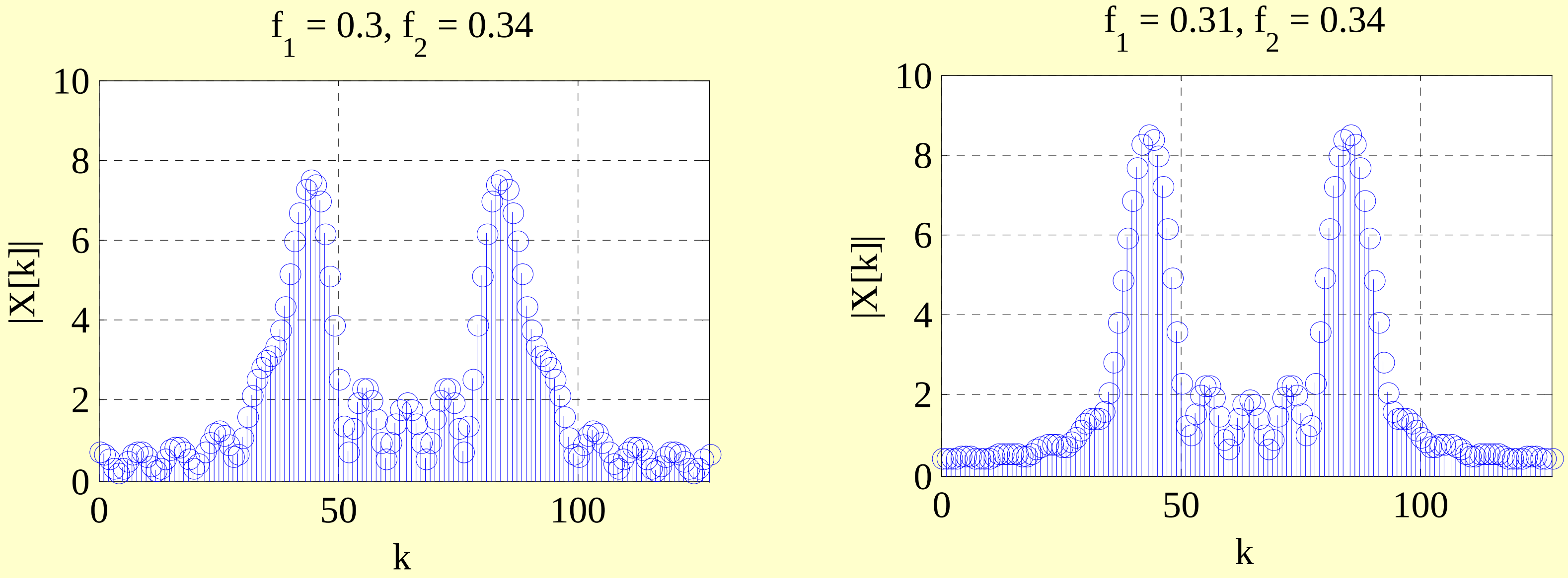
- Example - $N = 16,\ R = 128,\ f_2 = 0.34$

  f_1 varied from 0.28 to 0.31



- The two sinusoids are clearly resolved in both cases

Spectral Analysis of Sinusoidal Signals



- The two sinusoids cannot be resolved in both cases

Spectral Analysis of Sinusoidal Signals

- Reduced resolution occurs when the difference between the two frequencies becomes less than 0.4
- The DTFT $\Gamma(e^{j\omega})$ is obtained by summing the DTFTs of the two sinusoids
- As the difference between the two frequencies get smaller, the main lobes of the individual DTFTs get closer and eventually overlap

Spectral Analysis of Sinusoidal Signals

- If there is significant overlap, it will be difficult to resolve the two peaks
- Frequency resolution is determined by the width Δ_{ML} of the main lobe of the DTFT of the window
- For a length-N rectangular window $\Delta_{ML} = \frac{4\pi}{2M+1}$
- In terms of normalized frequency, for $N =$ 16, main lobe width is 0.125

Spectral Analysis of Sinusoidal Signals

- Thus, two closely spaced sinusoids windowed by a length-16 rectangular window can be resolved if the difference in the frequencies is about half the main lobe width, i.e., 0.0625
- Rectangular window has the smallest main lobe width and has the smallest frequency resolution

