Chapter 6

6.1 $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$. Therefore, $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n}$

$= \lim_{z\to\infty} x[0] + \lim_{z\to\infty} \sum_{n=1}^{\infty} x[n]z^{-n} = x[0].$

6.2 **(a)** $\mathcal{Z}\delta\{[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = \delta[0] = 1,$ which converges everywhere in the z–plane.

(b) $x[n] = \alpha^n \mu[n]$. From Table 6.1,

$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{1-\alpha z^{-1}}, \ |z| > |\alpha|.$ Let $g[n] = nx[n]$. Then,

$\mathcal{Z}\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$. Now, $\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} ng[n]z^{-n-1}$. Hence,

$z\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -G(z),$ or, $G(z) = -z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \ |z| > |\alpha|.$

(c) $x[n] = r^n \sin(\omega_o n)\mu[n] = \frac{r^n}{2j}(e^{j\omega_o n} - e^{-j\omega_o n})\mu[n]$. Using the results of Example 6.1 and the linearity property of the z–transform we get

$$\mathcal{Z}\{r^n \sin(\omega_o n)\mu[n]\} = \frac{1}{2j}\left(\frac{1}{1-r\,e^{j\omega_o} z^{-1}}\right) - \frac{1}{2j}\left(\frac{1}{1-r\,e^{-j\omega_o} z^{-1}}\right)$$

$$= \frac{\frac{r}{2j}(e^{j\omega_o} - e^{-j\omega_o})z^{-1}}{1-r(e^{j\omega_o} + e^{-j\omega_o})z^{-1} + r^2 z^{-2}} = \frac{r\sin(\omega_o)z^{-1}}{1-2r\cos(\omega_o)z^{-1} + r^2 z^{-2}}, \qquad \forall\ |z| > |r|.$$

6.3 **(a)** $x_1[n] = \alpha^n \mu[n-2]$. Note, $x_1[n]$ is a right-sided sequence. Hence, the ROC of its z–transform is exterior to a circle. Therefore, $X_1(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n-2]z^{-n} = \sum_{n=2}^{\infty} \alpha^n z^{-n}$

$= \sum_{n=0}^{\infty} \alpha^n z^{-n} - 1 - \alpha z^{-1}, \ |\alpha / z| < 1.$ Simplifying we get

$X_1(z) = \frac{1}{1-\alpha z^{-1}} - 1 - \alpha z^{-1} = \frac{\alpha^2 z^{-2}}{1-\alpha z^{-1}}$ whose ROC is given by $|z| > |\alpha|$.

(b) $x_2[n] = -\alpha^n \mu[-n-3]$. Note, $x_2[n]$ is a left-sided sequence. Hence, the ROC of its z–transform is interior to a circle. Therefore,

$$X_2(z) = -\sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-3] z^{-n} = -\sum_{n=-\infty}^{-3} \alpha^n z^{-n} = -\sum_{m=3}^{\infty} \alpha^{-m} z^m = -\sum_{m=3}^{\infty} (z/\alpha)^m$$

$= \sum_{m=0}^{\infty} (z/\alpha)^m - 1 - (z/\alpha) - (z/\alpha)^2$, $|z/\alpha| < 1$. Simplifying we get

$X_2(z) = \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ whose ROC is given by $|z| < |\alpha|$.

(c) $x_3[n] = \alpha^n \mu[n+4]$. Note, $x_3[n]$ is a right-sided sequence. Hence, the ROC of its z–transform is exterior to a circle. Therefore, $X_3(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+4] z^{-n} = \sum_{n=-4}^{\infty} \alpha^n z^{-n}$

$= \sum_{n=0}^{\infty} (\alpha/z)^n + (\alpha/z)^{-1} + (\alpha/z)^{-2} + (\alpha/z)^{-3} = \dfrac{1}{1-(\alpha/z)} + (\alpha/z)^{-1} + (\alpha/z)^{-2} + (\alpha/z)^{-3}$,

$|\alpha/z| < 1$. Simplifying we get $X_3(z) = \dfrac{(\alpha/z)^{-3}}{1-(\alpha/z)}$ whose ROC is given by $|z| > |\alpha|$.

(d) $x_4[n] = \alpha^n \mu[-n]$. Note, $x_4[n]$ is a left-sided sequence. Hence, the ROC of its z–transform is interior to a circle. Therefore, $X_4(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n] z^{-n} = \sum_{n=-\infty}^{0} \alpha^n z^{-n}$

$= \sum_{m=0}^{\infty} \alpha^{-m} z^m = \dfrac{1}{1-(z/\alpha)}$, $|z/\alpha| < 1$. Therefore the ROC of $X_4(z)$ is given by $|z| < |\alpha|$.

6.4 $\mathcal{Z}\{(0.4)^n \mu[n]\} = \dfrac{1}{1-0.4z^{-1}}$, $|z| > 0.4$; $\mathcal{Z}\{(-0.6)^n \mu[n]\} = \dfrac{1}{1+0.6z^{-1}}$, $|z| > 0.6$;

$\mathcal{Z}\{(0.4)^n \mu[-n-1]\} = -\dfrac{1}{1-0.4z^{-1}}$, $|z| < 0.4$;

$\mathcal{Z}\{(-0.6)^n \mu[-n-1]\} = -\dfrac{1}{1+0.6z^{-1}}$, $|z| < 0.6$;

(a) $\mathcal{Z}\{x_1[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $|z| > 0.6$.

(b) $\mathcal{Z}\{x_2[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $0.4 < |z| < 0.6$.

(c) $\mathcal{Z}\{x_3[n]\} = \dfrac{1}{1-0.4z^{-1}} + \dfrac{1}{1+0.6z^{-1}} = \dfrac{1+0.2z^{-1}}{(1-0.4z^{-1})(1+0.6z^{-1})}$, $|z| < 0.4$.

$= \dfrac{(\alpha^{-2}+\beta^{-2})-(\alpha\beta^{-2}+\alpha^{-2}\beta)z^{-1}}{z^{-2}(1-\alpha z^{-1})(1-\beta z^{-1})}$ with its ROC given by $|z|>|\beta|$.

(b) $x_2[n]=\alpha^n\mu[-n-2]+\beta^n\mu[n-1]$ with $|\beta|>|\alpha|$. Note that $x_2[n]$ is a two-sided sequence. Now,

$\mathcal{Z}\{\alpha^n\mu[-n-2]\} = \sum_{n=-\infty}^{-2}\alpha^n z^{-n} = \sum_{m=2}^{\infty}\alpha^{-m}z^m = \sum_{m=0}^{\infty}(z/\alpha)^m - 1-(z/\alpha)-(z/\alpha)^2$

$= \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ with its ROC given by $|z|<|\alpha|$. Likewise,

$\mathcal{Z}\{\beta^n\mu[n-1]\} = \sum_{n=1}^{\infty}\beta^n z^{-n} = \sum_{n=0}^{\infty}\beta^n z^{-n} - 1 = \dfrac{1}{1-\beta z^{-1}} - 1 = \dfrac{\beta z^{-1}}{1-\beta z^{-1}}$ with its ROC given by $|z|>|\beta|$. Since the two ROCs do not intersect, $\mathcal{Z}\{x_2[n]\}$ does not converge.

(c) $x_3[n]=\alpha^n\mu[n+1]+\beta^n\mu[-n-2]$ with $|\beta|>|\alpha|$. Note that $x_3[n]$ is a two-sided sequence. Now,

$\mathcal{Z}\{\alpha^n\mu[-n-2]\} = \sum_{n=-\infty}^{-2}\alpha^n z^{-n} = \sum_{m=2}^{\infty}\alpha^{-m}z^m = \sum_{m=0}^{\infty}(z/\alpha)^m - 1-(z/\alpha)-(z/\alpha)^2$

$= \dfrac{(z/\alpha)^3}{1-(z/\alpha)}$ with its ROC given by $|z|<|\alpha|$. Likewise,

$\mathcal{Z}\{\beta^n\mu[n-1]\} = \sum_{n=1}^{\infty}\beta^n z^{-n} = \sum_{n=0}^{\infty}\beta^n z^{-n} - 1 = \dfrac{1}{1-\beta z^{-1}} - 1 = \dfrac{\beta z^{-1}}{1-\beta z^{-1}}$ with its ROC given by $|z|>|\beta|$.

6.8 The denominator factor $(z^2+0.3z-0.18)=(z+0.6)(z-0.3)$ has poles at $z=-0.6$ and at $z=0.3$, and the factor (z^2-2z+4) has poles with a magnitude 2. Hence, the four ROCs are defined by the regions: $\mathcal{R}_1$: $0<|z|<0.3$, $\mathcal{R}_2$: $0.3<|z|<0.6$, $\mathcal{R}_3 = 0.6<|z|<2$, and $\mathcal{R}_4$: $|z|>2$. The inverse z–transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence, the inverse z–transforms associated with the ROCs $\mathcal{R}_2$ and $\mathcal{R}_3$ are two-sided sequences, and the inverse z–transform associated with the ROC $\mathcal{R}_4$ is a right-sided sequence.

6.9 $X(z) = \mathcal{Z}\{x[n]\}$ with an ROC given by $\mathcal{R}_x$. Using the conjugation property of the – transform given in Table 6.2, we observe that $\mathcal{Z}\{x^*[n]\} = X^*(z^*)$ whose ROC is given by $\mathcal{R}_x$. Now, $\text{Re}\{x[n]\} = \frac{1}{2}(x[n]+x^*[n])$. Hence, $\mathcal{Z}\{\text{Re}\, x[n]\}$ $= \frac{1}{2}\left(X(z)+X^*(z)\right)$ whose ROC is also $\mathcal{R}_x$. Likewise, $\text{Im}\{x[n]\} = \frac{1}{2j}(x[n]-x^*[n])$.

Thus, $\mathcal{Z}\{\text{Im}\, x[n]\} = \frac{1}{2j}\left(X(z) - X^*(z)\right)$ whose ROC is again $\mathcal{R}_x$.

6.10 $\{x[n]\} = \{2, \;\; 3, \;\; -1, \;\; 0, \;\; -4, \;\; 3, \;\; 1, \;\; 2, \;\; 4\}, -2 \le n \le 6$. Then, $\tilde{X}[k] = X(z)\big|_{z=e^{j\pi k/3}} = X(z)\big|_{z=e^{j2\pi k/6}} = X(e^{j\omega})\big|_{\omega=2\pi k/6}$. Note that $\tilde{X}[k]$ is a periodic sequence with a period 6. Hence, from Eq. (5.49), the inverse of the discrete Fourier series $\tilde{X}[k]$ is given by $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+6r] = x[n-6] + x[n] + x[n+6]$ for $0 \le n \le 5$. Let $y[n] = x[n-6] + x[n] + x[n+6], -2 \le n \le 6$. Now,
$\{x[n-6]\} = \{0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 2, \;\; 3, \;\; -1\}$ and
$\{x[n+6] = \{1, \;\; 2, \;\; 4, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0\}$. Therefore,
$\{y[n]\} = \{3, \;\; 5, \;\; 3, \;\; 0, \;\; -4, \;\; 3, \;\; 3, \;\; 5, \;\; 3\}, -2 \le n \le 6$. Hence,
$\{\tilde{x}[n]\} = \{3, \;\; 0, \;\; -4, \;\; 3, \;\; 3, \;\; 5\}, 0 \le n \le 5$.

