Chapter 4

4.1 Let $\phi(t)$ be an arbitrary continuous-time function with a CTFT $\Phi(j\Omega)$, where $\Phi(j\Omega) = \int_{-\infty}^{\infty} \phi(t) e^{-j\Omega t} dt$. Let $\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT)$ denote the periodic continuous-time function with a period T obtained by a periodic extension of $\phi(t)$. Note that $\tilde{\phi}_T(t)$ is also given by the convolution of $\phi(t)$ with the periodic impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t + nT)$, i.e., $\tilde{\phi}_T(t) = \int_{-\infty}^{\infty} \phi(\tau) p(t - \tau) d\tau$.

Tthe CTFT of $\tilde{\phi}_T(t)$ is then given by $\mathcal{F}\{\tilde{\phi}_T(t)\} = \Phi(j\Omega) \cdot \mathcal{F}\{p(t)\}$

$$= \Phi(j\Omega) \cdot \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\big(j(\Omega - n\Omega_T)\big) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \Phi(jn\Omega_T) \delta\big(j(\Omega - n\Omega_T)\big), \quad (4\text{-}1)$$

where $\Omega_T = \frac{2\pi}{T}$.

Now, the Fourier series expansion of $\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT)$ is given by

$\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\Omega_T t}$. A CTFT of both sides of this equation is then

$$\mathcal{F}\{\tilde{\phi}_T(t)\} = \sum_{n=-\infty}^{\infty} a_n \cdot 2\pi \delta\big(j(\Omega - n\Omega_T)\big). \quad (4\text{-}2)$$

Comparing Eqs. (4-1) and (4-2) we arrive at $a_n = \frac{1}{T} \Phi(jn\Omega_T)$. Substituting this expression in the Fourier expansion of $\tilde{\phi}_T(t)$ we therefore arrive at the Poisson's sum formula $\tilde{\phi}_T(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \Phi(jn\Omega_T) e^{jn\Omega_T t}$.

4.2 Consider the continuous-time signal $g_a(t) = \sin(\Omega_m t)$ which is bandlimited to Ω_m. If we sample $g_a(t)$ at a rate $\Omega_T = 2\Omega_m$ starting at $t = 0$, then all its samples are zero. Hence, $g_a(t)$ cannot be recovered from its samples obtained by sampling it at the Nyquist rate $\Omega_T = 2\Omega_m$. As a result, $g_a(t) = \sin(\Omega_m t)$ must be sampled at a rate $\Omega_T > 2\Omega_m$ to recover it fully from its samples.

4.3 **(a)** Now, the CTFT of $y_1(t)$ is given by $Y_1(j\Omega) = \frac{1}{2\pi} G_a(j\Omega) \circledast G_a(j\Omega)$ where $G_a(j\Omega)$ denotes the CTFT of $g_a(t)$ and $\circledast$ denotes the frequency-domain convolution. The highest frequency present in $y_1(t)$ is therefore twice that of $g_a(t)$ and hence, the Nyquist frequency of $y_1(t)$ is $2\Omega_m$.

(b) The CTFT of $y_2(t)$ is given by $Y_2(j\Omega) = \int_{-\infty}^{\infty} g_a(\frac{t}{3}) e^{-j\Omega t} dt$

$= 3\int_{-\infty}^{\infty} g_a(\tau) e^{-j3\Omega\tau} d\tau = 3G_a(j3\Omega).$ The highest frequency present in $y_2(t)$ is therefore one-third of that of $g_a(t)$ and hence, the Nyquist frequency of $y_2(t)$ is $\Omega_m / 3.$

(c) The CTFT of $y_3(t)$ is given by $Y_3(j\Omega) = \int_{-\infty}^{\infty} g_a(3t) e^{-j\Omega t} dt$

$= \frac{1}{3}\int_{-\infty}^{\infty} g_a(\tau) e^{-j\Omega\tau/3} d\tau = \frac{1}{3} G_a(j\frac{\Omega}{3}).$ The highest frequency present in $y_3(t)$ is therefore three times of that of $g_a(t)$ and hence, the Nyquist frequency of $y_3(t)$ is $3\Omega_m.$

(d) The CTFT of $y_4(t)$ is given by

$$Y_4(j\Omega) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_a(t-\tau) g_a(\tau) d\tau\right] e^{-j\Omega t} dt = \int_{-\infty}^{\infty} g_a(\tau)\left[\int_{-\infty}^{\infty} g_a(t-\tau) e^{-j\Omega t} dt\right] d\tau$$

$= \int_{-\infty}^{\infty} g_a(\tau) e^{-j\Omega\tau} G_a(j\Omega) d\tau = G_a(j\Omega)\int_{-\infty}^{\infty} g_a(\tau) e^{-j\Omega\tau} d\tau = G_a(j\Omega) G_a(j\Omega).$ The highest frequency present in $y_4(t)$ is therefore the same as that of $g_a(t)$ and hence, the Nyquist frequency of $y_4(t)$ is $\Omega_m.$

(e) Now $g_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_a(j\Omega) e^{j\Omega t} d\Omega.$ Differentiating both sides of this equation we get $\frac{dg_a(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\Omega G_a(j\Omega) e^{j\Omega t} d\Omega.$ Hence, it follows that the CTFT of $y_5(t) = \frac{dg_a(t)}{dt}$ is simply $j\Omega G_a(j\Omega).$ The highest frequency present in $y_5(t)$ is therefore the same as that of $g_a(t)$ and hence, the Nyquist frequency of $y_5(t)$ is $\Omega_m.$

4.4 By Parseval's relation, the total energy of $g_a(t)$ is given by

$\mathcal{E}_{g_a}(t) = \int_{-\infty}^{\infty} |g_a(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_a(j\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\Omega_m}^{\Omega_m} |G_a(j\Omega)|^2 d\Omega.$ Likewise, the total energy of $g[n]$ is given by $\mathcal{E}_{g[n]} = \sum_{-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega$

$$= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left| \frac{1}{T} G_a(j\Omega) \right|^2 d(\Omega T) = \frac{1}{2\pi T} \int_{-\pi/T}^{\pi/T} |G_a(j\Omega)|^2 d\Omega = \frac{1}{2\pi T} \int_{-\Omega_m}^{\Omega_m} |G_a(j\Omega)|^2 d\Omega$$

$$= \frac{1}{T} \mathrm{E}_{g_a(t)}.$$

4.5 Sampling period $T = \frac{2.5}{5000} =$ sec. Hence, the sampling frequency is $F_T = \frac{1}{T} = 2000$ Hz. Therefore, the highest frequency component that could be present in the continuous-time signal has a frequency $\frac{20000}{2} = 1000$ Hz.

4.6 Since the continuous-time signal $x_a(t)$ is being sampled at 2 kHz rate, the sampled version of its i-th sinusoidal component with a frequency F_i will generate discrete-time sinusoidal signals with frequencies $F_i \pm 2000n, -\infty < n < \infty.$ Hence, the frequencies F_{im} generated in the sampled version associated with the sinusoidal components present in are as follows:

$$F_1 = 300 \text{ Hz} \Rightarrow F_{1m} = 300, 1700, 2300, \ldots \text{ Hz}$$
$$F_2 = 500 \text{ Hz} \Rightarrow F_{2m} = 500, 1500, 2500, \ldots \text{ Hz}$$
$$F_3 = 1200 \text{ Hz} \Rightarrow F_{3m} = 1200, 800, 3200, \ldots \text{ Hz}$$
$$F_4 = 2150 \text{ Hz} \Rightarrow F_{4m} = 2150, 150, 4150, \ldots \text{ Hz}$$
$$F_5 = 3500 \text{ Hz} \Rightarrow F_{5m} = 3500, 1500, 5500, 500, 7500, \ldots \text{ Hz}$$

After filtering by a lowpass filter with a cutoff at 900 Hz, the frequencies of the sinusoidal components in $y_a(t)$ are $150, 300, 500, 800$ Hz.

4.7 One possible set of values for the frequencies present in $y_a(t)$ are: $F_1 = 350$ Hz, $F_2 = 575$ Hz, $F_3 = 815$ Hz, and $F_4 = 9650$ Hz. Another possible set of values for the frequencies present in $y_a(t)$ are: $F_1 = 350$ Hz, $F_2 = 575$ Hz, $F_3 = 815$ Hz, and $F_4 = 10575$ Hz. Hence, the solution is not unique.

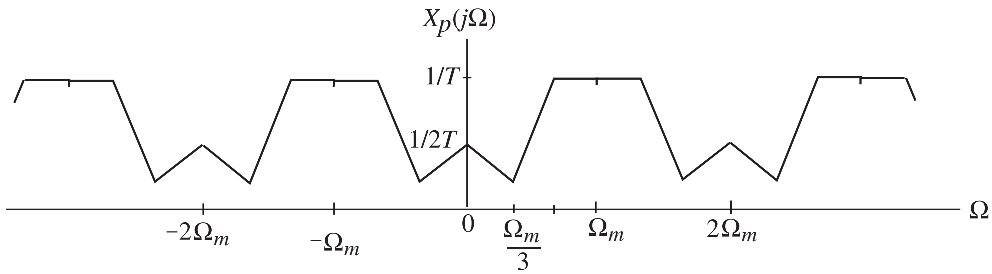
4.8 $t = nT = \frac{n}{50}.$ Therefore,

$$x[n] = 4\sin\left(\frac{20\pi n}{50}\right) - 5\cos\left(\frac{24\pi n}{50}\right) + 3\sin\left(\frac{120\pi n}{50}\right) + 2\cos\left(\frac{176\pi n}{50}\right)$$
$$= 4\sin\left(\frac{2\pi n}{5}\right) - 5\cos\left(\frac{12\pi n}{25}\right) + 3\sin\left(\frac{(10+2)\pi n}{5}\right) + 2\cos\left(\frac{(100-12)\pi n}{25}\right)$$
$$= 4\sin\left(\frac{2\pi n}{5}\right) - 5\cos\left(\frac{12\pi n}{25}\right) + 3\sin\left(\frac{2\pi n}{5}\right) + 2\cos\left(\frac{(12\pi n}{25}\right).$$

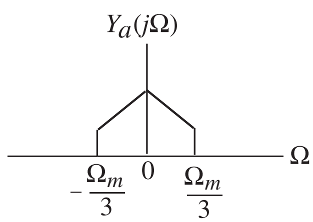
4.9 Both channels are being sampled at 45 kHz. Therefore, there are a total of $2\times 45000 = 90000$ samples/sec. Each sample is quantized using 12 bits. Hence, the total bit rate of the two channels after sampling and digitization is 108 kpbs.