Spectral Analysis of Sinusoidal Signals

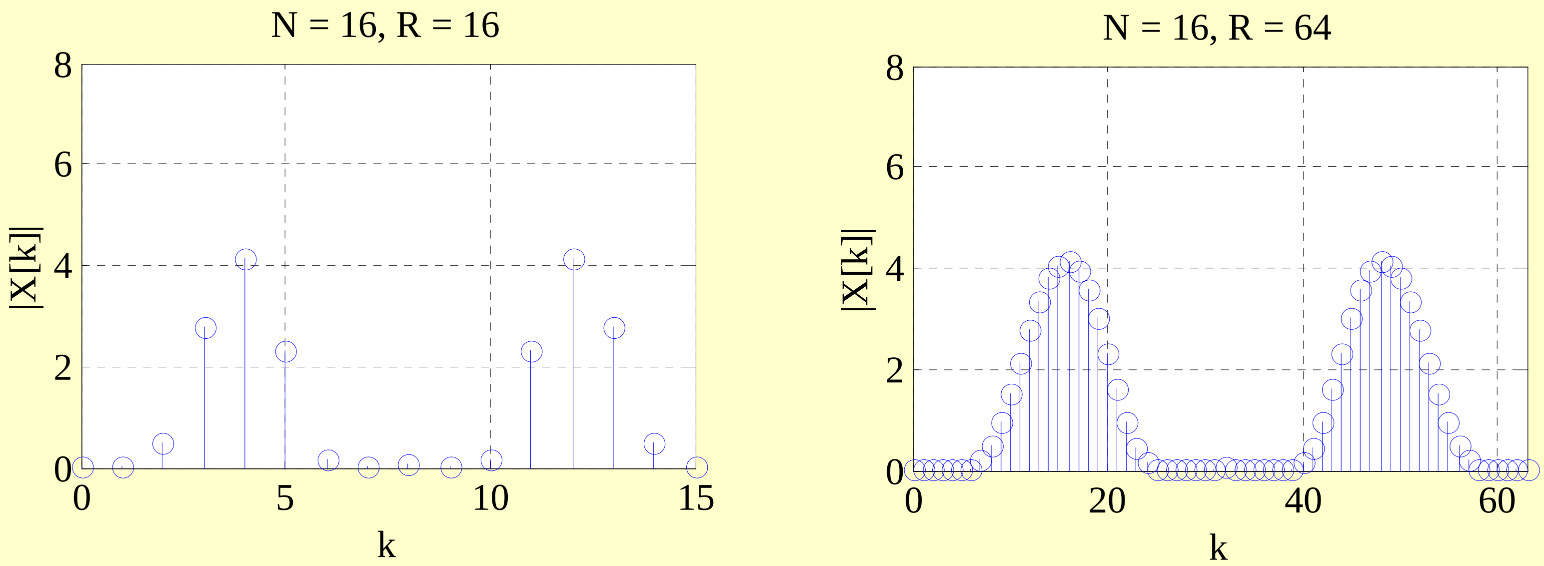
- But the rectangular window has the largest sidelobe amplitude causing considerable leakage
- From the previous two examples, it can be seen that the large amount of leakage results in minor peaks that may be identified falsely as sinusoids
- Leakage can be reduced by using other types of windows

Spectral Analysis of Sinusoidal Signals

- Example -

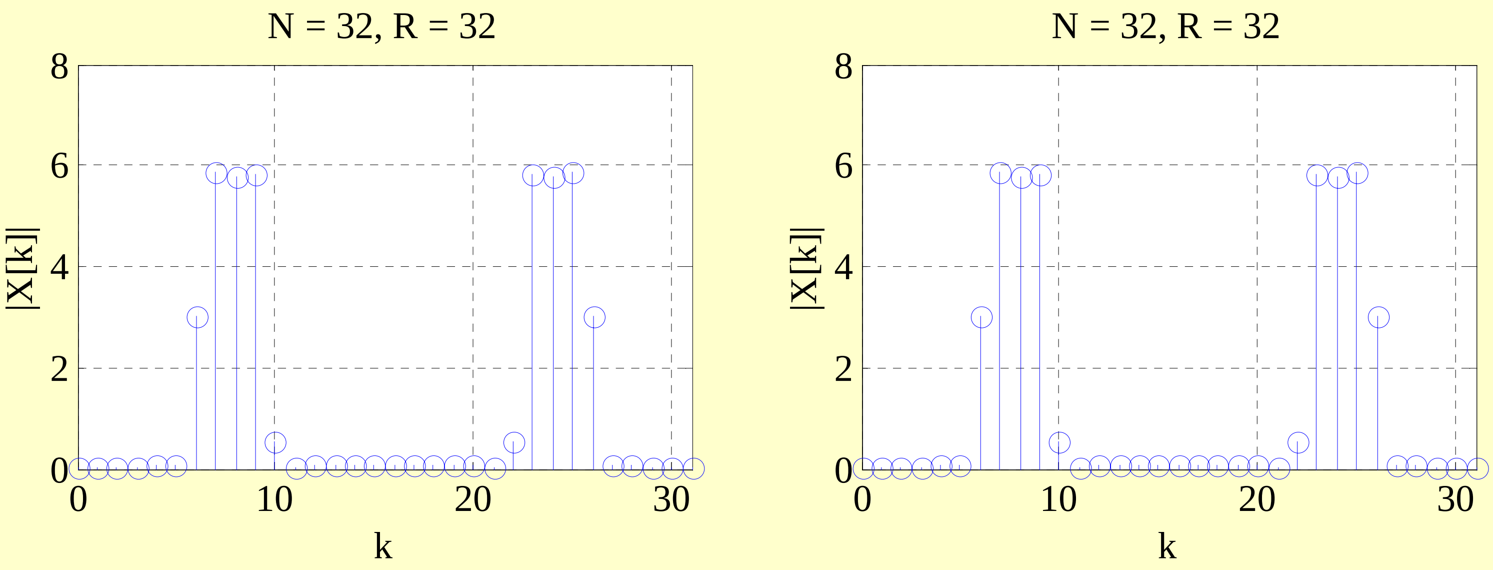
$$x[n] = 0.85\sin(2\pi \times 0.22) + \sin(2\pi \times 0.26)$$

windowed by a length-*R* Hamming window



- Leakage has been reduced considerably, but it is difficult to resolve the two sinusoids