6.11 $\{x[n]\} = \{4, \;\; 2, \;\; -1, \;\; 5, \;\; -3, \;\; 1, \;\; -2, \;\; 4, \;\; 2\}, -6 \le n \le 2$. Then, $\tilde{X}[k] = X(z)\big|_{z=e^{j\pi k/3}} = X(z)\big|_{z=e^{j2\pi k/6}} = X(e^{j\omega})\big|_{\omega=2\pi k/6}$. Note that $\tilde{X}[k]$ is a periodic sequence with a period 6. Hence, from Eq. (5.49), the inverse of the discrete Fourier series $\tilde{X}[k]$ is given by $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+6r] = x[n-6] + x[n] + x[n+6]$ for $0 \le n \le 5$. Let $y[n] = x[n-6] + x[n] + x[n+6], -6 \le n \le 5$. Now,
$\{x[n-6]\} = \{0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 4, \;\; 2, \;\; -1, \;\; 5, \;\; -3, \;\; 1\}, -6 \le n \le 5$,
$\{x[n+6] = \{-2, \;\; 4, \;\; 2, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0, \;\; 0\}, -6 \le n \le 5$.
Therefore, $\{y[n]\} = \{2, \;\; 6, \;\; 1, \;\; 5, \;\; -3, \;\; 1, \;\; 2, \;\; 6, \;\; 5, \;\; -3, \;\; 1\}, -6 \le n \le 5$.
Hence, $\{\tilde{x}[n]\} = \{2, \;\; 6, \;\; 1, \;\; 5, \;\; -3, \;\; 1\}, 0 \le n \le 5$.

6.12 $X(z) = \sum_{n=0}^{11} x[n] z^{-n}$. $X_0[k] = X(z)\big|_{z=e^{(2\pi k/9}} = \sum_{n=0}^{11} x[n] e^{-j2\pi kn/9}, 0 \le k \le 8$.

Therefore, $x_0[n] = \frac{1}{9}\sum_{k=0}^{8} X_0[k] e^{j2\pi kn/9} = \frac{1}{9}\sum_{k=0}^{8}\left(\sum_{r=0}^{11} x[r] e^{-j2\pi kr/9}\right) e^{j2\pi kn/9}$

$$= \frac{1}{9}\sum_{k=0}^{8}\sum_{r=0}^{11} x[r] e^{j2\pi k(n-r)/9} = \frac{1}{9}\sum_{r=0}^{11} x[r] \sum_{k=0}^{9} e^{j2\pi k(n-r)/9} = \frac{1}{9}\sum_{r=0}^{11} x[r] \sum_{k=0}^{9} W_9^{-(r-n)k}.$$

From Eq. (5.11), $\frac{1}{9}\sum_{k-0}^{8} W_9^{-(r-n)k} = \begin{cases} 1, & \text{for } r-n = 9i, \\ 0, & \text{otherwise.} \end{cases}$ Hence,

$x_0[n] = \begin{cases} x[0] + x[9], & \text{for } n = 0 \\ x[n], & \text{otherwise,} \end{cases}$ i.e.,

$\{x_0[n]\} = \{20 \;\; 5 \;\; 45 \;\; -15 \;\; -9 \;\; -19 \;\; -8 \;\; 21 \;\; -10\}, 0 \le n \le 8$.

6.13 **(a)** $X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$. Hence, $X(z^3)=\sum_{n=-\infty}^{\infty}x[n]z^{-3n}=\sum_{\substack{r=-\infty\\ m=3r}}^{\infty}x[m/3]z^{-m}$. Define a new sequence $g[m]=\begin{cases}x[m/3], & m=0,\pm 3,\pm 6,\ldots,\\ 0, & \text{otherwise.}\end{cases}$ We can then express $X(z^3)=\sum_{n=-\infty}^{\infty}g[n]z^{-n}$. Thus, the inverse z–transform of $X(z^3)$ is given by $g[n]$.

For $x[n]=(-0.5)^n\mu[n]$, $g[n]=\begin{cases}(-0.5)^{n/3}, & n=0,3,6,\ldots,\\ 0, & \text{otherwise.}\end{cases}$

(b) $Y(z)=(1+z^{-1})X(z^3)=X(z^3)+z^{-1}X(z^3)$. Therefore,
$y[n]=\mathcal{Z}^{-1}\{Y(z)\}=\mathcal{Z}^{-1}X(z^3)+\mathcal{Z}^{-1}z^{-1}X(z^3)=g[n]+g[n-1]$, where $g[n]=\mathcal{Z}^{-1}X(z^3)$. From Part (a), $g[n]=\begin{cases}(-0.5)^{n/3}, & n=0,3,6,\ldots,\\ 0, & \text{otherwise.}\end{cases}$ Hence,
$g[n-1]=\begin{cases}(-0.5)^{(n-1)/3}, & n=1,4,7,\ldots,\\ 0, & \text{otherwise.}\end{cases}$ Therefore,

$$y[n]=\begin{cases}(-0.5)^{n/3}, & n=0,3,6,\ldots,\\ (-0.5)^{(n-1)/3}, & n=1,4,7,\ldots,\\ 0, & \text{otherwise.}\end{cases}$$

6.14 **(a)** $X_a(z)=\mathcal{Z}\{\mu[n]-\mu[n-5]\}=\dfrac{1}{1-z^{-1}}-\dfrac{z^{-5}}{1-z^{-1}}=\dfrac{1-z^{-5}}{1-z^{-1}}$
$=1+z^{-1}+z^{-2}+z^{-3}+z^{-4}$. Since has all poles at the origin, the ROC is the entire z–plane except the point $z=0$, and hence includes the unit circle. On the unit circle,
$X_a(z)\big|_{z=e^{j\omega}}=X_a(e^{j\omega})=1+e^{-j\omega}+e^{-j2\omega}+e^{-j3\omega}+e^{-j4\omega}=\dfrac{1-e^{-j5\omega}}{1-e^{-j\omega}}$.

(b) $x_b[n]=\alpha^n\mu[n]-\alpha^n\mu[n-8]$, $|\alpha|<1$. From Table 6.1,
$X_b(z)=\dfrac{1}{1-\alpha z^{-1}}-\dfrac{z^{-8}}{1-\alpha z^{-1}}=\dfrac{1-z^{-8}}{1-\alpha z^{-1}}$. The ROC is exterior to the circle at $|z|=|\alpha|<1$. Hence, the ROC includes the unit circle. On the unit circle,
$X_b(z)\big|_{z=e^{j\omega}}=X_b(e^{j\omega})=\dfrac{1-e^{j8\omega}}{1-\alpha e^{j\omega}}$.

(c) $x_c[n]=(n+1)\alpha^n\mu[n]=n\alpha^n\mu[n]+\alpha^n\mu[n]$, $|\alpha|<1$. From Table 6.1,

$X_c(z) = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha z^{-1}}$. The ROC is exterior to the circle at $|z| = |\alpha| < 1$. Hence, the ROC includes the unit circle. On the unit circle,

$$X_c(e^{j\omega}) = \frac{1+\alpha e^{j\omega}}{1-e^{j\omega}}.$$

6.15 **(a)** $Y_1(z) = \sum_{n=-N}^{N} z^{-n} = z^N\left(\sum_{n=0}^{2N} z^{-n}\right) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})}$. $Y_1(z)$ has N poles at $z=0$ and N poles at $z=\infty$. Hence, the ROC is the entire z–plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit circle,

$$Y_1(z)\big|_{z=e^{j\omega}} = Y_1(e^{j\omega}) = \frac{1-e^{j(2N+1)\omega}}{e^{-jN\omega}(1-e^{-j\omega})} = \frac{\sin\left(\omega(N+\frac{1}{2})\right)}{\sin(\omega/2)}.$$

(b) $Y_2(z) = \sum_{n=0}^{N} z^{-n} = \frac{1-z^{-(N+1)}}{1-z^{-1}}$. $Y_2(z)$ has N poles at $z=0$. Hence, the ROC is the entire z–plane excluding the point $z=0$. On the unit circle,

$$Y_2(z)\big|_{z=e^{j\omega}} = Y_2(e^{j\omega}) = \frac{1-e^{j(N+1)\omega}}{1-e^{-j\omega}} = e^{-N\omega/2}\,\frac{\sin\left(\frac{N+1}{2}\omega\right)}{\sin(\omega/2)}.$$

(c) $y_3[n] = \begin{cases} 1-\frac{|n|}{N}, & -N \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$ Now, $y_3[n] = y_0[n] \circledast y_0[n]$ where

$y_0[n] = \begin{cases} 1, & -\frac{N}{2} \le n \le \frac{N}{2}, \\ 0, & \text{otherwise.} \end{cases}$ Therefore, $Y_3(z) = Y_0^2(z) = \frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}$.

$Y_3(z)$ has $\frac{N}{2}$ poles at $z=0$ and $\frac{N}{2}$ poles at $z=\infty$. Hence, the ROC is the entire z–plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit circle, $Y_3(e^{j\omega}) = Y_0^2(e^{j\omega}) = \frac{\sin^2\left(\omega\left(\frac{N+1}{2}\right)\right)}{\sin^2(\omega/2)}$.

(d) $y_4[n] = \begin{cases} N+1-|n|, & -N \le n \le N, \\ 0, & \text{otherwise,} \end{cases} = y_1[n] + N\cdot y_3[n]$, where $y_1[n]$ is the sequence of Part (a) and $y_3[n]$ is the sequence of Part (c). Therefore,

$Y_4(z) = Y_1(z) + N\cdot Y_3(z) = \frac{1-z^{-(2N+1)}}{z^{-N}(1-z^{-1})} + N\,\frac{(1-z^{-(N+1)})^2}{z^{-N}(1-z^{-1})^2}$. Since the ROCs of both $Y_1(z)$ and $Y_3(z)$ include the unit circle, the ROC of $Y_4(z)$ also includes the unit circle. On the unit circle, $Y_4(e^{j\omega}) = \dfrac{\sin\left(\omega(N+\frac{1}{2})\right)}{\sin(\omega/2)} + \dfrac{\sin^2\left(\omega\left(\frac{N+1}{2}\right)\right)}{\sin^2(\omega/2)}$.

(e) $y_f[n] = \begin{cases}\cos(\pi n/2N), & -N \le n \le N,\\ 0, & \text{otherwise.}\end{cases}$ Therefore,

$$Y_f(z) = \frac{1}{2}\sum_{n=-N}^{N} e^{-j(\pi n/2N)} z^{-n} + \frac{1}{2}\sum_{n=-N}^{N} e^{j(\pi n/2N)} z^{-n}$$

$$= \frac{e^{j(\pi/2)} z^{N}}{2}\left(\frac{1-e^{-j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1-e^{-j(\pi/2N)} z^{-1}}\right) + \frac{e^{-j(\pi/2)} z^{N}}{2}\left(\frac{1-e^{j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1-e^{j(\pi/2N)} z^{-1}}\right).$$

$Y_f(z)$ has N poles at $z=0$ and N poles at $z=\infty$. Hence, the ROC is the entire $z-$plane excluding the points $z=0$ and $z=\infty$, and includes the unit circle. On the unit circle, $Y_f(e^{j\omega}) = \dfrac{1}{2}\dfrac{\sin\left(\left(\omega-\frac{\pi}{2N}\right)\left(N+\frac{1}{2}\right)\right)}{\sin\left(\left(\omega-\frac{\pi}{2N}\right)/2\right)} + \dfrac{1}{2}\dfrac{\sin\left(\left(\omega+\frac{\pi}{2N}\right)\left(N+\frac{1}{2}\right)\right)}{\sin\left(\left(\omega+\frac{\pi}{2N}\right)/2\right)}$.