4.10 $h_r(t) = \dfrac{\sin(\Omega_c t)}{\Omega_T t/2}$. Therefore, $h_r(nT) = \dfrac{\sin(\Omega_c nT)}{\Omega_T nT/2}$. Since $T = 2\pi/\Omega_T$, we have $h_r(nT) = \dfrac{\sin\left(\dfrac{2\pi\Omega_c n}{\Omega_T}\right)}{\pi n}$. For $\Omega_c = \Omega_T/2$, we thus have $h_r(nT) = \dfrac{\sin(\pi n)}{\pi n} = \delta[n]$.

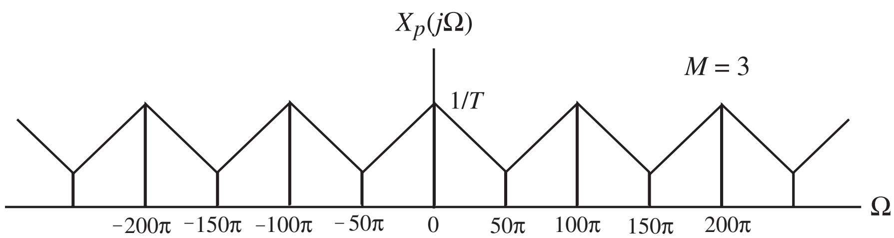
4.11 The spectrum of the sampled signal is as shown below:

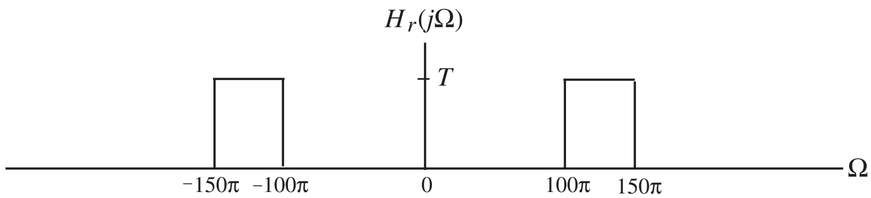


Now, $T = \dfrac{2\pi}{2\Omega_m} = \dfrac{\pi}{\Omega_m}$. As a result, $\omega_c = \dfrac{\Omega_m \pi}{3\Omega_m} = \dfrac{\pi}{3}$. Hence after lowpass filtering the spectrum of the output continuous-time signal $y_a(t)$ will be as shown below:

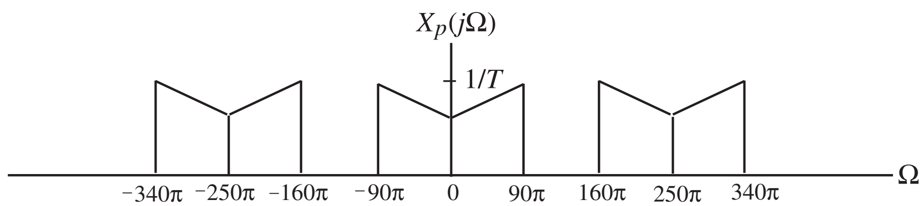


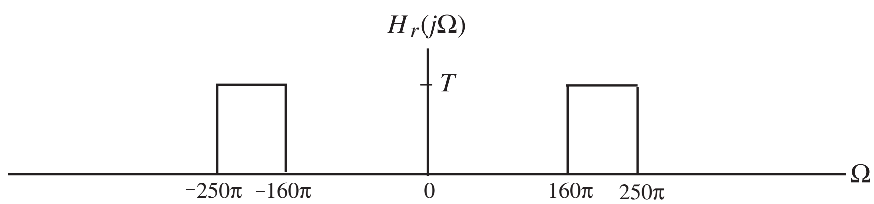
4.12 **(a)** $\Omega_1 = 100\pi, \Omega_1 = 150\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 50\pi$. Note $\Delta\Omega$ is an integer multiple of Ω_2. Hence, we choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 100\pi = \dfrac{2\times 150\pi}{M}$, which is satisfied for $M = 3$. The sampling frequency is therefore 50 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.



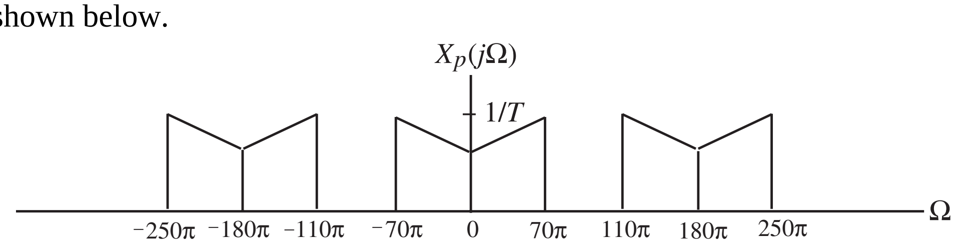


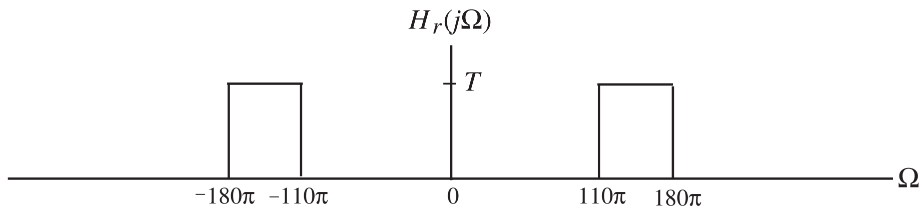
(b) $\Omega_1 = 160\pi, \Omega_1 = 250\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 90\pi$. Note $\Delta\Omega$ is not an integer multiple of Ω_2. Hence, we extend the bandwidth to the left by assuming the lowest frequency to be Ω_0 and choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_0) = \frac{2\times 250\pi}{M}$, which is satisfied for $\Omega_0 = 125\pi$ and $M = 2$. The sampling frequency is therefore 125 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.





(c) $\Omega_1 = 110\pi, \Omega_1 = 180\pi$. Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 70\pi$. Note $\Delta\Omega$ is not an integer multiple of Ω_2. Hence, we extend the bandwidth to the left by assuming the lowest frequency to be Ω_0 and choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_0) = \frac{2\times 180\pi}{M}$, which is satisfied for $\Omega_0 = 90\pi$ and $M = 2$. The sampling frequency is therefore 90 Hz. The CTFT $X_p(j\Omega)$ of the sampled sequence and the frequency response $H_r(j\Omega)$ of the desired reconstruction filter are shown below.





4.13 $\alpha_p = -20\log_{10}(1-\delta_p)$ dB and $\alpha_s = -20\log_{10}\delta_s$ dB. Therefore,

$\delta_p = 1 - 10^{-\alpha_p/20}$ and $\delta_s = 10^{-\alpha_s/20}$.

(a) $\alpha_p = 0.21$ dB and $\alpha_s = 52$ dB. Hence, $\delta_p = 0.0239$ and $\delta_s = 0.025$.

(b) $\alpha_p = 0.03$ dB and $\alpha_s = 69$ dB. Hence, $\delta_p = 0.0034$ and $\delta_s = 0.00355$.

(c) $\alpha_p = 0.33$ dB and $\alpha_s = 57$ dB. Hence, $\delta_p = 0.0373$ and $\delta_s = 0.0014$.

4.14 $H_a(s) = \dfrac{a}{s+a}$. Thus, $H_a(j\Omega) = \dfrac{a}{j\Omega + a}$, and hence,

$|H_a(j\Omega)|^2 = H_a(j\Omega)H_a(-j\Omega) = \dfrac{a}{j\Omega+a}\cdot\dfrac{a}{-j\Omega+a} = \dfrac{a^2}{\Omega^2+a^2}$. As Ω increases from 0 to ∞, it can be seen that the square-magnitude function $|H_a(j\Omega)|^2$ and hence, the magnitude function $|H_a(j\Omega)| = \dfrac{a}{\sqrt{\Omega^2+a^2}}$ decreases monotonically from $|H_a(j0)| = 1$ to $|H_a(j\infty)| = 0$. Let Ω_c denote the 3–dB cutoff frequency. Then $|H_a(j\Omega_c)|^2 = \dfrac{a^2}{\Omega_c^2+a^2} = \dfrac{1}{2}$, which implies $\Omega_c = a$.

4.15 $G_a(s) = \dfrac{s}{s+a}$. Thus, $G_a(j\Omega) = \dfrac{j\Omega}{j\Omega + a}$, and hence,

$|G_a(j\Omega)|^2 = G_a(j\Omega)G_a(-j\Omega) = \dfrac{j\Omega}{j\Omega+a}\cdot\dfrac{-j\Omega}{-j\Omega+a} = \dfrac{\Omega^2}{\Omega^2+a^2}$. As Ω increases from 0 to ∞, it can be seen that the square-magnitude function $|G_a(j\Omega)|^2$ and hence, the magnitude function $|G_a(j\Omega)| = \dfrac{\Omega}{\sqrt{\Omega^2+a^2}}$ increases monotonically from $|G_a(j0)| = 0$ to $|G_a(j\infty)| = 1$. Let Ω_c denote the 3–dB cutoff frequency. Then $|G_a(j\Omega_c)|^2 = \dfrac{\Omega_c^2}{\Omega_c^2+a^2} = \dfrac{1}{2}$, which implies $\Omega_c = a$.