Spectral Analysis of Sinusoidal Signals



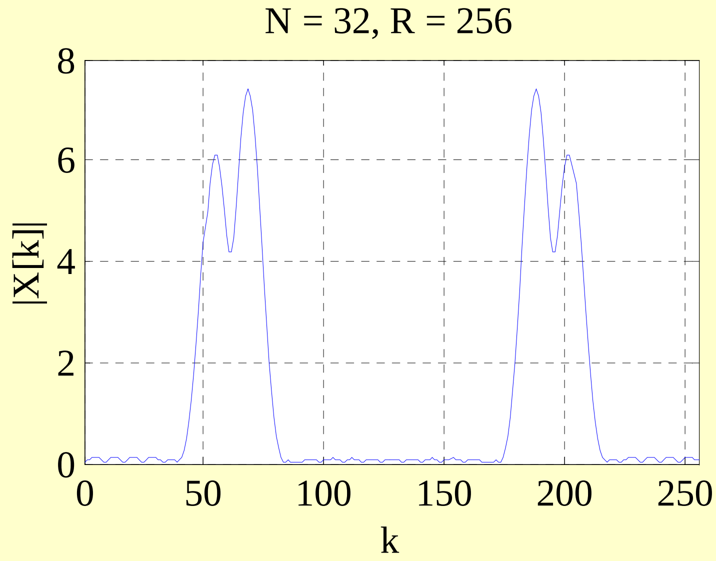
- An increase in the DFT length results in a substantial reduction of leakage, but the two sinusoids still cannot be resolved

Spectral Analysis of Sinusoidal Signals

- The main lobe width Δ_{ML} of a length-N Hamming window is $8\pi/N$
- For $N = 16$, normalized main lobe width is 0.25
- Two frequencies can thus be resolved if their difference is of the order of half of the main lobe width, i.e., 0.125
- In the example considered, the difference is 0.04, which is much smaller

Spectral Analysis of Sinusoidal Signals

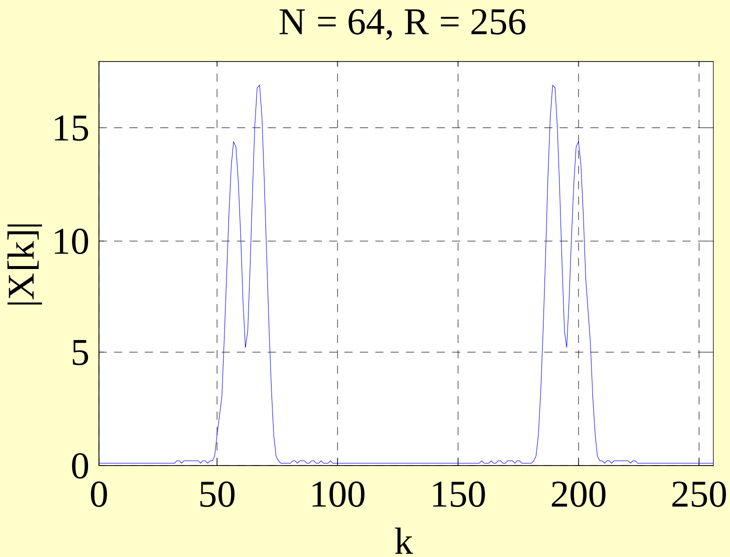
- To increase the resolution, increase the window length to $R = 32$ which reduces the main lobe width by half



- There now appears to be two peaks

Spectral Analysis of Sinusoidal Signals

- An increase of the DFT size to $R = 64$ clearly separates the two peaks
- Separation is more visible for $R = 256$



Spectral Analysis of Sinusoidal Signals

- General conclusions - Performance of DFT-based spectral analysis depends on three factors: (1) Type of window, (2) Window length, and (3) Size of the DFT
- Frequency resolution is increased by using a window with a very small main lobe width
- Leakage is reduced by using a window with a very small relative sidelobe level

Spectral Analysis of Sinusoidal Signals

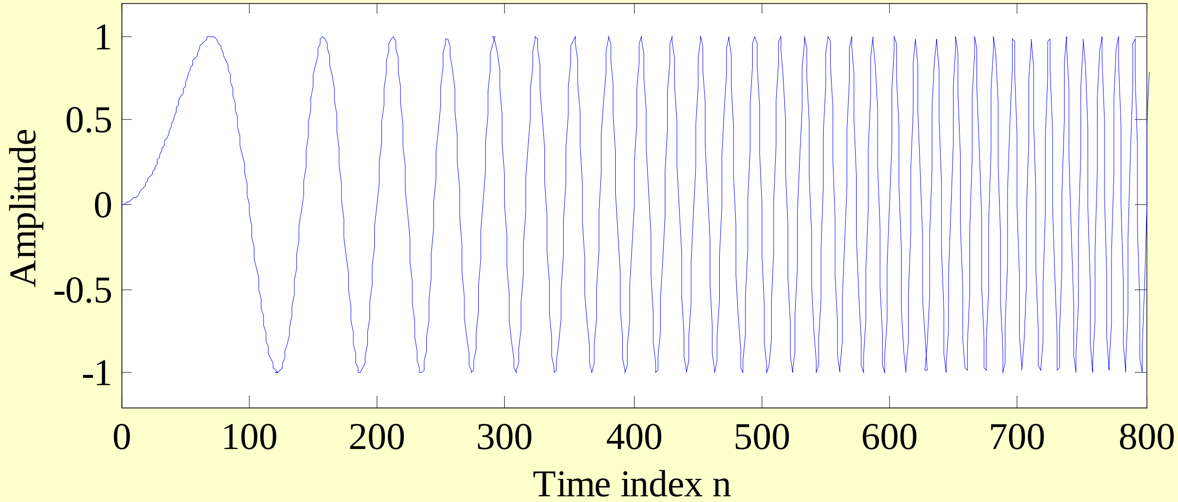
- Main lobe width can be reduced by increasing the window length
- An increase in the accuracy of locating peaks is obtained by increasing the DFT length
- It is preferable to use a DFT length that is a power of 2 so that very efficient FFT algorithms can be employed
- An increase in DFT size increases the computational complexity