6.16 $X(z) = -4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3}$, $Y(z) = 6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5}$, $W(z) = 3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8}$.

(a) $U(z) = X(z)Y(z)$

$= (-4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3})(6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5})$

$= -24z^4 + 42z^3 - 5z^2 - 20z - 45 + 23z^{-1} + 66z^{-2} - 25z^{-3} - 42z^{-4} - 17z^{-5}$
$+ 22z^{-6} + 14z^{-7} - 4z^{-8}$. Hence,

$\{u[n]\} = \{-24,\ 42,\ -5,\ -20,\ -45,\ 23,\ 66,\ -25,\ -42,\ -17,\ 22,\ 14,\ -4\}, -4 \le n \le 8.$

(b) $V(z) = X(z)W(z)$

$= (-4z^3 + 5z^2 + z - 2 - 3z^{-1} + 2z^{-3})(3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8})$

$= -12z + 7 + 5z^{-1} + 10z^{-2} - 16z^{-3} - 3z^{-4} - 28z^{-5} + 30z^{-6} + 13z^{-7} - 6z^{-8}$
$-15z^{-4} - 4z^{-10} + 10z^{-11}$. Hence,

$\{v[n]\} = \{-12,\ 7,\ 5,\ 10,\ -16,\ -3,\ -28,\ 30,\ 13,\ -6,\ -15,\ -4,\ 10\}, -1 \le n \le 11.$

(c) $G(z) = W(z)Y(z)$

$= (3z^{-2} + 2z^{-3} + 2z^{-4} - z^{-5} - 2z^{-7} + 5z^{-8})(6z - 3 - z^{-1} + 8z^{-3} + 7z^{-4} - 2z^{-5})$

$= 18z^{-1} + 3z^{-2} + 3z^{-3} - 14z^{-4} + 25z^{-5} + 26z^{-6} + 60z^{-7} - 11z^{-8} - 16z^{-9} - 14z^{-10}$

$+26z^{-11}+39z^{-12}-10z^{-13}$. Hence,
$\{g[n]\}=\{18,\ 3,\ 3,\ -14,\ 25,\ 26,\ 60,\ -11,\ -16,\ -14,\ 26,\ 39,\ -10\},\ 1\le n\le 13.$

6.17 $Y_L(z)=\sum_{n=0}^{2N-1}\left(\sum_{m=0}^{N-1}x[m]h[n-m]\right)z^{-n}$ and $Y_C(z)=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[\langle n-m\rangle_N]\right)z^{-n}$.

Now, $Y_L(z)$ can be rewritten as

$$Y_L(z)=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[n-m]\right)z^{-n}+\sum_{n=N}^{2N-1}\left(\sum_{m=0}^{N-1}x[m]h[n-m]\right)z^{-n}$$

$$=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[n-m]\right)z^{-n}+\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[k-m-N]\right)z^{-(k-N)}.$$ Therefore,

$$\langle Y_L(z)\rangle_{(z^{-N}-1)}=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[n-m]\right)z^{-n}+\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[\langle k-m\rangle_N]\right)z^{-k}$$

$$=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]h[\langle n-m\rangle_N]\right)z^{-n}=Y_C(z).$$

6.18 $G(z)=2-z^{-1}+3z^{-2}$, $H(z)=-2+4z^{-1}+2z^{-2}-z^{-3}$. Now,

$Y_L(z)=G(z)H(z)=(2-z^{-1}+3z^{-2})(-2+4z^{-1}+2z^{-2}-z^{-3})$

$=-4+10z^{-1}-6z^{-2}+8z^{-3}+7z^{-4}-3z^{-5}$. Therefore,

$y_L[n]=\{-4\quad 6\quad 2\quad 12\quad 5\quad -3\},\ 0\le n\le 5.$

Using the MATLAB statement `y = conv([2 -1 3], [-2 4 2 -1]);` we obtain

```
y =
-4     10     -6      8      7     -3
```

which is seen to be the same result as given above.

$Y_C(z)=\langle Y_C(z)\rangle_{(z^{-4}-1)}=\langle -4+10z^{-1}-6z^{-2}+8z^{-3}+7z^{-4}-3z^{-5}\rangle_{(z^{-4}-1)}$

$=-4+10z^{-1}-6z^{-2}+8z^{-3}+7-3z^{-1}=3+7z^{-1}-6z^{-2}+8z^{-3}$. Therefore,

$y_C[n]=\{1\quad 3\quad 2\quad 12\},\ 0\le n\le 3.$

Using the MATLAB statement `y = circonv([2 -1 3 0],[-2 4 2 -1]);` we obtain

```
y =
3      7     -6      8
```

which is seen to be the same result as given above.

6.19 $G(z)=\dfrac{P(z)}{D(z)}=\dfrac{P(z)}{(1-\lambda_\ell z^{-1})R(z)}$. The residue ρ_ℓ of $G(z)$ at the pole is given by

$\rho_\ell = \left.\dfrac{P(z)}{R(z)}\right|_{z=\lambda_\ell}$. Now,

$$D'(z) = \frac{dD(z)}{dz^{-1}} = \frac{d[(1-\lambda_\ell z^{-1})R(z)]}{dz^{-1}} = -\lambda_\ell R(z) + (1-\lambda_\ell z^{-1})\frac{dR(z)}{dz^{-1}}.$$ Hence,

$\left.D'(z)\right|_{z=\lambda_\ell} = -\lambda_\ell \left.R(z)\right|_{z=\lambda_\ell}$. Therefore, $\rho_\ell = -\lambda_\ell \left.\dfrac{P(z)}{D'(z)}\right|_{z=\lambda_\ell}$.

6.20 (a) $X_a(z) = \dfrac{3z}{(z+0.6)(z-0.3)} = \dfrac{3z^{-1}}{(1+0.6z^{-1})(1-0.3z^{-1})} = \dfrac{\rho_1}{1+0.6z^{-1}} + \dfrac{\rho_2}{1-0.3z^{-1}}$,

where $\rho_1 = \left.\dfrac{3}{z-0.3}\right|_{z=-0.6} = \dfrac{3}{-0.9} = -\dfrac{10}{3}$, $\rho_2 = \left.\dfrac{3}{z+0.6}\right|_{z=0.3} = \dfrac{3}{0.9} = \dfrac{10}{3}$.

Therefore, $X_a(z) = -\dfrac{10/3}{1+0.6z^{-1}} + \dfrac{10/3}{1-0.3z^{-1}}$.

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z–transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3}\left((-0.6)^n - (0.3)^n\right)\mu[-n-1]$.

The inverse z–transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = -\frac{10}{3}(-0.6)^n\mu[-n-1] + \frac{10}{3}(0.3)^n\mu[n]$.

The inverse z–transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

$\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \frac{10}{3}\left(-(-0.6)^n + (0.3)^n\right)\mu[n]$.

(b) $X_b(z) = \dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1+0.6z^{-1})(1-0.3z^{-1})^2} = K + \dfrac{\rho_1}{1+0.6z^{-1}} + \dfrac{\gamma_1}{1-0.3z^{-1}} + \dfrac{\gamma_2}{(1-0.3z^{-1})^2}$.

$K = X_b(0) = 0$, $\rho_1 = \left.\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{(1-0.3z^{-1})^2}\right|_{z=-0.6} = 2.7279$,

$\gamma_2 = \left.\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}}\right|_{z=0.3} = 0.6190$,

$\gamma_1 = \dfrac{1}{-0.3} \cdot \left.\dfrac{d}{dz^{-1}}\left(\dfrac{3z^{-1} + 0.1z^{-2} + 0.87z^{-3}}{1+0.6z^{-1}}\right)\right|_{z=0.3} = -0.3469$. Hence,

$X_b(z) = \dfrac{2.7279}{1+0.6z^{-1}} - \dfrac{0.3469}{1-0.3z^{-1}} + \dfrac{0.6190}{(1-0.3z^{-1})^2}$.

There are three ROCs - $\mathcal{R}_1: |z| < 0.3$, $\mathcal{R}_2: 0.3 < |z| < 0.6$, $\mathcal{R}_3: |z| > 0.6$.

The inverse z–transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(n+1)(-0.6)^n \mu[-n-1]$$
$$+\left(-0.3469+0.6190(n+1)\right)(0.3)^n \mu[-n-1].$$

The inverse z–transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = 2.7279(n+1)(-0.6)^n \mu[-n-1]$$
$$+\left(-0.3469+0.6190(n+1)\right)(0.3)^n \mu[n].$$

The inverse z–transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

$$\mathcal{Z}^{-1}\{X_b(z)\} = x_b[n] = 2.7279(-0.6)^n \mu[n] + \left(-0.3469+0.6190(n+1)\right)(0.3)^n \mu[n].$$

6.21 $G(z) = \dfrac{P(z)}{D(z)} = \dfrac{p_0 + p_1 z^{-1} + \cdots + p_M z^{-M}}{1 + d_1 z^{-1} + \cdots + d_N z^{-N}}$. Thus, $G(\infty) = \dfrac{P(\infty)}{D(\infty)}$. Now, a partial-fraction expansion of $G(z)$ in z^{-1} is given by $G(z) = \sum_{\ell=1}^{N} \dfrac{\rho_\ell}{1-\lambda_\ell z^{-1}}$, from which we obtain $G(\infty) = \sum_{\ell=1}^{N} \rho_\ell = \dfrac{p_0}{d_0}$.

6.22 $H(z) = \dfrac{1}{1-2r\cos\theta z^{-1} + r^2 z^{-2}}$, $|z| > r > 0$. By using partial-fraction expansion we write

$$H(z) = \frac{1}{(e^{j\theta} - e-j^{j\theta})}\left(\frac{e^{j\theta}}{1-re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1-re^{-j\theta}z^{-1}}\right) = \frac{1}{2\sin\theta}\left(\frac{e^{j\theta}}{1-re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1-re^{-j\theta}z^{-1}}\right).$$

Thus, $h[n] = \dfrac{1}{j2\sin\theta}\left(r^n e^{j\theta} e^{jn\theta}\mu[n] - r^n e^{-j\theta} e^{-jn\theta}\mu[n]\right) = \dfrac{r^n}{\sin\theta}\left(\dfrac{e^{j\theta(n+1)} - e^{-j\theta(n+1)}}{2j}\right)\mu[n]$

$$= \frac{r^n \sin\left((n+1)\theta\right)}{\sin\theta}\mu[n].$$

6.23 **(a)** $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{m=1}^{\infty} \alpha^{-m} z^m = \sum_{m=1}^{\infty} (z/\alpha)^m = \sum_{m=0}^{\infty} (z/\alpha)^m - 1$

$$= \frac{z/\alpha}{1-(z/\alpha)} = \frac{1}{1-\alpha z^{-1}}, \ |z| < |\alpha|.$$

(b) Using the differentiation property, we obtain from Part (a),

$Z\{nx[n]\} = -z\dfrac{dX(z)}{dz} = \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $|z| < |\alpha|$. Therefore, $Z\{y[n]\} = Z\{nx[n] + x[n]\}$

$$= \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} + \frac{1}{1-\alpha z^{-1}} = \frac{1}{(1-\alpha z^{-1})^2}, \ |z| < |\alpha|.$$