4.16 $H_a(s) = \dfrac{a}{s+a} = \dfrac{1}{2}\left(1 - \dfrac{s-a}{s+a}\right) = \dfrac{1}{2}\big(A_1(s) - A_2(s)\big)$ and $G_a(s) = \dfrac{s}{s+a}$

$= \dfrac{1}{2}\left(1 + \dfrac{s-a}{s+a}\right) = \dfrac{1}{2}\big(A_1(s) + A_2(s)\big)$, where $A_1(s) = 1$ and $A_2(s) = \dfrac{s-a}{s+a}$. Now,

$|A_1(j\Omega)| = 1$ and $|A_2(j\Omega)|^2 = \frac{j\Omega - a}{j\Omega + a} \cdot \frac{-j\Omega - a}{-j\Omega + a} = \frac{j\Omega - a}{j\Omega + a} \cdot \frac{j\Omega + a}{j\Omega - a} = 1$ for all values of Ω, $A_1(s)$ and $A_2(s)$ are allpass functions.

4.17 $H_a(s) = \frac{bs}{s^2 + bs + \Omega_o^2}$. Thus, $H_a(j\Omega) = \frac{jb\Omega}{jb\Omega + \Omega_o^2 - \Omega^2}$ and hence,

$$|H_a(j\Omega)|^2 = H_a(j\Omega)H_a(-j\Omega) = \frac{jb\Omega}{jb\Omega + \Omega_o^2 - \Omega^2} \cdot \frac{-jb\Omega}{-jb\Omega + \Omega_o^2 - \Omega^2}$$

$= \frac{b^2\Omega^2}{(\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2}$. At $\Omega = 0$, $|H_a(j0)| = 0$, at $\Omega = \infty$, $|H_a(j\infty)| = 0$, and at $\Omega = \Omega_o$, $|H_a(j\Omega)|$ has the maximum value of 1. Now,

$\frac{d|H_a(j\Omega)|^2}{d\Omega} = \frac{2b^2\Omega(\Omega_o^2 - \Omega^2)(\Omega_o^2 + \Omega^2)}{\left((\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2\right)^2}$. It therefore follows that in the frequency range $0 \le \Omega < \Omega_o$, $\frac{d|H_a(j\Omega)|^2}{d\Omega} > 0$, and in the frequency range $\Omega_o < \Omega < \infty$, $\frac{d|H_a(j\Omega)|^2}{d\Omega} < 0$. Hence, in the frequency range $0 \le \Omega < \Omega_o$, $|H_a(j\Omega)|^2$ is a monotonically increasing function of Ω and in the frequency range $\Omega_o < \Omega < \infty$, $|H_a(j\Omega)|^2$ is a monotonically decreasing function of Ω. Or in other words, $H_a(s)$ has a bandpass magnitude response. The 3–dB cutoff frequencies are given by the solution of $\frac{b^2\Omega_c^2}{(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2} = \frac{1}{2}$, or,

$(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2 = 2b^2\Omega_c^2$, i.e., $\Omega_c^4 - (b^2 + 2\Omega_o^2)\Omega_c^2 + \Omega_o^4 = 0$. Substituting $x = \Omega_c^2$ in the last equation we get $x^2 - (b^2 + 2\Omega_o^2)x + \Omega_o^4 = 0$. Let $x_1 = \Omega_1^2$ and $x_2 = \Omega_2^2$ be the two roots of this quadratic equation. Then, $x_1 x_2 = \Omega_1^2\Omega_2^2 = \Omega_o^4$ and $x_1 + x_2 = \Omega_1^2 + \Omega_2^2 = b^2 + 2\Omega_o^2$. Therefore, $\Omega_1\Omega_2 = \Omega_o^2$. From the last two equations we get $\Omega_1^2 + \Omega_2^2 - 2\Omega_1\Omega_2 = (\Omega_2 - \Omega_1)^2 = b^2 + 2\Omega_o^2 - 2\Omega_o^2 = b^2$. Hence, $\Omega_2 - \Omega_1 = b$.

4.18 $G_a(s) = \dfrac{s^2 + \Omega_o^2}{s^2 + bs + \Omega_o^2}$. Thus, $G_a(j\Omega) = \dfrac{\Omega_o^2 - \Omega^2}{jb\Omega + \Omega_o^2 - \Omega^2}$ and hence,

$$\left|G_a(j\Omega)\right|^2 = G_a(j\Omega)G_a(-j\Omega) = \frac{\Omega_o^2 - \Omega^2}{jb\Omega + \Omega_o^2 - \Omega^2} \cdot \frac{\Omega_o^2 - \Omega^2}{-jb\Omega + \Omega_o^2 - \Omega^2}$$

$= \dfrac{(\Omega_o^2 - \Omega^2)^2}{(\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2}$. Note $\left|G_a(j0)\right| = \left|G_a(j\infty)\right| = 1$ and $\left|G_a(j\Omega_o)\right| = 0$.

Now, $\dfrac{d\left|G_a(j\Omega)\right|^2}{d\Omega} = \dfrac{2b^2\Omega^3(\Omega^2 - \Omega_o^2)}{\left((\Omega_o^2 - \Omega^2)^2 + b^2\Omega^2\right)^2}$. It therefore follows that in the frequency range $0 \le \Omega < \Omega_o$, $\dfrac{d\left|G_a(j\Omega)\right|^2}{d\Omega} < 0$, and in the frequency range $\Omega_o < \Omega < \infty$, $\dfrac{d\left|G_a(j\Omega)\right|^2}{d\Omega} > 0$. Hence, in the frequency range $0 \le \Omega < \Omega_o$, $\left|G_a(j\Omega)\right|^2$ is a monotonically decreasing function of Ω and in the frequency range $\Omega_o < \Omega < \infty$, $\left|G_a(j\Omega)\right|^2$ is a monotonically increasing function of Ω. Or in other words, $G_a(s)$ has a bandstop magnitude response.

The $3-$dB cutoff frequencies are given by the solution of $\dfrac{(\Omega_o^2 - \Omega_c^2)^2}{(\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2} = \dfrac{1}{2}$, or, $2(\Omega_o^2 - \Omega_c^2)^2 = (\Omega_o^2 - \Omega_c^2)^2 + b^2\Omega_c^2$, i.e., $\Omega_c^4 - (b^2 + 2\Omega_o^2)\Omega_c^2 + \Omega_o^4 = 0$. This last equation is exactly the same as in solution of Problem 4.18 from which we get $\Omega_1\Omega_2 = \Omega_o^2$ and $\Omega_2 - \Omega_1 = b$.

4.19 $H_a(s) = \dfrac{bs}{s^2 + bs + \Omega_o^2} = \dfrac{1}{2}\left(1 - \dfrac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2}\right) = \dfrac{1}{2}\left(A_1(s) - A_2(s)\right)$ and

$G_a(s) = \dfrac{s^2 + \Omega_o^2}{s^2 + bs + \Omega_o^2} = \dfrac{1}{2}\left(1 + \dfrac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2}\right) = \dfrac{1}{2}\left(A_1(s) + A_2(s)\right)$, where

$A_1(s) = 1$ and $A_2(s) = \dfrac{s^2 - bs + \Omega_o^2}{s^2 + bs + \Omega_o^2}$. Now, $\left|A_1(j\Omega)\right| = 1$ and $\left|A_2(j\Omega)\right|^2$

$= A_2(j\Omega)A_2(-j\Omega) = \dfrac{-\Omega^2 - jb\Omega + \Omega_o^2}{-\Omega^2 + jb\Omega + \Omega_o^2} \cdot \dfrac{-\Omega^2 + jb\Omega + \Omega_o^2}{-\Omega^2 - jb\Omega + \Omega_o^2} = 1$, for all values of Ω, $A_1(s)$ and $A_2(s)$ are allpass functions.

4.20 **(a)** Let $A_i(s)=\dfrac{s+\lambda_i^*}{s-\lambda_i}$. Since the pole of $A_i(s)$ is strictly in the left-half s–plane and hence, $A_i(s)$ is causal and stable. Now $|A_i(j\Omega)|^2=A_i(j\Omega)A_i^*(j\Omega)$ $=\dfrac{j\Omega+\lambda_i^*}{j\Omega-\lambda_i}\cdot\dfrac{-j\Omega+\lambda_i}{-j\Omega-\lambda_i^*}=\dfrac{j\Omega+\lambda_i^*}{j\Omega-\lambda_i}\cdot\dfrac{j\Omega-\lambda_i}{j\Omega+\lambda_i^*}=1$. Hence, $A_i(s)$ is an allpass function. Since, $A(s)=\prod_{i=1}^{N}A_i(s)$, it is a product of causal, stable allpass functions, and as a result, is also a causal, stable allpass function.

(b) $|A_i(s)|^2=A_i(s)A_i^*(s)=\dfrac{s+\lambda_i^*}{s-\lambda_i}\cdot\dfrac{s^*+\lambda_i}{s^*-\lambda_i^*}=\dfrac{|s|^2+|\lambda_i|^2+2\operatorname{Re}\{s\lambda_i\}}{|s|^2+|\lambda_i|^2-2\operatorname{Re}\{s^*\lambda_i\}}$. Let $s=\sigma+j\Omega$ and $\lambda_i=a_i+jb_i$. Then $|A_i(s)|^2=\dfrac{(|s|^2+|\lambda_i|^2-2\Omega b_i)+2\sigma a_i}{(|s|^2+|\lambda_i|^2-2\Omega b_i)-2\sigma a_i}$.