Spectral Analysis of Nonstationary Signals

- DFT can be employed for spectral analysis of a length-N sinusoidal signal composed of sinusoidal signals as long as the frequency, amplitude and phase of each sinusoidal component are time-invariant and independent of N
- There are situations where the signal being analyzed is instead nonsationary, for which these parameters are time-varying

Spectral Analysis of Nonstationary Signals

- An example of a time-varying signal is the chirp signal $x[n] = A\cos(\omega_o n^2)$ and shown below for $\omega_o = 10\pi \times 10^{-5}$



- The instantaneous frequency of $x[n]$ is $2\omega_o n$

Spectral Analysis of Nonstationary Signals

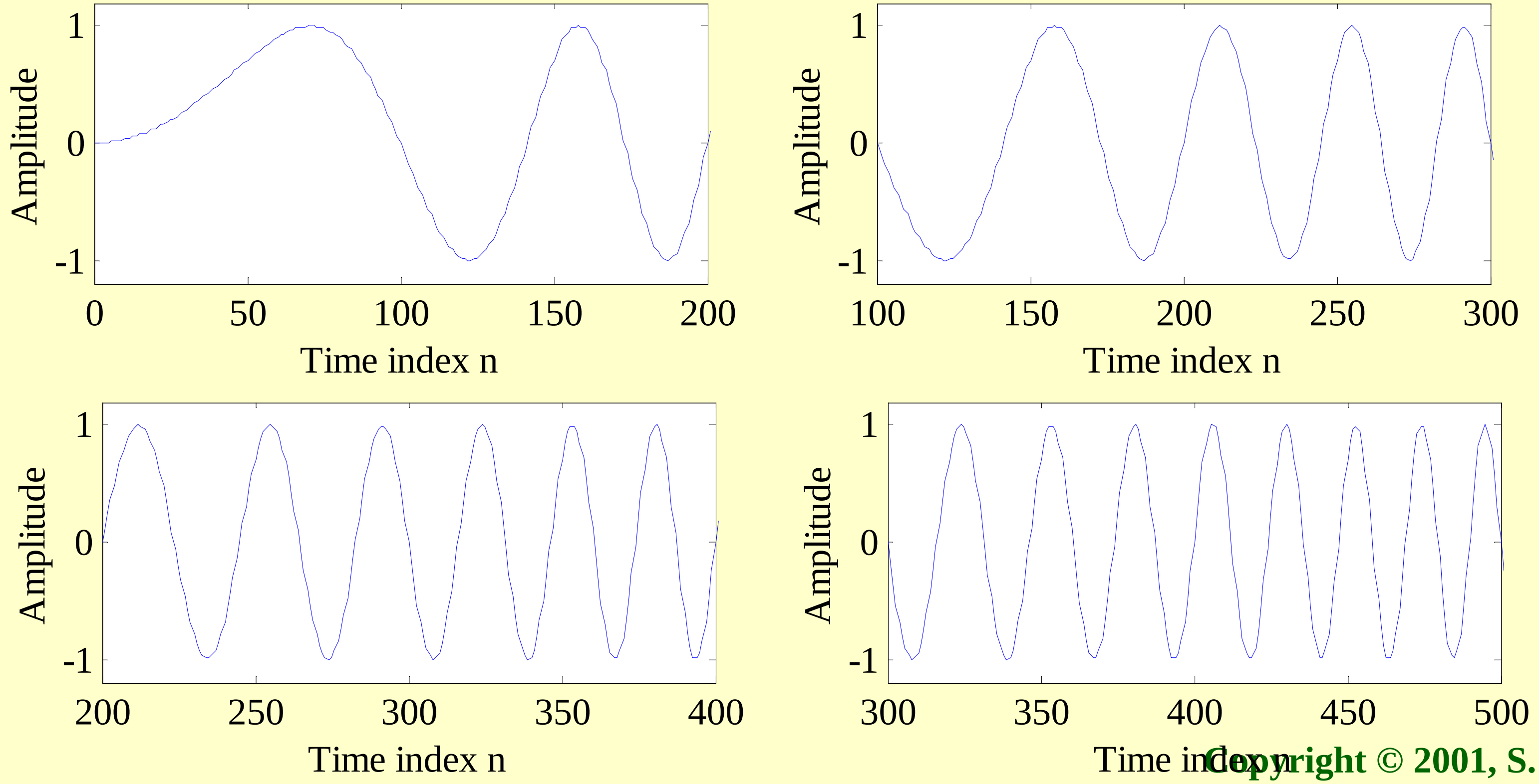
- Other examples of such nonstationary signals are speech, radar and sonar signals
- DFT of the complete signal will provide misleading results
- A practical approach would be to segment the signal into a set of subsequences of short length with each subsequence centered at uniform intervals of time and compute DFTs of each subsequence