6.24 **(a)** Expanding $X_1(z)$ in a power series we get $X_1(z) = \sum_{n=0}^{\infty} z^{-3n}, \; |z| > 1$. Thus,

$x_1[n] = \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$ Alternately, using partial-fraction expansion we get

$X_1(z) = \dfrac{1}{1-z^{-3}} = \dfrac{\frac{1}{3}}{1-z^{-1}} + \dfrac{\frac{1}{3}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}} + \dfrac{\frac{1}{3}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}$. Therefore,

$x_1[n] = \frac{1}{3}\mu[n] + \frac{1}{3}\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\mu[n] + \frac{1}{3}\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\mu[n]$

$= \frac{1}{3}\mu[n] + \frac{1}{3}e^{-j2\pi n/3}\mu[n] + \frac{1}{3}e^{j2\pi n/3}\mu[n] = \frac{1}{3}\mu[n] + \frac{2}{3}\cos(2\pi n/3)\mu[n]$. Thus,

$x_1[n] = \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(b) Expanding $X_2(z)$ in a power series we get $X_2(z) = \sum_{n=0}^{\infty} z^{-4n}, \; |z| > 1$. Thus,

$x_2[n] = \begin{cases} 1, & \text{if } n = 4k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$ Alternately, using partial-fraction expansion we get

$X_2(z) = \dfrac{\frac{1}{4}}{1-z^{-1}} + \dfrac{\frac{1}{4}}{1+z^{-1}} + \dfrac{\frac{1}{4}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}} + \dfrac{\frac{1}{4}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}$. Thus,

$x_2[n] = \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{4}\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\mu[n] + \frac{1}{4}\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\mu[n]$

$= \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{4}e^{-j2\pi n/3}\mu[n] + \frac{1}{4}e^{j2\pi n/3}\mu[n]$

$= \frac{1}{4}\mu[n] + \frac{1}{4}(-1)^n\mu[n] + \frac{1}{2}\cos(2\pi n/3)\mu[n]$. Thus, $x_2[n] = \begin{cases} 1, & \text{if } n = 4k \text{ and } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

6.25 **(a)** $X_1(z) = \log(1-\alpha z^{-1}), \; |z| > |\alpha|$. Expanding $X_1(z)$ in a power series we get

$X_1(z) = -\alpha z^{-1} - \dfrac{\alpha^2 z^{-2}}{2} - \dfrac{\alpha^3 z^{-3}}{3} - \ldots = -\sum_{n=1}^{\infty}\dfrac{\alpha^n}{n}z^{-n}$. Therefore,

$x_1[n] = -\dfrac{\alpha^n}{n}\mu[n-1]$.

(b) $X_2(z) = \log\left(\dfrac{\alpha - z^{-1}}{\alpha}\right) = \log\left(1-(\alpha z)^{-1}\right), \; |z| < |\alpha|$. Expanding $X_2(z)$ in a power series

we get $X_2(z) = -(\alpha z)^{-1} - \dfrac{(\alpha z)^{-2}}{2} - \dfrac{(\alpha z)^{-3}}{3} - \ldots = -\sum_{n=1}^{\infty}\dfrac{(\alpha z)^{-n}}{n}$. Therefore,

$x_2[n] = -\frac{\alpha^{-n}}{n}\mu[n-1].$

(c) $X_3(z) = \log\left(\frac{1}{1-\alpha z^{-1}}\right), |z| > |\alpha|.$ Expanding $X_3(z)$ in a power series we get

$X_3(z) = \alpha z^{-1} + \frac{\alpha^2 z^{-2}}{2} + \frac{\alpha^3 z^{-3}}{3} \ldots = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}.$ Therefore, $x_3[n] = \frac{\alpha^n}{n}\mu[n-1].$

(d) $X_4(z) = \log\left(\frac{\alpha}{\alpha - z^{-1}}\right) = -\log\left(1-(\alpha z)^{-1}\right), |z| < |\alpha|.$ Expanding $X_4(z)$ in a power series we get $X_4(z) = (\alpha z)^{-1} + \frac{(\alpha z)^{-2}}{2} + \frac{(\alpha z)^{-3}}{3} - \ldots = \sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}.$ Therefore,

$x_4[n] = \frac{\alpha^{-n}}{n}\mu[n-1].$

6.26 $H(z) = \frac{z^{-1} + 1.7z^{-2}}{(1-0.3z^{-1})(1+0.5z^{-1})} = k + \frac{\rho_1}{1-0.3z^{-1}} + \frac{\rho_2}{1+0.5z^{-1}}$, where $k = H(0) = -\frac{34}{3}$,

$\rho_1 = \left.\frac{z^{-1} + 1.7z^{-2}}{1+0.5z^{-1}}\right|_{z=0.3} = \frac{25}{3}, \quad \rho_2 = \left.\frac{z^{-1} + 1.7z^{-2}}{1-0.3z^{-1}}\right|_{z=-0.5} = 3.$

The statement `[r,p,k]=residuez([0 1 1.7],conv([1 -0.3],[1 0.6]);` yields

```
r =
 3.0000
 8.3333

p =
-0.5000
 0.3000

k =
-11.3333
```

Thus, $H(z) = -\frac{34}{3} + \frac{25/3}{1-0.3z^{-1}} + \frac{3}{1+0.5z^{-1}}$. Hence, its inverse z–transform is given by

$h[n] = -\frac{34}{3}\delta[n] + \frac{25}{3}(0.3)^n\mu[n] + 3(-0.5)^n\mu[n].$

6.27 $G(z) = \mathcal{Z}\{g[n] = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ with a ROC given by $\mathcal{R}_g$ and $H(z) = \mathcal{Z}\{h[n] = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ with a ROC given by $\mathcal{R}_h$.

(a) $G^*(z) = \sum_{n=-\infty}^{\infty} g^*[n](z^*)^{-n}$ and $G^*(z^*) = \sum_{n=-\infty}^{\infty} g^*[n]z^{-n}$. Therefore,

$\mathcal{Z}\{g^*[n]\} = G^*(z^*)$ with a ROC given by $\mathcal{R}_g$.

(b) Replace n by $-m$ in the sum defining $G(z)$. Then
$\sum_{m=-\infty}^{\infty} g[-m]z^m = \sum_{m=-\infty}^{\infty} g[-m](1/z)^{-m} = G(1/z)$. Thus, $\mathcal{Z}\{g[-n]\} = G(1/z)$. Since z has been replaced by $1/z$, the ROC of is given by $1/\mathcal{R}_g$.

(c) Let $y[n] = \alpha g[n] + \beta h[n]$. Then
$Y(z) = \mathcal{Z}\{\alpha g[n] + \beta h[n] = \alpha\mathcal{Z}\{g[n] + \beta\mathcal{Z}\{h[n] = \alpha G(z) + \beta H(z)$. In this case $Y(z)$ will converge wherever both $G(z)$ and $H(z)$ converge. Hence the ROC of $Y(z)$ is $\mathcal{R}_g \cap \mathcal{R}_h$.

(d) $y[n] = g[n - n_o]$. Hence,

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n - n_0]z^{-n} = \sum_{m=-\infty}^{\infty} y[m]z^{-(m+n_o)}$$

$= z^{-n_o} \sum_{m=-\infty}^{\infty} g[n]z^{-m} = z^{-n_o} G(z)$. In this case, the ROC of $Y(z)$ is same as that of $G(z)$ except for the possible addition or elimination of the point $z = 0$ or $z = \infty$ (due to the factor of z^{-n_o}).

(e) $y[n] = \alpha^n g[n]$. Hence, $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \alpha g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n](z/\alpha)^{-n}$ $= G(z/\alpha)$. The ROC of $Y(z)$ is $|\alpha|\mathcal{R}_g$.

(f) $y[n] = n\,g[n]$. Hence, $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} ng[n]z^{-n}$. Now,
$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$. Thus, $\dfrac{dG(z)}{dz} = -\sum_{n=-\infty}^{\infty} ng[n]z^{-n-1}$. Hence,
$-z\dfrac{dG(z)}{dz} = \sum_{n=-\infty}^{\infty} ng[n]z^{-n-1}$. Thus, $Y(z) = \mathcal{Z}\{ng[n]\} = -\dfrac{dG(z)}{dz}$. In this case, the ROC of $Y(z)$ is same as that of $G(z)$ except possibly the point $z = 0$ or $z = \infty$.

6.28 From Eq. (6.111), for $N = 3$, we get
$\mathbf{D}_3 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{bmatrix}$. The determinant of $\mathbf{D}_3$ is given by

$$\det(\mathbf{D}_3) = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix} = \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} \\ 1 & z_2^{-1} + z_0^{-1} \end{vmatrix} = (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1})$$

$$= \prod_{2 \ge k \ge \ell \ge 0} (z_k^{-1} - z_\ell^{-1}).$$

From Eq. (6.111), for $N = 4$, we get

$$\mathbf{D}_4 = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{vmatrix}.$$ The determinant of $\mathbf{D}_4$ is given by

$$\det(\mathbf{D}_4) = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-3} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-3} \\ 0 & z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-3} \end{vmatrix}$$

$$= \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-2} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-2} \\ z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-2} \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1} z_0^{-1} + z_0^{-2} \\ 1 & z_2^{-1} + z_0^{-1} & z_2^{-2} + z_2^{-1} z_0^{-1} + z_0^{-2} \\ 1 & z_3^{-1} + z_0^{-1} & z_3^{-2} + z_3^{-1} z_0^{-1} + z_0^{-2} \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1} z_0^{-1} + z_0^{-2} \\ 0 & z_2^{-1} - z_0^{-1} & (z_2^{-1} - z_0^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ 0 & z_3^{-1} - z_0^{-1} & (z_3^{-1} - z_0^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} z_2^{-1} - z_1^{-1} & (z_2^{-1} - z_1^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ z_3^{-1} - z_1^{-1} & (z_3^{-1} - z_1^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix}$$

$$= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1})(z_3^{-1} - z_1^{-1})(z_3^{-1} - z_2^{-1})$$

$= \prod_{3 \ge k \ge \ell \ge 0} (z_k^{-1} - z_\ell^{-1})$. Hence, in the general case, $\det(\mathbf{D}_N) = \prod_{N-1 \ge k \ge \ell \ge 0} (z_k^{-1} - z_\ell^{-1})$. It follows from this expression that the determinant $\det(\mathbf{D}_N)$ is non-zero, i.e., $\mathbf{D}_N$ is non-singular, if the sampling points z_k are distinct.