Since $a_i<0$, it follows from the above that $|A_i(s)|^2<1$ for $\sigma>0$, $|A_i(s)|^2=1$ for $\sigma=0$, and $|A_i(s)|^2>1$ for $\sigma<0$.

4.21 $|H_a(j\Omega)|^2=\dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$. Define $D(\Omega)=\dfrac{1}{|H_a(j\Omega)|^2}=1+(\Omega/\Omega_c)^{2N}$. It follows then $\dfrac{d^kD(\Omega)}{d\Omega^k}=2N(2N-1)\cdots(2N-k+1)\dfrac{\Omega^{2N-k}}{\Omega_c^{2N}}$. Therefore,

$\left.\dfrac{d^kD(\Omega)}{d\Omega^k}\right|_{\Omega=0}=0$ for $k=1,2,\ldots,N-1$, or, equivalently, $\left.\dfrac{d^k|H_a(j\Omega)|}{d\Omega^k}\right|_{\Omega=0}=0$ for $k=1,2,\ldots,N-1$.

4.22 $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right)=-0.25$. Therefore, $\varepsilon^2=10^{0.025}-1=0.0593$. Next, from $10\log_{10}\left(\dfrac{1}{A^2}\right)=-25$, we get $A^2=10^{2.5}=316.2278$. Now, $\dfrac{1}{k}=\dfrac{\Omega_s}{\Omega_p}=\dfrac{6}{1.5}=4$ and $\dfrac{1}{k_1}=\sqrt{\dfrac{A^2-1}{\varepsilon^2}}=\sqrt{\dfrac{315.2278}{0.0593}}=72.9381$. Hence, $N=\dfrac{\log_{10}(1/k_1)}{\log_{10}(1/k)}=3.0943$.
We choose $N=4$ as the filter order.
To verify using MATLAB, we use the code fragment

```
[N,Wn]=buttord(2*pi*1500,2*pi*6000,0.25,25,'s');
```

which yields `N = 4` and `Wn = 18365.51286`.

4.23 The poles are given by $p_\ell = e^{j\pi(5+2\ell)/}$, $1 \le \ell \le 6$. Hence,

$p_1 = e^{j(7\pi/12)} = -0.2588 + j0.9659,\ p_2 = e^{j(9\pi/12)} = -0.7071 + j0.7071,$

$p_3 = e^{j(9\pi/12)} = -0.9659 + j0.2588,\ p_4 = e^{j(11\pi/12)} = p_3^* = -0.9659 - j0.2588,$

$p_5 = e^{j(13\pi/12)} = p_2^* = -0.7071 - j0.7071,\ p_6 = e^{j(15\pi/12)} = p_1^* = -0.2588 - j0.9659.$

The poles can also be determined in MATLAB using the statement
`[z,p,k]=buttap(6)` which yields

```
p =
  -0.2588 + 0.9659i
  -0.2588 - 0.9659i
  -0.7071 + 0.7071i
  -0.7071 - 0.7071i
  -0.9659 + 0.2588i
  -0.9659 - 0.2588i
```

4.24 From Eq. (4.41) of text, $T_N(\Omega) = 2\Omega T_{N-1}(\Omega) - T_{N-2}(\Omega)$, where $T_N(\Omega)$ is defined in Eq. (4.40).

<u>Case 1</u>: $|\Omega| \le 1$. Making use of Eq. (4.40) in Eq. (4.41) we get

$$\begin{aligned}
T_N(\Omega) &= 2\Omega\cos\left((N-1)\cdot\cos^{-1}\Omega\right) - \cos\left((N-2)\cdot\cos^{-1}\Omega\right) \\
&= 2\Omega\cos\left(N\cos^{-1}\Omega - \cos^{-1}\Omega\right) - \cos\left(N\cos^{-1}\Omega - 2\cos^{-1}\Omega\right) \\
&= 2\Omega\left[\cos(N\cos^{-1}\Omega)\cos(\cos^{-1}\Omega) + \sin(N\cos^{-1}\Omega)\sin(\cos^{-1}\Omega)\right] \\
&\qquad - \left[\cos(N\cos^{-1}\Omega)\cos(2\cos^{-1}\Omega) + \sin(N\cos^{-1}\Omega)\sin(2\cos^{-1}\Omega)\right] \\
&= 2\Omega\cos(N\cos^{-1}\Omega)\cos(\cos^{-1}\Omega) - \cos(N\cos^{-1}\Omega)\cos(2\cos^{-1}\Omega) \\
&= 2\Omega^2\cos(N\cos^{-1}\Omega) - \cos(N\cos^{-1}\Omega)\left[2\cos^2(\cos^{-1}\Omega) - 1\right] \\
&= \cos(N\cos^{-1}\Omega)\left[2\Omega^2 - 2\Omega^2 + 1\right] = \cos(N\cos^{-1}\Omega).
\end{aligned}$$

<u>Case 2</u>: $|\Omega| > 1$. Making use of Eq. (4.40) in Eq. (4.41) we get

$T_N(\Omega) = 2\Omega\cosh\left((N-1)\cdot\cosh^{-1}\Omega\right) - \cosh\left((N-2)\cdot\cosh^{-1}\Omega\right)$ Using the trigonometric identities

$\cosh(A-B) = \cosh(A)\cosh(B) - \sinh(A)\sinh(B)$, $\sinh(2A) = 2\sinh(A)\cosh(A)$, and

$\cosh(2A) = 2\cosh^2(A) - 1$, and following a similar algebra as in Case 1, we can show

$T_N(\Omega) = \cosh(N\cosh^{-1}\Omega).$

4.25 From the solution of Problem 4.22, we have $\dfrac{1}{k} = 4$ and $\dfrac{1}{k_1} = 72.9381$. Hence,

$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.4151.$ We choose the filter order as $N = 3.$

The filter order obtained using the MATLAB statement
`[N,Wn]=cheb1ord(2*pi*1500,2*pi*6000,0.25, 25, 's')` results in `N=3`.

4.26 $10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.25,$ which yields $\varepsilon = 0.2434.$ $10\log_{10}\left(\frac{1}{A^2}\right) = -25,$ which yields $A^2 = 316.2278.$ Now, $k = \frac{\Omega_p}{\Omega_s} = \frac{1500}{6000} = 0.25$ and $k_1 = \frac{\varepsilon}{\sqrt{A^2-1}} = \frac{0.2434}{\sqrt{315.2278}} = = 0.0137.$ Substituting the value of k in Eq. (4..55a) we get $k' = 0.9682.$ Then from Eq. (4.55b) we get $\rho_0 = 0.004.$ Substituting the value ρ_0 in Eq. (4.55c) we get $\rho = 0.004.$ Finally, from Eq. (4.54) we arrive at $N = 2.0591.$ We choose the next higher integer as the filter order $N = 3.$

The filter order obtained using the MATLAB statement
`[N,Wn]=ellipord(2*pi*1500,2*pi*6000,0.25, 25, 's')` results in `N=3`.

4.27 $B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s),$ where $B_1(s) = s+1$ and $B_2(s) = s^2+3s+3.$

(a) Thus, $B_3(s) = 5B_2(s) + s^2 B_1(s) = 5(s^2+3s+3) + s^2(s+1) = s^3 + 6s^2 + 15s + 15,$

$B_4(s) = 7B_3(s) + s^2 B_2(s) = 7(s^3+6s^2+15s+15) + s^2(s^2+3s+3)$

$= s^4 + 10s^3 + 45s^2 + 105s + 105.$

(b) $B_5(s) = 9B_4(s) + s^2 B_3(s) = 9(s^4+10s^3+45s^2+105s+105) + s^2(s^3+6s^2+15s+15)$

$= s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945.$

4.28 $\Omega_p = 2\pi\times 0.24$ and $\hat{\Omega}_p = 2\pi\times 3.$ The mapping is thus $s = \frac{\Omega_p\hat{\Omega}_p}{\hat{s}} = \frac{4\pi^2\times 0.72}{\hat{s}}.$

Denote $K = 4\pi^2\times 0.72 = 28.4245.$ Hence, the desired highpass transfer function is given by $H_{HP}(\hat{s}) = H_{LP}(s)\big|_{s=K/\hat{s}} = \frac{10}{\left(\frac{K}{\hat{s}}\right)^3 + 4.309\left(\frac{K}{\hat{s}}\right)^2 + 9.2835\left(\frac{K}{\hat{s}}\right) + 10}$

$$= \frac{10\hat{s}^3}{K^3 + 4.309K^2\hat{s} + 9.2835K\hat{s}^2 + 10\hat{s}^3} = \frac{10\hat{s}^3}{10\hat{s}^3 + 263.8785\hat{s}^2 + 3481.5\hat{s} + 22966}$$

$$= \frac{\hat{s}^3}{\hat{s}^3 + 26.38785\hat{s}^2 + 348.15\hat{s} + 2296.6}.$$

4.29 $\Omega_p = 2\pi\times 0.9$ and $\hat{\Omega}_p = 2\pi\times 3.$ The mapping is thus $s = \frac{\Omega_p\hat{\Omega}_p}{\hat{s}} = \frac{4\pi^2\times 2.7}{\hat{s}}.$ Denote $K = 4\pi^2\times 2.7 = 106.5917.$ Hence, the desired lowpass transfer function is given by

$$H_{LP}(s) = H_{LP}(s)\Big|_{s=K/\hat{s}} = \frac{\left(\frac{K}{\hat{s}}\right)^3}{\left(\frac{K}{\hat{s}}\right)^3 + 9.238\left(\frac{K}{\hat{s}}\right)^2 + 40.087\left(\frac{K}{\hat{s}}\right) + 100}$$

$$= \frac{K^3}{K^3 + 9.238K^2\hat{s} + 40.087K\hat{s}^2 + 100\,\hat{s}^3}$$

$$= \frac{12110.735}{\hat{s}^3 + 42.729\hat{s}^2 + 1049.602\hat{s} + 12110.735}.$$