Spectral Analysis of Nonstationary Signals

- The frequency-domain description of the long sequence is then given by a set of short-length DFTs, i.e., a **time-dependent DFT**
- To represent a nonstationary $x[n]$ in terms of a set of short-length subsequences, $x[n]$ is multiplied by a window $w[n]$ that is stationary with respect to time and move $x[n]$ through the window

Spectral Analysis of Nonstationary Signals

- Four segments of the chirp signal as seen through a stationary length-200 rectangular window



Short-Time Fourier Transform

- **Short-time Fourier transform** (STFT), also known as **time-dependent Fourier transform** of a signal $x[n]$ is defined by

$$X_{\mathrm{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m]\,w[m]\,e^{-j\omega m}$$

  where $w[m]$ is a suitably chosen window sequence

Short-Time Fourier Transform

- The STFT is also defined as given below:

$$X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-j\omega m}$$

- Here, if $w[n] = 1$ for all values of n, the STFT reduces to DTFT of $x[n]$

Short-Time Fourier Transform

- Even though DTFT of $x[n]$ exists under certain well-defined conditions, windowed $x[n]$ being of finite length ensures the existence of any $x[n]$
- Function of $w[n]$ is to extract a finite-length portion of $x[n]$ such that the spectral characteristics of the extracted section are approximately stationary

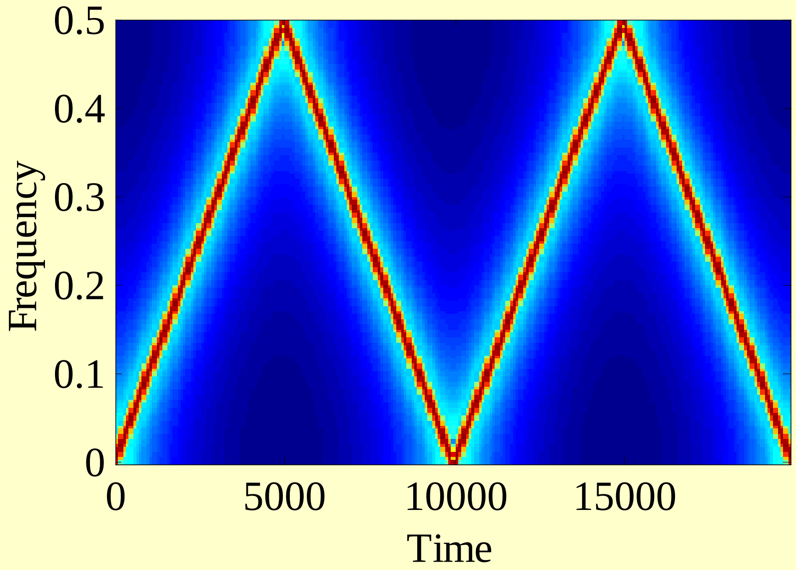
Short-Time Fourier Transform

- $X_{\text{STFT}}(e^{j\omega}, n)$ is a function of 2 variables: integer variable time index n and continuous frequency variable ω
- $X_{\text{STFT}}(e^{j\omega}, n)$ is a periodic function of ω with a period 2π
- Display of $\left| X_{\text{STFT}}(e^{j\omega}, n) \right|$ is usually referred to as **spectrogram**
- Display of spectrogram requires normally three dimensions

Short-Time Fourier Transform

- Often, STFT magnitude is plotted in two dimensions with the magnitude represented by the darkness of the plot
- Plot of STFT magnitude of chirp sequence $x[n] = A\cos(\omega_o n^2)$ with $\omega_o = 10\pi \times 10^{-5}$ for a length of 20,000 samples computed using a Hamming window of length 200 shown next

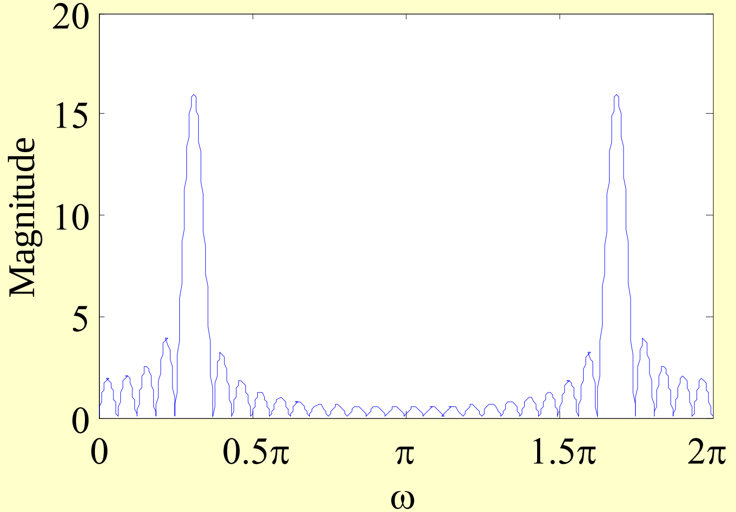
Short-Time Fourier Transform



- STFT for a given value of n is essentially the DFT of a segment of an almost sinusoidal sequence

Short-Time Fourier Transform

- Shape of the DFT of such a sequence is similar to that shown below
- Large nonzero-valued DFT samples around the frequency of the sinusoid
- Smaller nonzero-valued DFT samples at other frequency points