6.29 $X(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}$. Thus,

$$X_{\text{NDFT}}[0] = X(z_0) = 1 - 2\left(-\frac{1}{2}\right)^{-1} + 3\left(-\frac{1}{2}\right)^{-2} - 4\left(-\frac{1}{2}\right)^{-3} = 1 + 4 + 3\cdot 4 + 4\cdot 8 = 49,$$

$$X_{\text{NDFT}}[1] = X(z_1) = 1 - 2 + 3 - 4 = -2,$$

$$X_{\text{NDFT}}[2] = X(z_2) = 1 - 2\left(\frac{1}{2}\right)^{-1} + 3\left(\frac{1}{2}\right)^{-2} - 4\left(\frac{1}{2}\right)^{-3} = 1 - 4 + 3\cdot 4 - 4\cdot 8 = -23,$$

$$X_{\text{NDFT}}[3] = X(z_3) = 1 - 2\left(\frac{1}{3}\right)^{-1} + 3\left(\frac{1}{3}\right)^{-2} - 4\left(\frac{1}{3}\right)^{-3} = 1 - 2\cdot 3 + 3\cdot 9 - 4\cdot 27 = -86.$$

$$I_0(z) = (1 - z^{-1})(1 - \tfrac{1}{2}z^{-1})(1 - \tfrac{1}{3}z^{-1}) = 1 - \frac{11}{6}z^{-1} + z^{-2} - \tfrac{1}{6}z^{-3} \Rightarrow I_0(-\tfrac{1}{2}) = 10,$$

$$I_1(z) = (1 + \tfrac{1}{2}z^{-1})(1 - \tfrac{1}{2}z^{-1})(1 - \tfrac{1}{3}z^{-1}) = 1 - \tfrac{1}{3}z^{-1} - \tfrac{1}{4}z^{-2} + \tfrac{1}{12}z^{-3} \Rightarrow I_1(1) = \frac{1}{2},$$

$$I_2(z) = (1 + \tfrac{1}{2}z^{-1})(1 - z^{-1})(1 - \tfrac{1}{3}z^{-1}) = 1 - \tfrac{5}{6}z^{-1} - \tfrac{1}{3}z^{-2} + \tfrac{1}{6}z^{-3} \Rightarrow I_2(\tfrac{1}{2}) = -\frac{2}{3},$$

$$I_3(z) = (1 + \tfrac{1}{2}z^{-1})(1 - z^{-1})(1 - \tfrac{1}{2}z^{-1}) = 1 - z^{-1} - \tfrac{1}{4}z^{-2} + \tfrac{1}{4}z^{-3} \Rightarrow I_3(\tfrac{1}{3}) = \frac{5}{2}.$$

Therefore, $X(z) = \dfrac{49}{10}I_0(z) + \dfrac{-2}{1/2}I_1(z) + \dfrac{-23}{-2/3}I_2(z) + \dfrac{-86}{5/2}I_3(z)$

$$= 4.9I_0(z) - 4I_1(z) + 34.5I_2(z) - 34.4I_3(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}.$$

6.30 **(a)** $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Let $\hat{X}(z) = \log(X(z)) \Rightarrow X(z) = e^{\hat{X}(z)}$. Thus,

$X(e^{j\omega}) = e^{\hat{X}(e^{j\omega})}$.

(b) $\hat{x}[n] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X(e^{j\omega})\right)e^{j\omega n}d\omega$. If $x[n]$ is real, then $X(e^{j\omega}) = X^*(e^{-j\omega})$.

Therefore, $\log\left(X(e^{j\omega})\right) = \log\left(X^*(e-j^{j\omega})\right)$.

$$\hat{x}^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X^*(e^{j\omega})\right)e^{-j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X(e^{-j\omega})\right)e^{-j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X(e^{j\omega})\right)e^{j\omega n}d\omega = \hat{x}[n].$$

(c) $\hat{x}_{ev}[n] = \dfrac{\hat{x}[n] + \hat{x}[-n]}{2} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X(e^{j\omega})\right)\left(\dfrac{e^{j\omega n} + e^{-j\omega n}}{2}\right)d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(X(e^{j\omega})\right)\cos(\omega n)d\omega.$$

Similarly, $\hat{x}_{ev}[n] = \dfrac{\hat{x}[n] - \hat{x}[-n]}{2} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\left(\dfrac{e^{j\omega n} - e^{-j\omega n}}{2}\right)d\omega$

$$= \frac{j}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\sin(\omega n)d\omega.$$

6.31 $x[n] = a\delta[n] + b\delta[n-1]$. Thus, $X(z) = \mathcal{Z}\{x[n]\} = a + bz^{-1}$. Also,

$$\hat{X}(z) = \log(a + bz^{-1}) = \log(a) + \log(1 + b/az^{-1}) = \log(a) + \sum_{n=-\infty}^{\infty}(-1)^{n-1}\frac{(b/a)^n}{n}z^{-n}.$$

Therefore, $\hat{x}[n] = \begin{cases} \log(a), & \text{if } n = 0, \\ (-1)^{n-1}\dfrac{(b/a)^n}{n}, & \text{for } n > 0, \\ 0, & \text{otherwise.} \end{cases}$

6.32 (a) $\hat{X}(z) = \log(K) + \sum_{k=1}^{N_\alpha}\log(1 - \alpha_k z^{-1}) + \sum_{k=1}^{N_\gamma}\log(1 - \gamma_k z) - \sum_{k=1}^{N_\beta}\log(1 - \beta_k z^{-1}) - \sum_{k=1}^{N_\delta}\log(1 - \delta_k z)$

$$= \log(K) - \sum_{k=1}^{N_\alpha}\sum_{n=1}^{\infty}\frac{\alpha_k^n}{n}z^{-n} - \sum_{k=1}^{N_\lambda}\sum_{n=1}^{\infty}\frac{\gamma_k^n}{n}z^{n} - \sum_{k=1}^{N_\beta}\sum_{n=1}^{\infty}\frac{\beta_k^n}{n}z^{-n} - \sum_{k=1}^{N_\delta}\sum_{n=1}^{\infty}\frac{\delta_k^n}{n}z^{n}.$$

Thus, $\hat{x}[n] = \begin{cases} \log(K), & n = 0, \\ \sum_{k=1}^{N_\beta}\sum_{n=1}^{\infty}\dfrac{\beta_k^n}{n} - \sum_{k=1}^{N_\alpha}\sum_{n=1}^{\infty}\dfrac{\alpha_k^n}{n}, & n > 0, \\ \sum_{k=1}^{N_\gamma}\sum_{n=1}^{\infty}\dfrac{\gamma_k^{-n}}{n} - \sum_{k=1}^{N_\delta}\sum_{n=1}^{\infty}\dfrac{\delta_k^{-n}}{n}, & n < 0. \end{cases}$

(b) $\left|\hat{x}[n]\right| < N\dfrac{|r|^n}{|n|}$ as $n \to \infty$, where r is the maximum value of $\alpha_k, \beta_k, \gamma_k$, and δ_k for all values of k, and N is a constant. Thus, $\hat{x}[n]$ is a decaying bounded sequence as $n \to \infty$.

(c) From Part (a) if $\alpha_k = \beta_k = 0$, then $\hat{x}[n] = 0$ for all $n > 0$, and is thus anti-causal.

(d) If $\gamma_k = \delta_k = 0$, then $\hat{x}[n] = 0$ for all $n < 0$, and is thus causal.

6.33 If $X(z)$ has no poles and zeros on the unit circle, then from Part (b) of Problem 6.32, $\gamma_k = \delta_k = 0$, then $\hat{x}[n] = 0$ for all $n < 0$.

$\hat{X}(z) = \log\left(X(z)\right)$. Therefore, $\dfrac{d\hat{X}(z)}{dz} = \dfrac{1}{X(z)}\dfrac{dX(z)}{dz}$. Thus, $z\dfrac{dX(z)}{dz} = zX(z)\dfrac{d\hat{X}(z)}{dz}$.

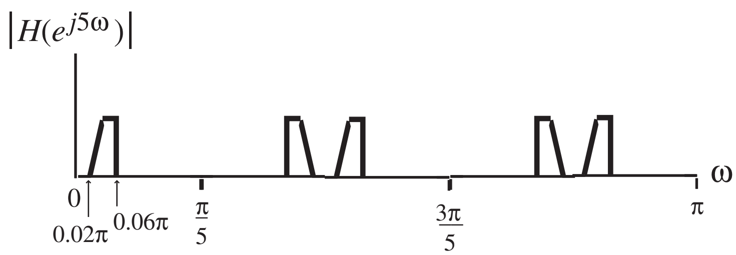
Taking the inverse –transform we get $nx[n] = \sum_{k=0}^{n} k\,\hat{x}[k]x[n-k],\ n \neq 0.$ Or,

$x[n] = \sum_{k=0}^{n-1} \frac{k}{n}\hat{x}[k]x[n-k] + \hat{x}[n]x[0].$ Hence, $\hat{x}[n] = \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1}\left(\frac{k}{n}\right)\frac{\hat{x}[k]x[n-k]}{x[0]},\ n \neq 0.$

For $n = 0,\ \hat{x}[0] = \hat{X}(z)\big|_{z=\infty} = X(z)\big|_{z=\infty} = \log(x[0]).$ Thus,

$$\hat{x}[n] = \begin{cases} 0, & n < 0, \\ \log(x[0]), & n = 0, \\ \frac{x[n]}{x[0]} - \sum_{n=0}^{n-1}\left(\frac{k}{n}\right)\frac{\hat{x}[k]x[n-k]}{x[0]}, & n > 0. \end{cases}$$

6.34



6.35 Given the real part of a real, stable transfer function

$$H_{re}(e^{j\omega}) = \frac{\sum_{i=0}^{N} a_i \cos(i\omega)}{\sum_{i=0}^{N} b_i \cos(i\omega)} = \frac{A(e^{j\omega})}{B(e^{j\omega})}, \qquad (6\text{-}1)$$

the problem is to determine the transfer function $H(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{N} p_i z^{-i}}{\sum_{i=0}^{N} p_i z^{-i}}.$

(a) $H_{\text{re}}(e^{j\omega}) = \frac{1}{2}[H(e^{j\omega}) + H^*(e^{j\omega})] = \frac{1}{2}[H(e^{j\omega}) + H(e^{-j\omega})]$

$$= \frac{1}{2}[H(z) + H(z^{-1})]\Big|_{z=e^{j\omega}}.$$

Substituting $H(z) = P(z)/D(z)$ in the above we get

$$H_{\text{re}}(e^{j\omega}) = \frac{1}{2}\frac{P(z)D(z^{-1}) + P(z^{-1})D(z)}{D(z)D(z^{-1})}\Bigg|_{z=e^{j\omega}}, \qquad (6\text{-}2)$$

which is Eq. (6.117).

(b) Comparing Eqs. (6-1) and (6-2) we get $B(e^{j\omega}) = D(z)D(z^{-1})\big|_{z=e^{j\omega}}$, (6-3)

$$A(e^{j\omega}) = \frac{1}{2}[P(z)D(z^{-1}) + P(z^{-1})D(z)]\Big|_{z=e^{j\omega}}. \qquad (6\text{-}4)$$

Now, $D(z)$ is of the form $D(z) = Kz^{-N}\prod_{i=1}^{N}(z - z_i),$ (6-5)

where the z_i's are the roots of $B(z) = B(e^{j\omega})\Big|_{e^{j\omega}=z}$ inside the unit circle and K is a scalar constant. Putting $\omega = 0$ in Eq. (6-3) we get $B(1) = [D(1)]^2$, or $\sqrt{B(1)} = K\prod_{i=1}^{N}(1 - z_i)$. Hence,

$$K = \sqrt{B(1)}\,/\prod_{i=1}^{N}(1 - z_i). \quad (6\text{-}6)$$

(c) By analytic continuation, Eq. (6-4) yields $A(z) = \frac{1}{2}[P(z)D(z^{-1}) + P(z^{-1})D(z)].$ (6-7)

Substituting $A(z) = \frac{1}{2}\sum_{i=0}^{N} a_i(z^i + z^{-i})$ and the polynomial forms of $P(z)$ and $D(z)$, we get $\sum_{i=0}^{N} a_i(z^i + z^{-i}) = \left(\sum_{i=0}^{N} p_i z^{-i}\right)\left(\sum_{i=0}^{N} d_i z^{i}\right) + \left(\sum_{i=0}^{N} p_i z^{i}\right)\left(\sum_{i=0}^{N} d_i z^{-i}\right)$ and equating the coefficients of $(z^i + z^{-i})$ on both sides, we arrive at a set of N equations which can be solved for the numerator coefficients p_i of $H(z)$.