4.30 $\Omega_p = 2\pi \times 0.25 = 0.5\pi,\ \hat{\Omega}_o = 2\pi \times 3 = 6\pi,\ \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi(0.5) = \pi.$ The mapping is thus

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} = 0.5\pi\left(\frac{\hat{s}^2 + 36\pi^2}{\pi\hat{s}}\right) = \frac{\hat{s}^2 + 36\pi^2}{2\hat{s}}.$$

$$H_{BP}(\hat{s}) = H_{LP}(s)\Big|_{s=(\hat{s}^2+36\pi^2)/2\hat{s}} = \frac{0.01\left[\left(\frac{\hat{s}^2+36\pi^2}{2\hat{s}}\right)^2 + 367.93\right]}{\left(\frac{\hat{s}^2+36\pi^2}{2\hat{s}}\right)^2 + 2.269\left(\frac{\hat{s}^2+36\pi^2}{2\hat{s}}\right) + 3.895}$$

$$= \frac{0.01(\hat{s}^4 + 2182.33\,\hat{s}^2 + 126242.18)}{\hat{s}^4 + 4.538\,\hat{s}^3 + 726.19\,\hat{s}^2 + 1612.38\,\hat{s} + 126242.18}.$$

4.31 $\hat{\Omega}_p = 2\pi \times 6.5 \times 10^3$ and $\hat{\Omega}_s = 2\pi \times 1.5 \times 10^3.$

$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.5,$ and hence, $\varepsilon^2 = 10^{0.05} - 1 = 0.122.$

$10\log_{10}\left(\frac{1}{A^2}\right) = -40,$ and hence, $A^2 = 10^4.$ Therefore, $\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\varepsilon^2}} = 286.2632.$

Set $\Omega_p = 1.$ Then $\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = \frac{6.5}{1.5} = \frac{13}{3}.$ Thus, $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{13}{3}.$ The order of the prototype lowpass filter is thus given by $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.8579.$ As a result, we choose the filter order as $N = 4.$

The order of the prototype lowpass filter obtained using the MATLAB statement `[N,Wn]=buttord(1,13/3,0.5, 40, 's')` results in `N=4`.

The order of the desired highpass filter is also 4.

4.32 $\hat{F}_{p1} = 20 \times 10^3,\ \hat{F}_{p2} = 45 \times 10^3,\ \hat{F}_{s1} = 10 \times 10^3,$ and $\hat{F}_{s2} = 50 \times 10^3.$ Thus,

$\hat{F}_{p1}\hat{F}_{p2} = 9\times10^8$ and $\hat{F}_{s1}\hat{F}_{s2} = 7.5\times10^8$. Since $\hat{F}_{p1}\hat{F}_{p2} > \hat{F}_{s1}\hat{F}_{s2}$, we can either increase left stopband edge $\hat{F}_{s1}$ or decrease the left passband edge $\hat{F}_{p1}$ to make $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2}$. We choose to increase $\hat{F}_{s1}$ to a new value given by $\hat{F}_{s1} = 18\times10^3$, in which case $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = \hat{F}_o^2 = 9\times10^8$. The center angular frequency of the bandpass filter is therefore $\hat{\Omega}_o = 2\pi\times30\times10^3$. The passband width is $B_w = \hat{\Omega}_{p1} - \hat{\Omega}_{p2} = 2\pi\times25\times10^3$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_p = 1$ and thus,

$$\Omega_s = \Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1}B_w} = \frac{30^2 - 18^2}{18\times25} = 1.28.$$

Now, $k = \dfrac{\Omega_p}{\Omega_s} = \dfrac{1}{1.28} = 0.78125$. Hence, $k' = \sqrt{1-k^2} = 0.62421826$.

Next, $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right) = -0.25$ or equivalently, $\log_{10}(1+\varepsilon^2) = 0.025$ which yields $\varepsilon^2 = 10^{0.025} - 1 = 0.059253725$ or $\varepsilon = 0.243421$. Likewise, $10\log_{10}\left(\dfrac{1}{A^2}\right) = -50$ or, equivalently, $\log_{10}(A^2) = 5$ which yields $A^2 = 10^5 = 100000$. Therefore,

$k_1 = \dfrac{\varepsilon}{\sqrt{A^2-1}} = 7.69768\times10^{-4}$, $\rho_0 = \dfrac{1-\sqrt{k'}}{2(1+\sqrt{k'})} = 0.058635856$. As a result,

$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13} = 0.058637246$. Hence,

$N = \dfrac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 6.0328$. We choose $N = 7$ as the order of the prototype lowpass filter.

Note that the order can also estimated using the specifications of the bandpass filter. To this end, the statement to use is
`[N,Wn]=ellipord([20 45],[15 50],0.25,50,'s')` which also yields `N=7` as the order of the prototype lowpass filter. The order of the desired bandpass filter is therefore $7\times2 = 14$.

4.33 $\hat{F}_{p1} = 10\times10^6$, $\hat{F}_{p2} = 70\times10^6$, $\hat{F}_{s1} = 20\times10^6$, and $\hat{F}_{s2} = 45\times10^6$. Thus, $\hat{F}_{p1}\hat{F}_{p2} = 70\times10^{13}$ and $\hat{F}_{s1}\hat{F}_{s2} = 90\times10^{13}$. Since $\hat{F}_{p1}\hat{F}_{p2} < \hat{F}_{s1}\hat{F}_{s2}$, we can either increase left passband edge $\hat{F}_{p1}$ or decrease the left stopband edge $\hat{F}_{s1}$ to make $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2}$. We choose to increase $\hat{F}_{p1}$ to a new value given by

$\hat{F}_{p1}\hat{F}_{p2} = \dfrac{\hat{F}_{s1}\hat{F}_{s2}}{\hat{F}_{p2}} = 12.8571\times10^6$, in which case $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = F_o^2 = 700\times10^{12}$.

The width of the stopband is $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 2\pi\times25\times10^6$ and the center angular frequency of the stopband is $\Omega_o^2 = 4\pi^2\times700\times10^{12}$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_s = 1$ resulting in its passband edge $\Omega_p = \Omega_p \dfrac{\hat{\Omega}_{p1}B_w}{\hat{\Omega}_o^2 - \hat{\Omega}_{p1}^2} = 0.4375$.

Now, $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right) = -0.5$ or equivalently, $\log_{10}(1+\varepsilon^2) = 0.05$ which yields

$\varepsilon^2 = 10^{0.05} - 1 = 0.1220184543$ or $\varepsilon = 0.349114$. Likewise, $10\log_{10}\left(\dfrac{1}{A^2}\right) = -30$

or, equivalently, $\log_{10}(A^2) = 3$ which yields $A^2 = 10^3 = 1000$. Therefore,

$\dfrac{1}{k} = \dfrac{\Omega_s}{\Omega_p} = \dfrac{1}{0.4375} = 2.2857$ and $\dfrac{1}{k_1} = \dfrac{\sqrt{A^2-1}}{\varepsilon} = \dfrac{\sqrt{999}}{0.349114} = 90.4836236$.

Substituting the values of $\dfrac{1}{k}$ and $\dfrac{1}{k_1}$ in Eq. (4.43) we get

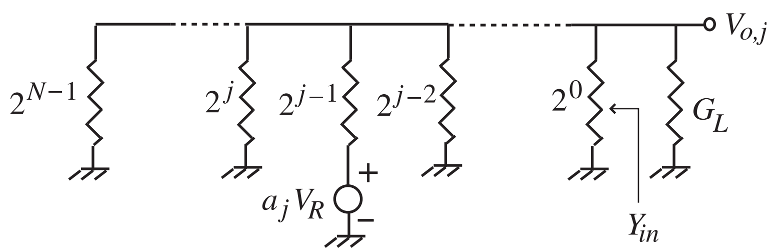
$N = \dfrac{\cosh^{-1}(90.4836236)}{\cosh^{-1}(2.2857)} = 3.5408$. We therefore choose $N = 4$ as the order of the prototype lowpass filter. The order of the desired bandstop filter is thus 8.

Using the statement `[N,Wn]=cheb1ord(0.4375,1,0.5,30,'s')` we get `N=4`.

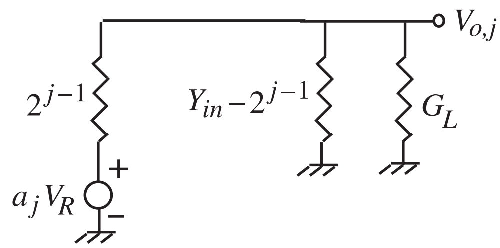
Note that the order can also estimated using the specifications of the bandstop filter. To This end, the statement to use is
`[N,Wn]=cheb1ord([10 70],[20 45],0.5,30,'s')` which also yields `N=4` as the order of the prototype lowpass filter.

4.34 From Eq. (4.71), the difference in dB in the attenuation levels at Ω_p and Ω_s is given by $20N\log_{10}(\Omega_p/\Omega_s)$. Hence, for $\Omega_o = 2\Omega_p$, the attenuation difference in dB is equal to $20N\log_{10}2 = 6.0206N$. Likewise, for $\Omega_o = 3\Omega_p$, the attenuation difference in dB is equal to $20N\log_{10}3 = 9.5424N$. Finally, for $\Omega_o = 4\Omega_p$, the attenuation difference in dB is equal to $20N\log_{10}4 = 12.0412N$.

4.35 The equivalent representation of the D/A converter of Figure 4.48 reduces to the circuit shown below if j-th bit is ON and the remaining bits are OFF, i.e., $a_j = 1$ and $a_k = 0,\ k \neq j$.