Short-Time Fourier Transform

- In the spectrogram plot, large-valued DFT samples show up as narrow very short dark vertical lines
- Other DFT samples show up as points
- As the instantaneous frequency of the chirp signal increases linearly with n, short dark line move up in the vertical direction
- Because of aliasing, dark line starts moving down in the vertical direction
- Spectrogram appears in a triangular shape

Short-Time Fourier Transform

- In practice, the STFT is computed at a finite set of discrete values of ω
- The STFT is accurately represented by its frequency samples as long as the number of frequency samples N is greater than window length R
- The portion of the sequence $x[n]$ inside the window can be fully recovered from the frequency samples of the STFT

Short-Time Fourier Transform

- Sampling $X_{\text{STFT}}(e^{j\omega}, n)$ at N equally spaced frequencies $\omega_k = 2\pi k / N$, with $N \geq R$ we get

$$X_{\text{STFT}}[k,n] = X_{\text{STFT}}\left(e^{j\omega}, n\right)\Big|_{\omega = 2\pi k/N}$$

$$= \sum_{m=0}^{R-1} x[n-m]w[m]e^{-j2\pi km/N}, \;\; 0 \leq k \leq N-1$$

- If $w[m] \neq 0$, $X_{\text{STFT}}[k,n]$ is simply the R-point DFT of $x[n-m]w[m]$

Short-Time Fourier Transform

- $X_{\mathrm{STFT}}[k,n]$ is a 2-D sequence and periodic in k with a period N
- Applying the IDFT we get

$$x[n-m]w[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[k,n]e^{j2\pi km/N},\ 0 \le m \le R-1$$

or

$$x[n-m] = \frac{1}{Nw[m]}\sum_{k=0}^{N-1} X[k,n]e^{j2\pi km/N},\ 0 \le m \le R-1$$

- Thus the sequence inside the window can be fully recovered from

Short-Time Fourier Transform

- The sampled STFT for a window defined in the region $0 \le m \le R-1$ is given by

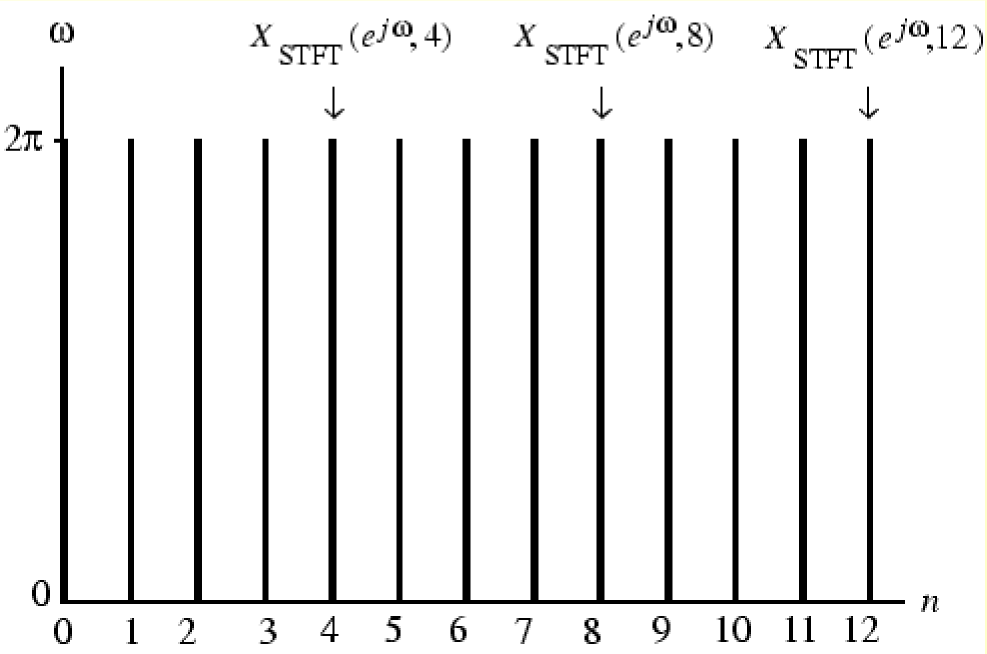
$$X_{\text{STFT}}[k, \ell L] = X_{\text{STFT}}(e^{j2\pi k/N}, \ell L)$$

$$= \sum_{m=0}^{R-1} x[\ell L - m] w[m] e^{-j2\pi km/N}$$

where ℓ and k are integers such that $-\infty < \ell < \infty$ and $0 \le k \le N-1$