For the given example, i.e., $H_{re}(e) = \dfrac{1 + \cos(\omega) + \cos(2\omega)}{17 - 8 + \cos(2\omega)}$, we observe

$$A(z) = 1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}). \quad (6\text{-}8)$$

Also, $B(z) = 17 - 4(z^2 + z^{-2})$, which has roots at $z = \pm\frac{1}{2}$ and $z = \pm 2$. Hence,

$$D(z) = Kz^{-2}(z - \tfrac{1}{2})(z + \tfrac{1}{2}) = K(z^2 - \tfrac{1}{4})z^{-2}. \quad (6\text{-}9)$$

Also, from Eq. (6-6) we have $K = \sqrt{17 - 8}\,/(1 - \frac{1}{4}) = 4$, so that $D(z) = 4 - z^{-2}$.

Substituting Eqs. (6-8), (6-9) and $P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2}$ in Eq. (6-7) we get

$$1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}) = \left[(p_0 + p_1 z^{-1} + p_2 z^{-2})(4 - z^2) + (p_0 + p_1 z + p_2 z^2)(4 - z^{-2})\right]$$

Equating the coefficients of $(z^i + z^{-i})/2, 0 \le i \le 2,$ on both sides we get $4p_0 - p_2 = 1,\ 3p_1 = 1, 4p_2 - p_0 = 1.$ Solving these equations we then arrive at $p_0 = p_1 = p_2 = 1/3.$ Therefore, $H(z) = \dfrac{1 + z^{-1} + z^{-2}}{3(4 - z^{-2})}.$

6.36 $A(z) = \dfrac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}} \Rightarrow A(1) = 1$ and $A(-1) = -1$ if M is odd. In which case, $G(1) = H(1)$ and $G(-1) = H(-1)$. If is even, then $G(1) = H(1)$ and $G(-1) = H(1)$..

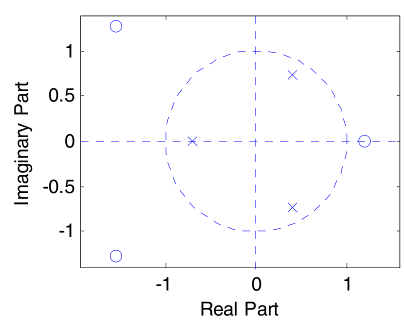
6.37 $H(z) = H_1(z)H_2(z) + H_3(z) = (1.2 + 3.3z^{-1} + 0.7z^{-2})(-4.1 - 2.5z^{-1} + 0.9z^{-2})$
$+ 2.3 + 4.3z^{-1} + 0.8z^{-2} = -2.62 - 12.23z^{-1} - 9.24z^{-2} + 1.22z^{-3} + 0.63z^{-4}.$

6.38 **(a)** $(1 - 0.1z^{-1} + 0.14z^{-2} + 0.49z^{-3})Y(z) = (5 + 9.5z^{-1} + 1.4z^{-2} - 24z^{-3})X(z) \Rightarrow$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{5 + 9.5z^{-1} + 1.4z^{-2} - 24z^{-3}}{1 - 0.1z^{-1} + 0.14z^{-2} + 0.49z^{-3}}$. Using Program 6_1.m we factorize $H(z)$ and develop is pole-zero plot shown below:

```
Numerator factors
1.00000000000000   3.10000000000000   4.00000000000000
1.00000000000000  -1.20000000000000                  0

Denominator factors
1.00000000000000  -0.80000000000000   0.70000000000000
1.00000000000000   0.70000000000000                  0

Gain constant
  5
```

The factored form of $H(z)$ is thus $H(z) = \dfrac{5(1 + 3.1z^{-1} + 4z^{-2})(1 - 1.2z^{-1})}{(1 - 0.81z^{-1} + 0.7z^{-2})(1 + 0.7z^{-1})}$



As all poles are inside the unit circle, $H(z)$ is BIBO stable.

(b) $(1 - 0.5z^{-1} + 0.1z^{-2} + 0.3z^{-3} - 0.0936\,z^{-4})Y(z)$
$= (5 + 16.5z^{-1} + 14.7z^{-2} - 22.4z^{-3} - 33.6z^{-4})X(z) \Rightarrow$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{5 + 16.5z^{-1} + 14.7z^{-2} - 22.4z^{-3} - 33.6z^{-4}}{1 - 0.5z^{-1} + 0.1z^{-2} + 0.3z^{-3} - 0.0936z^{-4}}$. Using Program 6_1.m we factorize $H(z)$ and develop is pole-zero plot shown below:

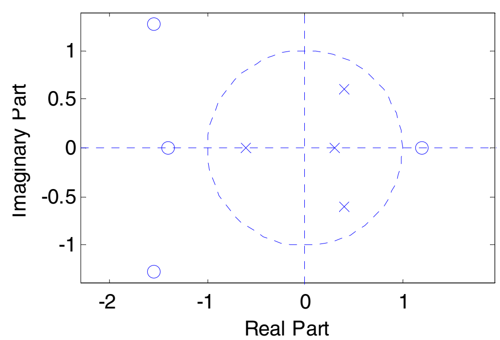
```
Numerator factors
   1.00000000000000  -1.20000000000000                  0
   1.00000000000000   3.10000000000001   4.00000000000001
```

```
   1.00000000000000   1.39999999999999                  0

Denominator factors
   1.00000000000000   0.59950918226500                  0
   1.00000000000000  -0.80131282790906   0.52021728677142
   1.00000000000000  -0.29819635435594                  0

Gain constant
     5
```

The factored form of $H(z)$ is thus

$$H(z) = \frac{5(1-1.2z^{-1})(1+3.1z^{-1}+4z^{-2})(1+1.4z^{-1})}{(1+0.5995z^{-1})(1-0.801313z^{-1}+0.520217z^{-2})(1-0.2982z^{-1})}$$

As all poles are inside the unit circle, $H(z)$ is BIBO stable.



6.39 A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields $H(z) = -\frac{1.21212}{1-0.4z^{-1}} + \frac{2.21212(1-0.81781z^{-1})}{1+0.5z^{-1}+0.3z^{-2}}$. Comparing the denominator of the quadratic factor with $1-2r\cos(\omega_o)z^{-1}+r^2z^{-2}$ we get $r = \sqrt{0.3} = 0.54772$ and $\cos(\omega_o) = -\frac{0.5}{2\sqrt{0.3}}$, or $\omega_o = 2.04478$. Hence, from Table 6.1 we have

$h[n] = -1.21212(0.4)^n \mu[n] + (\sqrt{0.3})^n \cos(2.04478n)\mu[n]$.

6.40 **(a)** A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields $H(z) = -2 + \frac{5}{1+0.6z^{-1}} - \frac{2}{1-0.3z^{-1}}$. Hence, from Table 6.1 we have

$h[n] = -2\delta[n] + 5(-0.6)^n \mu[n] - 2(0.3)^n \mu[n]$.

(b) $x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n]$. Its z–transform is thus given by

$X(z) = \frac{2.1}{1-0.4z^{-1}} + \frac{0.3}{1+0.3z^{-1}} = \frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}$, $|z| > 0.4$. The z–transform of the output is then given by $Y(z) = \left[\frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}\right]\cdot\left[\frac{1-3.3z^{-1}+0.36z^{-2}}{1+0.3z^{-1}-0.18z^{-2}}\right]$.

A partial-fraction expansion of $Y(z)$ in z^{-1} using the M-file `residuez` yields

$Y(z) = \frac{9.3}{1+0.6z^{-1}} - \frac{16.8}{1-0.4z^{-1}} + \frac{12.3}{1-0.3z^{-1}} - \frac{2.4}{1+0.3z^{-1}}$, $|z| > 0.6$. Hence, from Table 6.1 we have $y[n] = \left(9.3(-0.6)^n - 16.8(0.4)^n + 12.3(0.3)^n - 2.4(-0.3)^n\right)\mu[n]$.

6.41 **(a)** $H(z) = \mathcal{Z}\{h[n]\} = \frac{1}{1+0.4z^{-1}}$, $|z| > 0.4$, $X(z) = \mathcal{Z}\{x[n]\} = \frac{1}{1-0.2z^{-1}}$, $|z| > 0.4$. Thus,

$Y(z) = H(z)X(z) = \frac{1}{(1+0.4z^{-1})(1-0.2z^{-1})}$, $|z| > 0.4$. A partial-fraction expansion of using the M-file `residuez` yields $Y(z) = \frac{2/3}{1+0.4z^{-1}} + \frac{1/3}{1-0.2z^{-1}}$. Hence, from Table 6.1

$y[n] = \frac{2}{3}(-0.4)^n\mu[n] + \frac{1}{3}(0.2)^n\mu[n]$.

(b) $H(z) = \mathcal{Z}\{h[n]\} = \frac{1}{1+0.2z^{-1}}$, $|z| > 0.2$, $X(z) = \mathcal{Z}\{x[n]\} = \frac{1}{1+0.2z^{-1}}$, $|z| > 0.4$. Thus,

$Y(z) = H(z)X(z) = \frac{1}{(1+0.2z^{-1})^2}$, $|z| > 0.2$. Hence, from Table 6.1,

$y[n] = (n+1)(-0.2)^n\mu[n]$.

6.42 $Y(z) = \mathcal{Z}\{y[n]\} = \frac{2}{1+0.3z^{-1}}$, $|z| > 0.3$, $X(z) = Z\{x[n]\} = \frac{4}{1-0.6z^{-1}}$, $|z| > 0.2$. Thus,

$H(z) = \frac{Y(z)}{X(z)} = \frac{0.5(1-0.6z^{-1})}{1+0.3z^{-1}}$, $|z| > 0.3$. A partial-fraction expansion of using the M-file `residuez` yields $H(z) = -1 + \frac{1.5}{1+0.3z^{-1}}$. Hence, from Table 6.1,

$h[n] = -\delta[n] + 1.5(-0.3)^n\mu[n]$.

6.43 **(a)** Taking the –transform of both sides of the difference equation we get

$Y(z) = 0.2z^{-1}Y(z) + 0.08z^{-2}Y(z) + 2X(z)$. Hence, $H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.2z^{-1}-0.08z^{-2}}$.

(b) A partial-fraction expansion of using the M-file `residuez` yields

$H(z) = \frac{4/3}{1-0.4z^{-1}} + \frac{2/3}{1+0.2z^{-1}}$. Hence, from Table 6.1,

$h[n] = \frac{4}{3}(0.4)^n \mu[n] + \frac{2}{3}(-0.2)^n \mu[n]$.

(c) Now $S(z) = \mathcal{Z}\{s[n]\} = H(z) \cdot \mathcal{Z}\{\mu[n]\} = \frac{2}{(1-0.2z^{-1}-0.08z^{-2})(1-z^{-1})}$.

A partial-fraction expansion of using the M-file `residuez` yields

$S(z) = \frac{2.7778}{1-z^{-1}} - \frac{0.8889}{1-0.4z^{-1}} + \frac{0.1111}{1+0.2z^{-1}}$. Hence, from Table 6.1,

$s[n] = 2.7778\mu[n] - 0.8889(0.4)^n \mu[n] + 0.1111(-0.2)^n \mu[n]$.