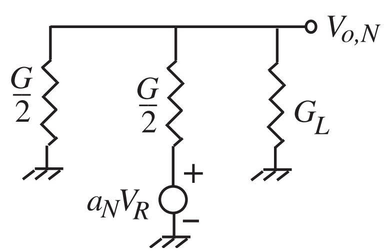
In the above circuit, Y_{in} is the total conductance seen by the load conductance G_L which is given by $Y_{in} = \sum_{i=0}^{N-1} 2^i = 2^N - 1$. The above circuit can be redrawn as indicated below:

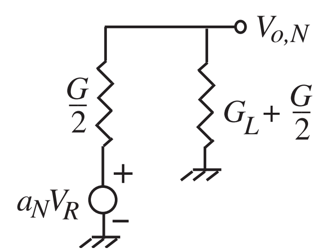


Using the voltage-divider relation we then get $V_{o,j} = \dfrac{2^{j-1}}{Y_{in} + G_L} \cdot a_j V_R$. Using the superposition theorem, the general expression for the output voltage V_o is thus given by

$$V_o = \sum_{j=1}^{N} \frac{2^{j-1}}{Y_{in} + G_L} \cdot a_j V_R = \sum_{j=1}^{N} 2^{j-1} a_j \left(\frac{R_L}{1 + (2^N - 1) R_L} \right) V_R.$$

4.36 The equivalent representation of the D/A converter of Figure 4.49 reduces to the circuit shown below on the left if N-th bit is ON and the remaining bits are OFF, i.e., $a_N = 1$ and $a_k = 0,\ k \neq N,$



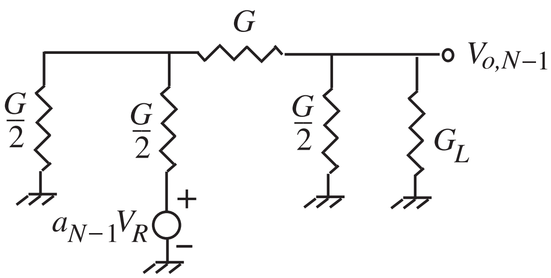


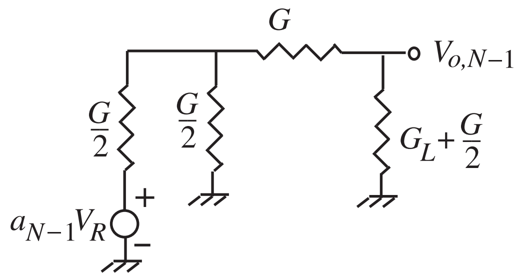
which simplifies to the circuit shown above on the right.

Using the voltage-divider relation we then get

$$V_{o,N} = \frac{\frac{G}{2}}{\frac{G}{2} + G_L + \frac{G}{2}} \cdot a_N V_R = \frac{R_L}{2(R + R_L)} \cdot a_N V_R.$$

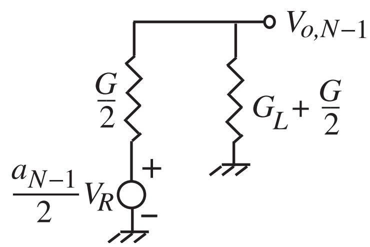
The equivalent representation of the D/A converter of Figure 4.49 reduces to the circuit shown below on the left if $(N-1)$-th bit is ON and the remaining bits are OFF, i.e., $a_{N-1} = 1$ and $a_k = 0,\ k \neq N-1,$





which simplifies to the circuit shown above on the right.

Its Thevenin equivalent circuit is indicated below:



from which we readily obtain

$$V_{o,N-1} = \frac{\frac{G}{2}}{G+G_L}\cdot\frac{a_{N-1}}{2}V_R = \frac{R_L}{2(R_L+R)}\cdot\frac{a_{N-1}}{2}V_R.$$

Following the same procedure we can show that if the ℓ–th bit is ON and the remaining bits are OFF, i.e., $a_\ell = 1$, and $a_k = 0,\ k \neq \ell$, then

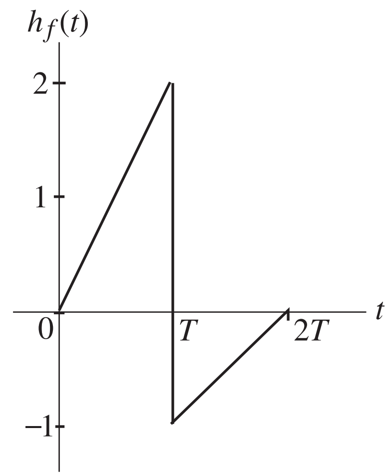
$$V_{o,\ell} = \frac{R_L}{2(R_L+R)}\cdot\frac{a_\ell}{2^{N-\ell}}V_R.$$

Hence, in general we have

$$V_o = \sum_{\ell=1}^{N}\frac{R_L}{2(R_L+R)}\cdot\frac{a_\ell}{2^{N-\ell}}V_R.$$

4.37 From the input-output relation of the first-order hold, we get the expression for the impulse response as $h_f(t) = \delta(nT) + \frac{\delta(nT)-\delta(nT-T)}{T}(t-nT),\ nT \le t < (n+1)T$. In the range $0 \le t < T$, the impulse response is given by $h_f(t) = \delta(0) + \frac{\delta(0)-\delta(-T)}{T}t = 1 + \frac{t}{T}$. Likewise, in the range $T \le t < 2T$, the impulse response is given by $h_f(t) = \delta(T) + \frac{\delta(T)-\delta(0)}{T}(t-T) = 1 - \frac{t}{T}$. Outside these two ranges, $h_f(t) = 0$. Hence we have

$$h_f(t) = \begin{cases} 1 + \frac{t}{T}, & 0 \leqslant t < T, \\ 1 - \frac{t}{T}, & T \leqslant t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$



Using the step function we can write

$$h_f(t) = \left(1 + \frac{t}{T}\right)[\mu(t) - \mu(t-T)] + \left(1 - \frac{t}{T}\right)[\mu(t-T) - \mu(t-2T)]$$

$$= \mu(t) + \frac{t}{T}\mu(t) - \frac{2(t-T)}{T}\mu(t-T) - 2\mu(t-T) - \mu(t-2T) + \frac{(t-2T)}{T}\mu(t-2T) + 2\mu(t-2T).$$

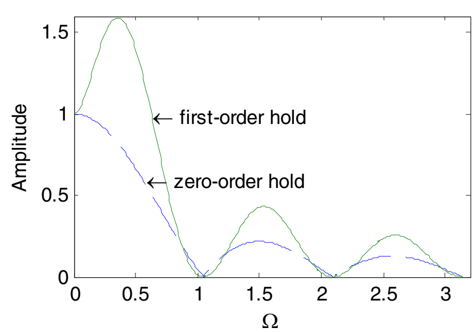
Taking the Laplace transform of the above equation we arrive at the transfer function

$$H_f(s) = \frac{1}{s} + \frac{1}{Ts^2} - \frac{2}{T}\cdot\frac{e^{-sT}}{s^2} - 2\frac{e^{-sT}}{s} - \frac{e^{-2sT}}{s} + \frac{1}{T}\cdot\frac{e^{-2sT}}{s^2} + 2\frac{e^{-2sT}}{s} = \left(\frac{1+sT}{T}\right)\left(\frac{1-e^{-sT}}{s}\right)^2.$$

Hence, the frequency response is given by

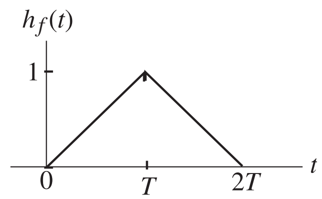
$$H_f(j\Omega) = \left(\frac{1+j\Omega T}{T}\right)\left(\frac{1-e^{-j\Omega T}}{j\Omega}\right)^2 = T\sqrt{1+\Omega^2T^2}\left(\frac{2\sin(\Omega T/2)}{\Omega T/2}\right)^2 e^{-j\Omega T} e^{j\tan^{-1}\Omega T}.$$ A

plot of the magnitude responses of the zero-order hold and the first-order hold is shown below:



4.37 From the input-output relation of thelinear interpolator, we get the expression for the impulse response as $h_f(t) = \delta(nT - T) + \frac{\delta(nT) - \delta(nT - T)}{T}(t - nT),\ nT \le t < (n+1)T$. In the range $0 \le t < T$, the impulse response is given by $h_f(t) = \delta(-T) + \frac{\delta(0) - \delta(-T)}{T}t$. Likewise, in the range $T \le t < 2T$, the impulse response is given by $h_f(t) = \delta(0) + \frac{\delta(T) - \delta(0)}{T}(t - T)$.