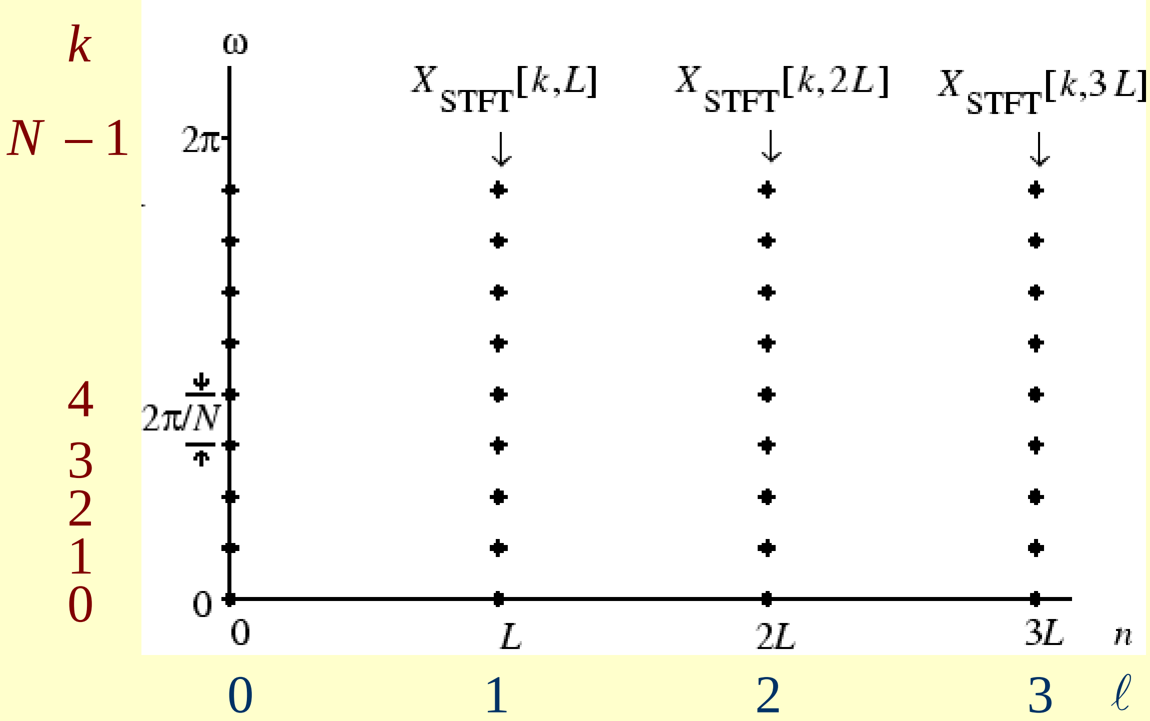
Short-Time Fourier Transform

- Figure below shows lines in the (ω,n)-plane corresponding to $X_{\mathrm{STFT}}(e^{j\omega},n)$ for $N = 9$ and $L = 4$



Short-Time Fourier Transform

- Figure below shows the grid of sampling points in (ω, n)-plane for $N = 9$ and $L = 4$



Short-Time Fourier Transform

- As we have shown it is possible to uniquely reconstruct the original signal from such a 2-D discrete representation provided

$$L \leq R \leq N$$

where N is the DFT length, R is the window length and L is the sampling period in time

STFT Window Selection

- The function of the window $w[n]$ is to extract a portion of the signal $x[n]$ and ensure that the extracted section is approximately stationary
- To the end, the window length L should be small, in particular for signals with widely varying spectral parameters

STFT Window Selection

- A decrease in the window length increases the time resolution property of the STFT
- On the other hand, the frequency resolution property of the STFT increases with an increase in the window length
- ⟹ A shorter window provides a **wideband spectrogram**, whereas, a longer window results in a **narrowband spectrogram**

STFT Window Selection

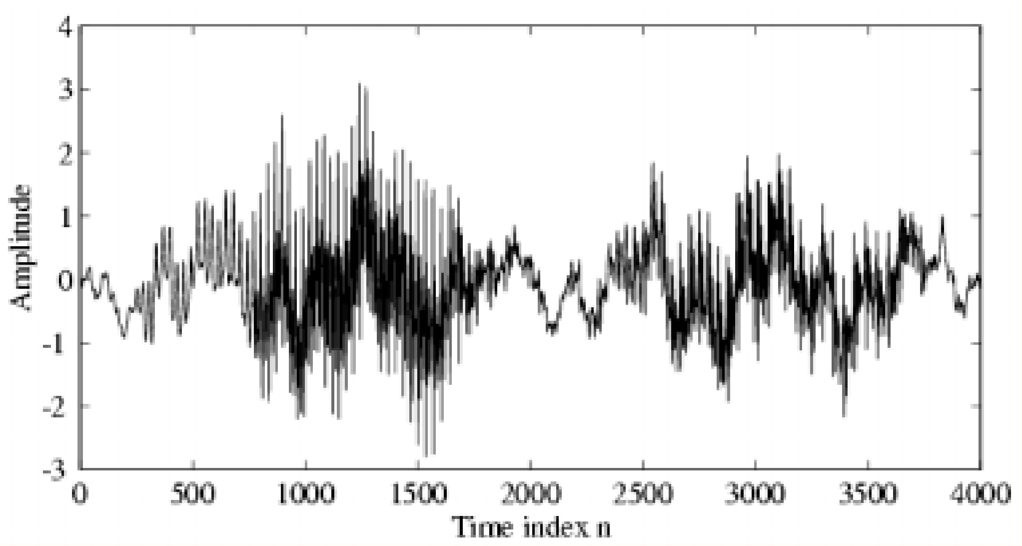
- Parameters characterizing the DTFT of a window are the main lobe width Δ_{ML} and the relative sidelobe amplitude $A_{s\ell}$
- Δ_{ML} determines the ability of the window to resolve two sinusoidal components in the vicinity of each other
- $A_{s\ell}$ controls the degree of leakage of one component into a nearby signal component