6.44 $H(z) = \frac{1-z^{-2}}{1-(1+\alpha)\cos(\omega_c)z^{-1} + \alpha z^{-2}}$. Thus,

$H(e^{j\omega}) = \frac{1-e^{j2\omega}}{1-(1+\alpha)\cos(\omega_c)e^{-j\omega} + \alpha e^{-j2\omega}}$.

$\left|H(e^{j\omega})\right|^2 = \frac{2(1-\cos 2\omega)}{1+(\cos\omega_c)^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos 2\omega - 2\cos\omega_c(1+\alpha)^2\cos\omega}$

$= \frac{4\sin^2\omega}{(1+\alpha)^2(\cos\omega - \cos\omega_c) + (1-\alpha)^2\sin^2\omega}$. Note that $\left|H(e^{j\omega})\right|^2$ is maximum when

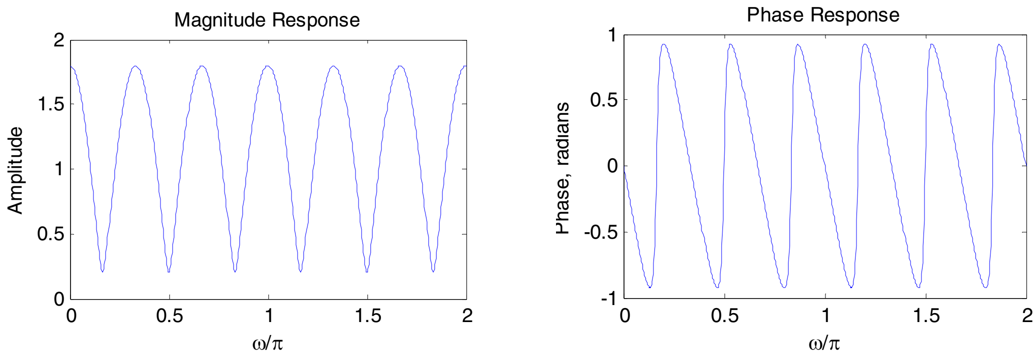
$\cos\omega = \cos\omega_c$, i.e, $\omega = \omega_c$. Then $\left|H(e^{j\omega_c})\right|^2 = \frac{4\sin^2\omega_c}{(1-\alpha)^2\sin^2\omega_c} = \frac{4}{(1-\alpha)^2}$, and hence,

$\left|H(e^{j\omega_c})\right| = 2/(1-\alpha)$.

6.45 $H(z) = 1-\alpha z^{-R} \Rightarrow \; H(e^{j\omega}) = 1-\alpha e^{-j\omega R}$. Then, $\left|H(e^{j\omega})\right| = \sqrt{1+\alpha^2 - 2\alpha\cos(\omega R)}$.

$\left|H(e^{j\omega})\right|$ is maximum when $\cos(\omega R) = -1$ and is minimum when $\cos(\omega R) = 1$. The maximum value of $\left|H(e^{j\omega})\right|$ is $1+|\alpha|$, and the minimum value is $1-|\alpha|$. $\left|H(e^{j\omega})\right|$ has R peaks and R dips in the range $0 \le \omega < 2\pi$.

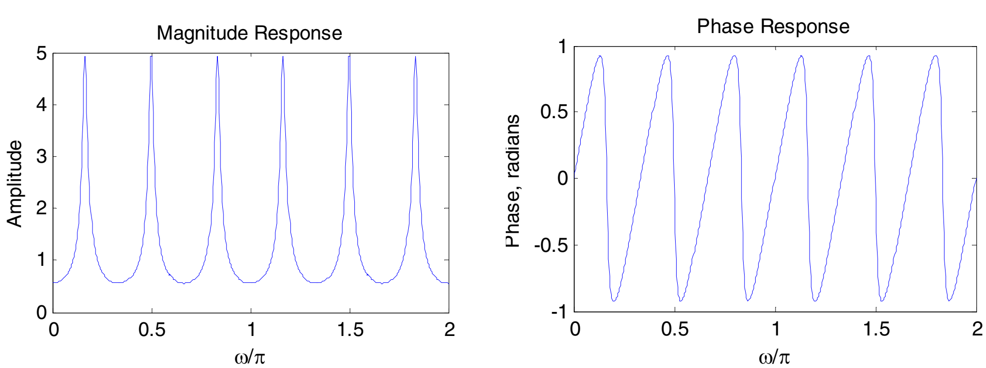
The peaks are located at $\omega = \omega_k = \frac{2\pi k}{R}$ and the dips are located at $\omega = \omega_k = \frac{(2k+1)\pi}{R}$, $0 \le k \le R-1$.



6.46 $G(e^{j\omega}) = \left(H(e^{j\omega})\right)^3 = (1 - \alpha e^{j\omega R})^3$.

6.47 $G(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-j\omega n} = \frac{1-\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$. Note that $G(e^{j\omega}) = H(e^{j\omega})$ for $\alpha^n = \frac{1}{M}$, $0 \le n \le M-1$. Now, $G(e^{j0}) = \frac{1-\alpha^M}{1-\alpha}$. Hence, to make the dc value of the magnitude response equal to unity, the impulse response should be multiplied by a constant $K = \left|(1-\alpha)/(1-\alpha^M)\right|$.

6.48 $Y(e^{j\omega}) = X(e^{j\omega}) + \alpha e^{-j\omega R} Y(e^{j\omega})$. Hence, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\alpha e^{-j\omega R}}$. Maximum value of $\left|H(e^{j\omega})\right|$ is $\frac{1}{1-|\alpha|}$ and the minimum value is $\frac{1}{1+|\alpha|}$. There are R peaks and dips in the range $0 \le \omega \le 2\pi$. The locations of the peaks and dips are given by $1 - \alpha e^{-j\omega R} = 1 \pm |\alpha|$ or, $e^{-j\omega R} = \pm \frac{|\alpha|}{\alpha}$. The locations of the peaks are given by $\omega = \omega_k = \frac{2\pi k}{R}$ and the locations of the dips are given by $\omega = \omega_k = \frac{(2\pi + 1)k}{R}$, $0 \le k \le R-1$. Plots of the magnitude and phase responses of $H(e^{j\omega})$ for $\alpha = 0.8$ and $R = 6$ are shown below:



6.49 $A(e^{j\omega}) = \dfrac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = \dfrac{(b_0 e^{j\omega} + b_2 e^{-j\omega}) + b_1}{(e^{j\omega} + a_2 e^{-j\omega}) + a_1}$

$= \dfrac{b_1 + (b_0 + b_2)\cos\omega + j(b_0 - b_2)\sin\omega}{a_1 + (1 + a_2)\cos\omega + j(1 - a_2)\sin\omega}$. Therefore,

$\left|A(e^{j\omega})\right|^2 = \dfrac{[b_1 + (b_0 + b_2)]^2 \cos^2\omega + (b_0 - b_2)^2 \sin^2\omega}{[a_1 + (1 + a_2)]^2 \cos^2\omega + (1 - a_2)^2 \sin^2\omega} = 1$. Hence, at $\omega = 0$, we have $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, and at $\omega = \pi/2$, we have $b_0 - b_2 = \pm(1 - a_2)$.

<u>Solution #1</u>: Consider $b_0 - b_2 = 1 - a_2$. Choose $b_0 = 1$, $-b_2 = 1 - a_2$, and $b_2 = a_2$. Substituting these values in $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, we get $b_1 = a_1$. In this case,

$A(e^{j\omega}) = \dfrac{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = 1$, a trivial solution.

<u>Solution #2</u>: Consider $b_0 - b_2 = a_2 - 1$. Choose $b_0 = a_2$ and $b_2 = 1$. Substituting these values in $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$, we get $b_1 = a_1$. In this case,

$A(e^{j\omega}) = \dfrac{a_2 + a_1 e^{-j\omega} + e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}$.

6.50 From Eq. (2.20), the input-output relation of a factor-of-2 up-sampler is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The DTFT of $x_u[n]$ is therefore given by

$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\omega m} = X(e^{j2\omega})$ where $X(e^{j\omega})$ is the DTFT of $x[n]$.

6.51 $H(e^{j\omega}) = \dfrac{1}{1 - 0.5 e^{-j\omega}} = \dfrac{1}{1 - 0.5\cos\omega + j0.5\sin\omega}$. Thus,

$H(e^{\pm j\pi/4}) = \dfrac{1}{1 - 0.5\cos(\pm\pi/4) + j0.5\sin(\pm\pi/4)} = \dfrac{1}{0.6464 \pm j0.3536} = 0.6512 \mp j1.1907.$

Therefore, $\left|H(e^{\pm j\pi/4})\right| = 1.3572$ and $\arg\{H(e^{\pm j\pi/4})\} = \theta(\pm j\pi/4) = \mp 1.0703.$

Now, for an input $x[n] = \sin(\omega_o n)\mu[n]$, the steady-state output is given by

$y[n] = \left|H(e^{j\omega_o})\right| \sin(\omega_o n + \theta(\omega_o))$ which for $\omega_o = \pi/4$ reduces to

$y[n] = \left|H(e^{j\pi/4})\right| \sin\left(\frac{\pi}{4} n + \theta(\pi/4)\right) = 1.3572 \sin\left(\frac{\pi}{4} n - 1.0703\right).$

6.52 To guarantee the stability of $G(z)$, the transformation $z \to F(z)$ should be such that the unit circle remains inside the ROC after the mapping. If the points inside the unit circle after the mapping remains inside the unit circle, $G(z)$ will be causal and stable. On the other hand, if the points inside the unit circle after the mapping move outside the unit circle, $G(z)$ will be stable but anti-causal. For example, the mapping $z \to -z$ will ensure that $G(z)$ will be causal and stable, whereas, the mapping $z \to z^{-1}$ will result in a $G(z)$ that is stable, but anti-causal.

6.53

6.54 $\sum_{n=0}^{K} |h[n]|^2 = 0.95 \sum_{n=0}^{\infty} |h[n]|^2$. Since $H(z) = 1/(1-\beta z^{-1})$, $h[n] = (\beta)^n \mu[n]$. Thus,

$\frac{1-|\beta|^{2K}}{1-|\beta|^2} = \frac{0.95}{1-|\beta|^2}$. Solving this equation for we get $K = 0.5\frac{\log(0.05)}{\log(|\alpha|)}$.

6.55 Let the output of the predictor of Figure P6.3(a) be denoted by $E(z)$. Then analysis of this structure yields $E(z) = P(z)[U(z)+E(z)]$ and $U(z) = X(z) - E(z)$. From the first equation we have $E(z) = \frac{P(z)}{1-P(z)}U(z)$ which when substituted in the second equation yields

$H(z) = \frac{U(z)}{X(z)} = 1 - P(z).$

Analyzing Figure P6.3(b) we get $Y(z) = V(z) + P(z)Y(z)$ which leads to

$G(z) = \frac{Y(z)}{V(z)} = \frac{1}{1-P(z)}$, which is seen to be the inverse of $H(z)$.

For $P(z) = h_1 z^{-1}$, $H(z) = 1 - h_1 z^{-1}$ and $G(z) = \frac{1}{1-h_1 z^{-1}}$. Similarly, for

$P(z) = h_1 z^{-1} + h_2 z^{-2}$, $H(z) = 1 - h_1 z^{-1} - h_2 z^{-2}$ and $G(z) = \frac{1}{1-h_1 z^{-1} - h_2 z^{-2}}$.