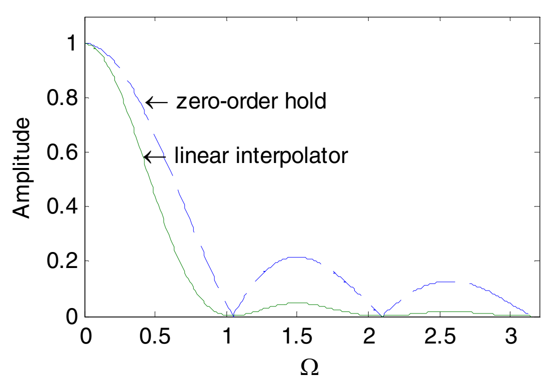
Outside these two ranges, $h_f(t) = 0$. Hence we have $h_f(t) = \begin{cases} \frac{t}{T}, & 0 \le t < T, \\ 2 - \frac{t}{T}, & T \le t < 2T, \\ 0, & \text{otherwise.} \end{cases}$



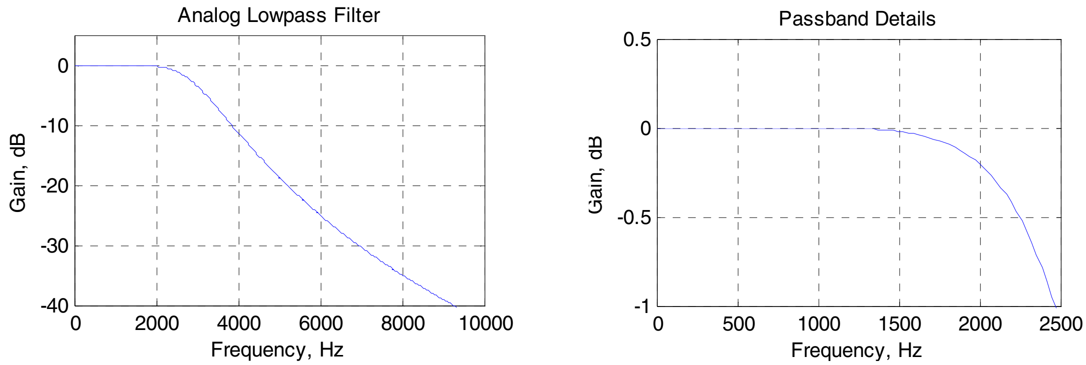
Using the step function we can write

$$h_f(t) = \frac{t}{T}[\mu(t) - \mu(t-T)] + \left(2 - \frac{t}{T}\right)[\mu(t-T) - \mu(t-2T)]$$
$$= \frac{t}{T}\mu(t) - \frac{2(t-T)}{T}\mu(t-T) + \frac{(t-2T)}{T}\mu(t-2T).$$

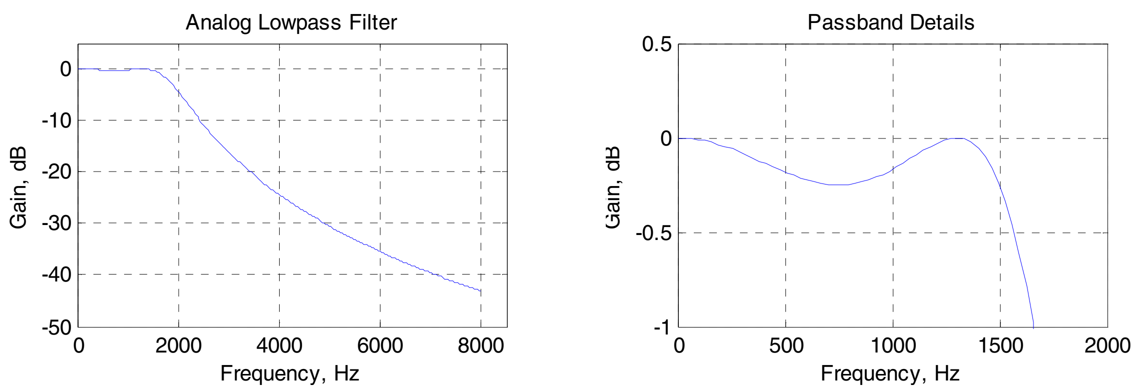
Taking the Laplace transform of the above equation we arrive at the transfer function $H_f(s) = \frac{1}{s^2 T} - \frac{2e^{-sT}}{s^2 T} + \frac{e^{-2sT}}{s^2 T} = T\left(\frac{1 - e^{-sT}}{sT}\right)^2$. Hence, the frequency response is given by $H_f(j\Omega) = T\left(\frac{1 - e^{-j\Omega T}}{j\Omega T}\right)^2 = T\left(\frac{\sin(\Omega T/2)}{\Omega T/2}\right)^2 e^{-j\Omega T}$. A plot of the magnitude responses of the ideal filter, zero-order hold and the first-order hold is shown below:



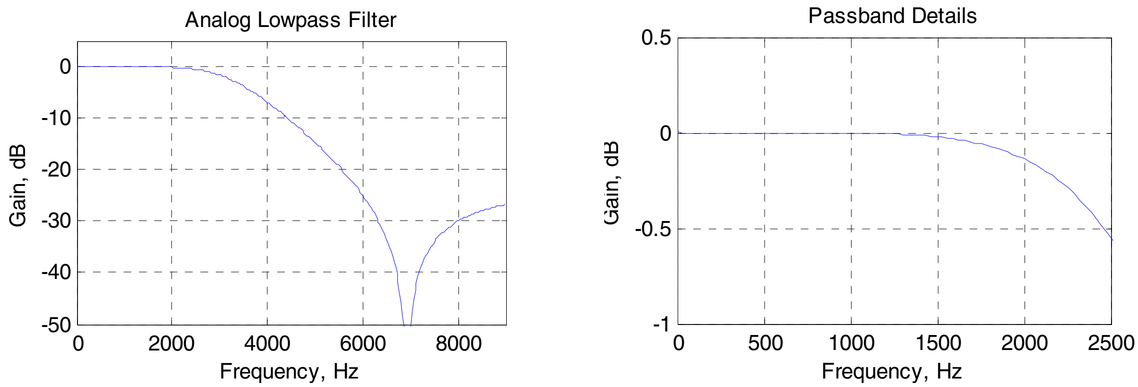
M4.1 We use `N = 4` and `Wn = 18365.512865` computed in Problem 4.22 and use omega = 0:2*pi:2*pi*10000; to evaluate the frequency points. The gain plot obtained using Program 4_2 is shown below.



M4.2 We use `N = 3` computed in Problem 4.23 and `Fp = 2*pi*1500` and `Rp = 0.25` and use omega = 0:2*pi:2*pi*10000; to evaluate the frequency points. The gain plot obtained using Program 4_3 is shown below.



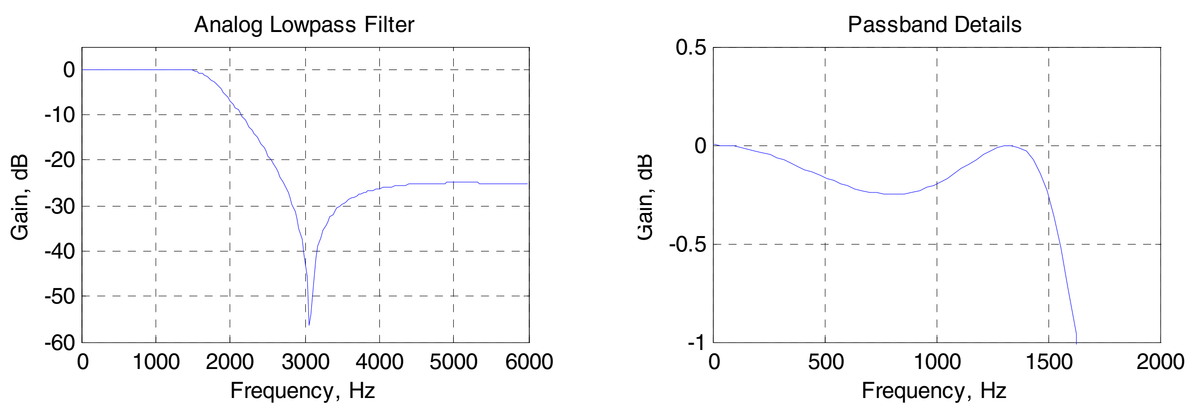
M4.3 We replace the statement
`Fp = input('Passband edge frequency in Hz = ');` with
`Fs = input('Stopband edge frequency in Hz = ');` replace
`Rp = input('Passband ripple in dB = ');`
with `Rs = input('Minimum stopband attenuation in dB = ');` and replace `[num,den] = cheby1(N,Rp,Fp,'s');` with `[num,den] = cheby2(N,Rs,Fs,'s');` to modify Program 4_3.
Next, we run the modified program using N = 3 and Rs = 25, and Fs = 2*pi*6000. The gain response plot generated by the modified program is shown below.



The numerator and the denominator coefficients of the 3rd order Type 2 Chebyshev lowpass filter can be obtained by typing `num` and `den` in the command window:

$$H_{LP}(s) = \frac{10138.1864\,s^2 + 4.8663294\times 10^{10}}{s^3 + 7030.255525\,s^2 + 2.4198332254\times 10^{7}\,s + 4.8663294\times 10^{10}}.$$

M4.4 We use `N = 3` and `Wn = 9424.777960769379` computed in Problem 4.26 in Program 4_4 and use omega = [0: 200: 12000*pi]; to evaluate the frequency points. The gain plot generated by running this program is shown below:

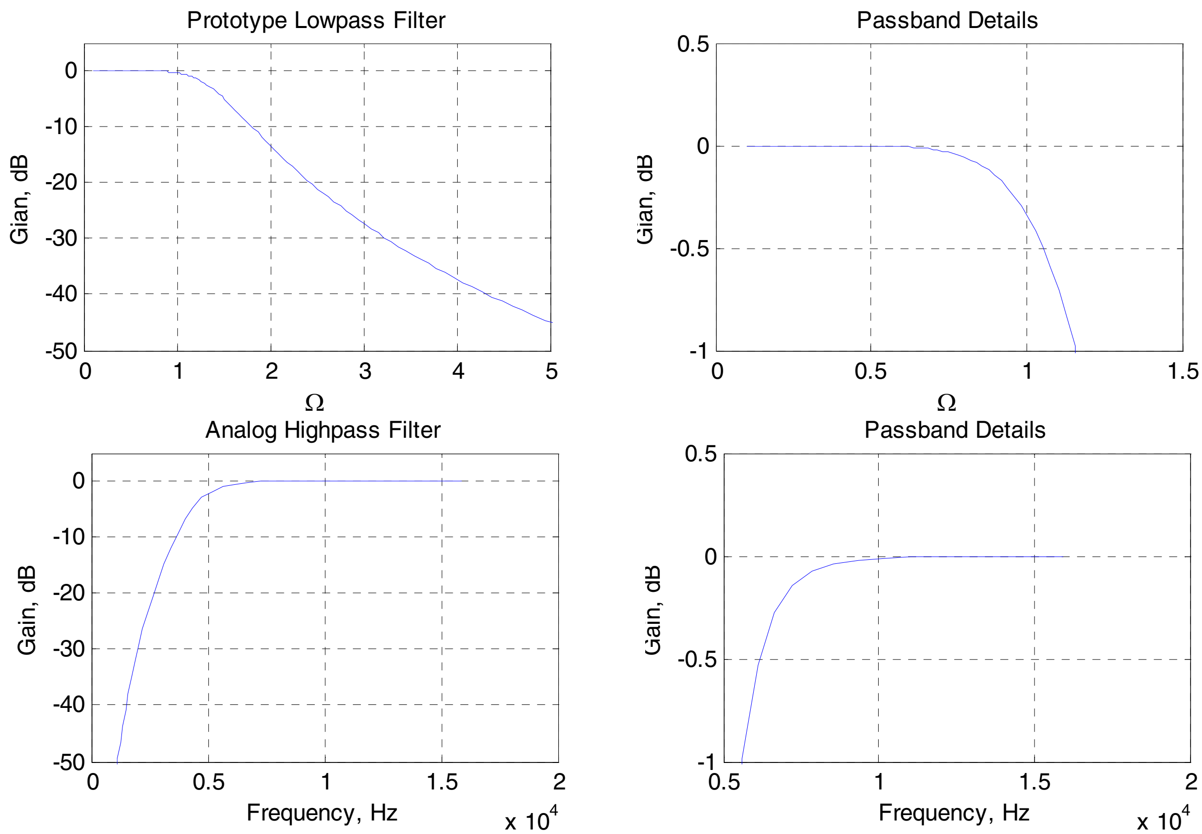


M4.5 The MATLAB program used is as given below:

```
[N,Wn]=buttord(1,13/3,0.5, 40, 's');
[B,A] = butter(N,Wn,'s');
[num,den]=lp2hp(B,A,2*pi*6500);
figure(1)
[h,w]=freqs(B,A);gain = 20*log10(abs(h));
plot(w,gain);grid
xlabel('\Omega');ylabel('Gain, dB');
title('Analog Lowpass Filter');
figure(2)
[h,w]=freqs(num,den);gain = 20*log10(abs(h));
plot(w/(2*pi),gain);grid
xlabel('Frequency, Hz');ylabel('Gain, dB');
title('Analog Highpass Filter');
```

$$H_{LP}(s) = \frac{3.5262257}{s^4 + 3.58086432 s^3 + 6.41129464 s^2 + 6.72423556 s + 3.5262257},$$

$$H_{HP}(s) = \frac{s^4}{s^4 + 7.7880 s^3 + 3.0326485 \times 10^9 s^2 + 6.91763168 \times 10^{13} s + 7.8897418 \times 10^{17}}.$$



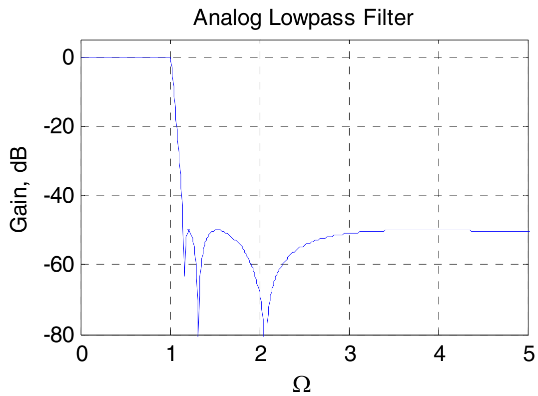
M4.6 The MATLAB program used is given below:

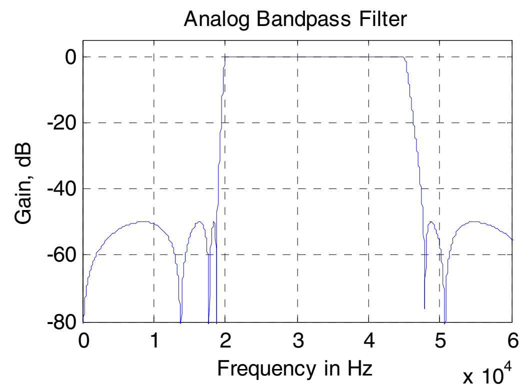
```
[N,Wn] = ellipord(1,1.28,0.25,50,'s');
[B,A] = ellip(N,0.25,50,Wn,'s');
[num,den] = lp2bp(B,A,2*pi*30e3,2*pi*25e3);

figure(1)
omega = [0:0.01:10];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega,gain); grid; axis([0 5 -80 5]);
xlabel('\Omega'); ylabel('Gain, dB');
title('Analog Lowpass Filter');

figure(2)
omega = [0:200:100e3*2*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
```

```
plot(omega/(2*pi),gain); grid; axis([0 60e3 -80 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB');
title('Analog Bandpass Filter');
```





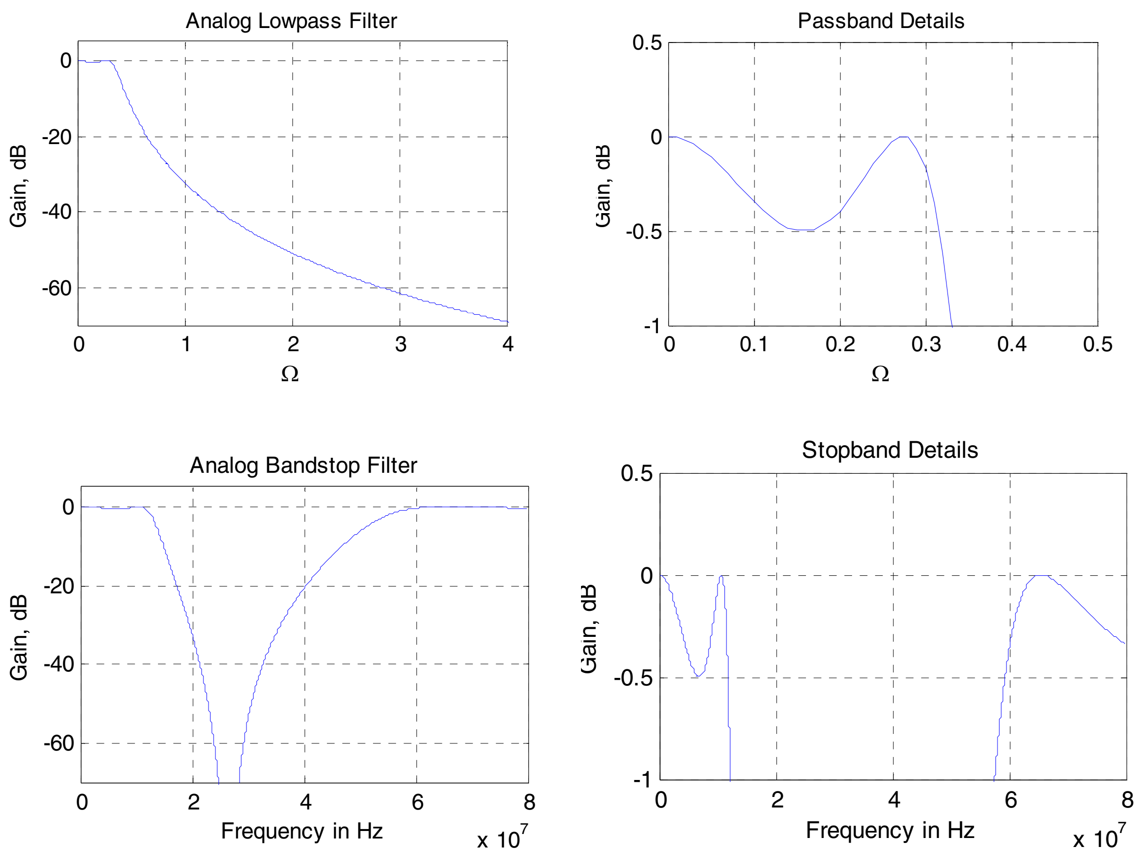
$$H_{LP}(s) = \frac{0.0185\,s^6 + 0.1364\,s^4 + 0.2887\,s^2 + 0.186835}{s^7 + 1.36426\,s^6 + 2.9795\,s^5 + 2.7545\,s^4 + 2.70025\,s^3 + 1.56215\,s^2 + 0.7275\,s + 0.1868}.$$

The numerator and denominator coefficients of can be obtained by typing `num` and `den` in the Command Window.

M4.7 The MATLAB program used is given below:

```
[N,Wn] = cheb1ord(0.3157894, 1, 0.5, 30,'s');
[B,A] = cheby1(N,0.5, Wn,'s');
[num,den] = lp2bs(B,A,2*pi*sqrt(700)*10^6, 2*pi*15e6);
figure(1)
omega = [0:0.01:10];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega, gain); grid; axis([0 4 -70 5]);
xlabel('\Omega'); ylabel('Gain, dB');
title('Analog Lowpass Filter');
figure(2)
omega = [0:10000:160e6*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot(omega/(2*pi), gain); grid; axis([0 80e6 -70 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB');
title('Analog Bandstop Filter');
```

$$H_{LP}(s) = \frac{0.02253823}{s^3 + 0.3956566\,s^2 + 0.1530643\,s + 0.02253823}.$$



M4.8 The MATLAB program to generate the plots of Figure 4.56 is given below:

```
% Droop Compensation
w = 0:pi/100:pi;
h1 = freqz([-1/16 9/8 -1/16],1,w);
h2 = freqz(9, [8 1], w);
w1 = 0;
for n = 1:101;
     h3(n) = sin(w1/2)/(w1/2);
     w1 = w1 + pi/100;
end
m1 = 20*log10(abs(h1));
m2 = 20*log10(abs(h2));
m3 = 20*log10(abs(h3));
plot(w/pi,m3,'-',w/pi,m1+m2,'--',w/pi,m2+m3,'-.');grid
xlabel('Normalized frequency');ylabel(Gain, dB');
```