STFT Window Selection

- ⟹ In order to obtain a reasonably good estimate of the frequency spectrum of a time-varying signal, the window should be chosen to have very small $A_{s\ell}$ with a length R chosen based on the acceptable accuracy of the frequency and time resolutions

STFT Computation Using MATLAB

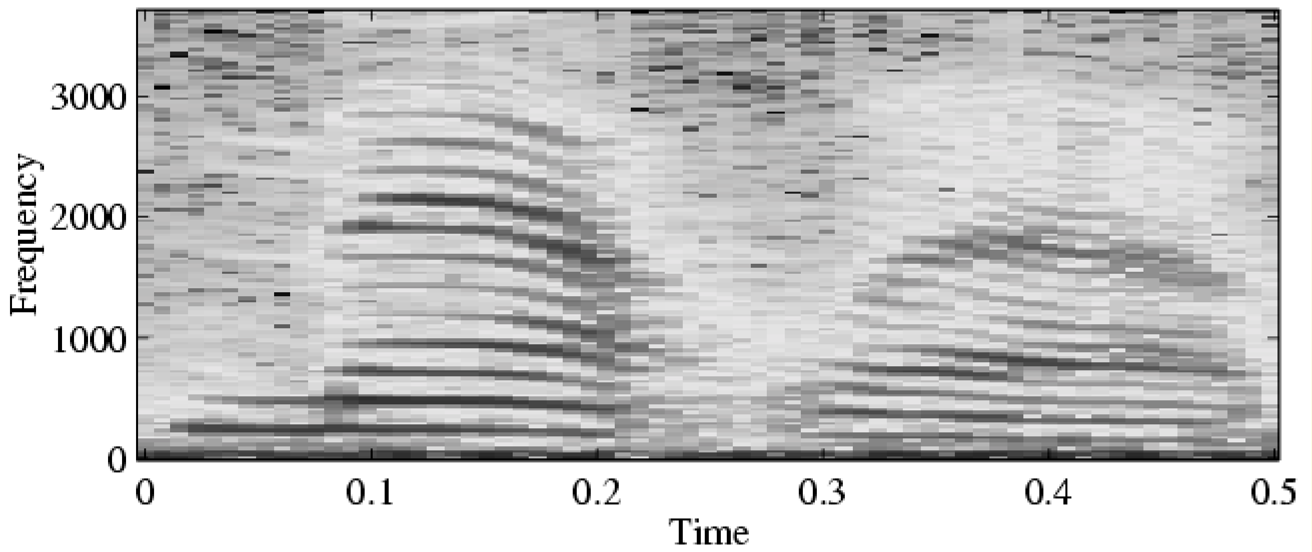
- The M-file `specgram` can be used to compute the STFT of a signal
- The application of `specgram` is illustrated next



A speech signal of duration 4001 samples

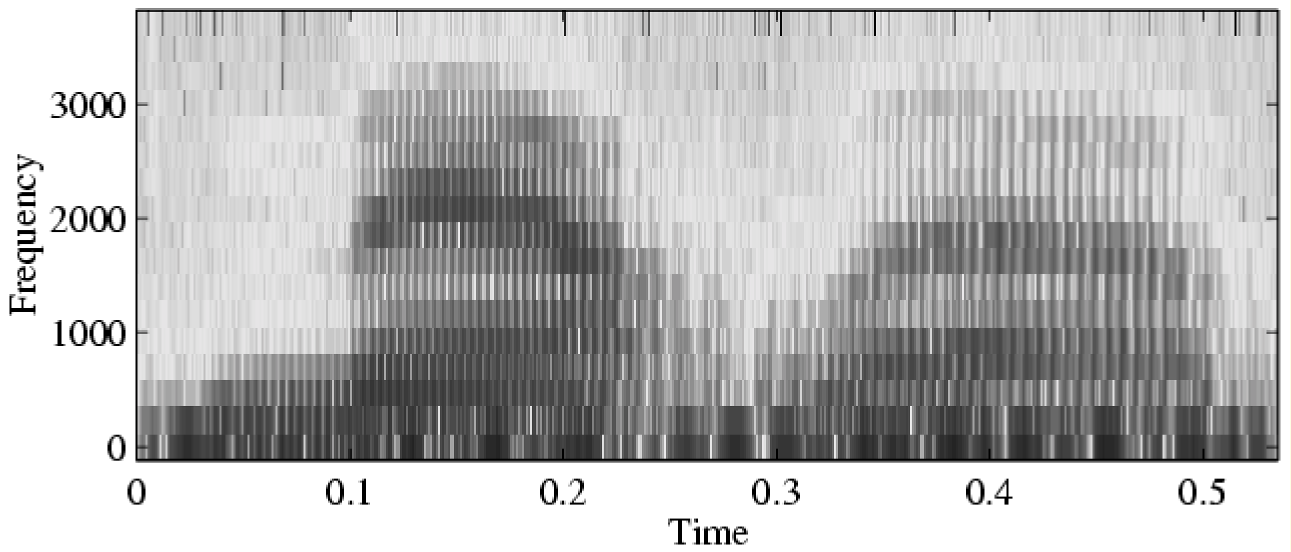
STFT Computation Using MATLAB

- Using Program 11_4 we compute the **narrowband spectrogram** of this speech signal



STFT Computation Using MATLAB

- The **wideband spectrogram** of the speech signal is shown below



- The frequency and time resolution tradeoff between the two spectrograms can be seen