6.56 $Y(z) = [H_0(z)F_0(z) - H_0(-z)F_0(-z)]X(z)$. Since the output is a delayed replica of the input, we must have $H_0(z)F_0(z) - H_0(-z)F_0(-z) = z^{-r}$. But, $H_0(z) = 1 + \alpha z^{-1}$. Hence, $(1+\alpha z^{-1})F_0(z) - (1-\alpha z^{-1})F_0(-z) = z^{-r}$. Let $F_0(z) = a_0 + a_1 z^{-1}$. This implies,

$2(a_0\alpha + a_1)z^{-1} = z^{-r}$.

The solution is therefore, $r = 1$ and $2(a_0\alpha + a_1) = 1$. One possible solution is thus $a_0 = 1/2$ and $a_1 = 1/4$. Hence, $F_0(z) = 0.25(1+z^{-1})$.

6.57 $H_1(z) = \mathcal{Z}\{h_1[n]\} = 1.2 + \frac{0.5}{1+0.5z^{-1}} - \frac{0.6}{1-0.2z^{-1}} = \frac{1.1-0.04z^{-1}-0.12z^{-2}}{1+0.3z^{-1}-0.1z^{-2}}$. The transfer function of the inverse transform is thus

$H_2(z) = \frac{1}{H_1(z)} = \frac{1+0.3z^{-1}-0.1z^{-2}}{1.1-0.04z^{-1}-0.12z^{-2}} = \frac{(1+0.5z^{-1})(1-0.2z^{-1})}{1.1(1-0.349z^{-1})(1+0.3126z^{-1})}$. As both poles are inside the unit circle, $H_2(z)$ is stable and causal with an ROC $|z| > 0.349$. A partial-fraction expansion in obtained using the M-file `residuez` is

$H_2(z) = \frac{1}{1.2} + \frac{0.498}{1-0.34897z^{-1}} - \frac{0.42224464}{1+0.31261z^{-1}}$. Hence,

$h_2[n] = \frac{1}{1.2}\delta[n] + 0.498(0.34897)^n \mu[n] - 0.42224464(-0.31261)^n \mu[n]$.

6.58 Now, $H(z)\big|_{z=e^{j\omega}} = H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$. Denote $H'(z) = \frac{dH(z)}{dz}$.

From the above we get $\ln H(z)\big|_{z=e^{j\omega}} = \ln|H(e^{j\omega})| + j\theta(\omega)$. Therefore,

$$\frac{H'(z)}{H(z)} \cdot \frac{dz}{d\omega}\bigg|_{z=e^{j\omega}} = \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|} + j\frac{d\theta(\omega)}{d\omega} = \frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|} - j\tau_g(\omega). \qquad \text{(6-A)}$$

Hence, $\tau_g(\omega) = -z\frac{H'(z)}{H(z)}\bigg|_{z=e^{j\omega}} - j\frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|}$. (6-B)

Replacing by in Eq, (6-A) we arrive at

$$\tau_g(\omega) = -z^{-1}\frac{H'(z)}{H(z)}\bigg|_{z=e^{j\omega}} + j\frac{d|H(e^{j\omega})|/d\omega}{|H(e^{j\omega})|}. \qquad \text{(6-C)}$$

Adding Eqs. (6-B) and (6-C), and making use of the notation $T(z) = z\frac{H'(z)}{H(z)}$ we finally get

$$\tau_g(\omega) = -\frac{T(z)+T(z^{-1})}{2}\bigg|_{z=e^{j\omega}}.$$

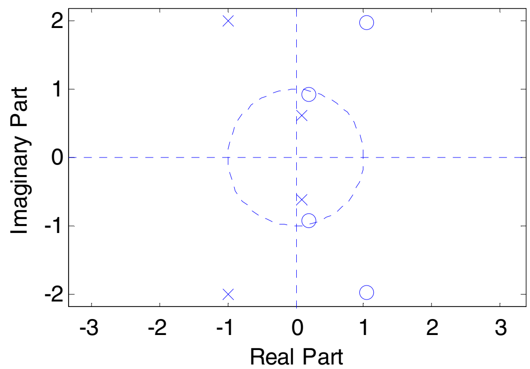
M6.1 **(a)** The output data generated by Program 6_1 is as follows:

```
Numerator factors
1.00000000000000  -2.10000000000000   5.00000000000000
1.00000000000000  -0.40000000000000   0.90000000000000

Denominator factors
1.00000000000000   2.00000000000000   4.99999999999999
1.00000000000000  -0.20000000000000   0.40000000000001
```

```
Gain constant
0.50000000000000
```

Hence, $G_1(z) = \dfrac{0.5(1-2.1z^{-1}+5z^{-2})(1+2z^{-1}+0.9z^{-2})}{(1+2z^{-1}+5z^{-2})(1-0.2z^{-1}+0.4z^{-2})}$.

The pole-zero plot of $G_1(z)$ is given below:



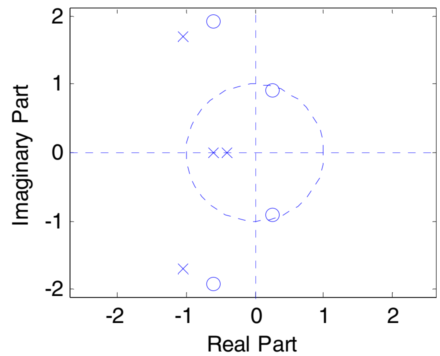
There are 3 ROCs associated with $G_1(z)$: $\mathcal{R}_1 : |z| < \sqrt{0.4}$, $\mathcal{R}_2 : \sqrt{0.4} < |z| < \sqrt{5}$, and $\mathcal{R}_3 : |z| > \sqrt{5}$. The inverse z–transform corresponding to the ROC $\mathcal{R}_1$ is a left-sided sequence, the inverse z–transform corresponding to the ROC $\mathcal{R}_2$ is a two-sided sequence, and the inverse z–transform corresponding to the ROC $\mathcal{R}_3$ is a right-sided sequence.

(b) The output data generated by Program 6_1 is as follows:

```
Numerator factors
1.00000000000000   1.20000000000000   3.99999999999999
1.00000000000000  -0.50000000000000   0.90000000000001

Denominator factors
1.00000000000000   2.10000000000000   4.00000000000001
1.00000000000000   0.60000000000003                  0
1.00000000000000   0.39999999999997                  0

Gain constant
1
```

Hence, $G_2(z) = \dfrac{(1+1.2z^{-1}+4z^{-2})(1-0.5z^{-1}+0.9z^{-2})}{(1+2.1z^{-1}+4z^{-2})(1+0.6z^{-1})(1+0.4z^{-1})}$

The pole-zero plot of $G_2(z)$ is given below:



There are 4 ROCs associated with $G_2(z)$: $\mathcal{R}_1 : |z| < 0.4$, $\mathcal{R}_2 : 0.4 < |z| < 0.6$, $\mathcal{R}_3 : 0.6 < |z| < 2$, and $\mathcal{R}_4 : |z| > 2$. The inverse z–transform corresponding to the ROC $\mathcal{R}_1$ is a left-sided sequence, the inverse z–transform corresponding to the ROC $\mathcal{R}_2$ is a two-sided sequence, the inverse z–transform corresponding to the ROC $\mathcal{R}_3$ is a two-sided sequence, and the inverse z–transform corresponding to the ROC $\mathcal{R}_4$ is a right-sided sequence.

M6.2 **(a)** The output data generated by Program 6_3 is as follows:
Residues
-3.33333333333333   3.33333333333333

Poles
-0.60000000000000   0.30000000000000

Constants
0
Hence, the partial-fraction expansion of $X_a(z)$ is given by

$X_a(z) = -\frac{10/3}{1+0.6z^{-1}} + \frac{10/3}{1-0.3z^{-1}}$. The z–transform has poles at $z = -0.6$ and at $z = 0.3$. Thus, it is associated with ROCs as given in the solution of Problem 6.20 which also shows their corresponding inverse z–transform.

(b) The output data generated by Program 6_3 is as follows:

```
Residues
Columns 1 through 2
2.33333333333333     -3.66666666666667 + 0.00000008829151i
Column 3
4.33333333333333 - 0.00000008829151i

Poles
Columns 1 through 2

-0.60000000000000    0.30000000000000 - 0.00000000722385i
Column 3
```

```
0.30000000000000 + 0.00000000722385i

Constants
```

Hence, the partial-fraction expansion of $X_b(z)$ is given by

$$X_b(z) = \frac{2.3333}{1+0.6z^{-1}} - \frac{3.66666}{1-0.3z^{-1}} + \frac{4.33333}{(1-0.3z^{-1})^2}.$$ The z–transform has two poles at $z = -0.6$ and one at $z = 0.3$. Thus, it is associated with ROCs as given in the solution of Problem 6.20 which also shows their corresponding inverse z–transform.

M6.3 **(a)** $X_1(z) = 3 - \frac{4}{5+z^{-1}} - \frac{7}{6+z^{-1}} = 3 - \frac{4/5}{1+(1/5)z^{-1}} - \frac{7/6}{1+(1/6)z^{-1}}.$

The output data generated by Program 6_4 is as follows:

```
Numerator polynomial coefficients
1.03333333333333   0.73333333333333  0.1000000000000

Denominator polynomial coefficients
1.00000000000000   0.36666666666667  0.03333333333333
```

Hence, $X_1(z) = \frac{31+22z^{-1}+3z^{-2}}{30+11z^{-1}+z^{-2}}.$

(b) $X_2(z) = -2.5 + \frac{3}{1+0.4z^{-1}} - \frac{1.4+z^{-1}}{1+0.6z^{-2}}$

$$= -2.5 + \frac{3}{1+0.4z^{-1}} - \frac{0.7 - j6454972243679}{1-j0.774596669z^{-1}} - \frac{0.7 + j6454972243679}{1+j0.774596669z^{-1}}.$$

The output data generated by Program 6_4 is as follows:

```
Numerator polynomial coefficients
-0.9000    -2.5600    -0.1000   -0.6000

Denominator polynomial coefficients
1.0000     0.4000     0.6000     0.2400
```

Hence, $X_b(z) = -\frac{0.9+2.56z^{-1}+0.1z^{-2}+0.6z^{-3}}{1.0+0.4z^{-1}+0.6z^{-2}+0.24z^{-3}}.$

(c) $X_3(z) = \frac{5}{1+0.64z^{-1}} + \frac{6}{4+2z^{-1}} + \frac{-4}{(4+2z^{-1})^2}$

$$= \frac{5}{1+0.64z^{-1}} + \frac{1.5}{1+0.5z^{-1}} + \frac{-0.25}{(1+0.5z^{-1})^2}.$$

The output data generated by Program 6_4 is as follows:

```
Numerator polynomial coefficients
    6.2500    6.5500    1.7300         0

Denominator polynomial coefficients
```

```
    1.0000    1.6400    0.8900    0.1600
```

Hence, $X_3(z) = \dfrac{6.25 + 6.55z^{-1} + 1.73z^{-2}}{1 + 1.64z^{-1} + 0.89z^{-2} + 0.16z^{-3}}$.

(d) $X_4(z) = -5 + \dfrac{2}{4 + 3z^{-1}} + \dfrac{z^{-1}}{4 + 3z^{-1} + 0.9z^{-2}}$

$$= -5 + \frac{0.5}{1 + 0.75z^{-1}} - \frac{j0.4303}{1 + (0.3750 - \mathrm{j}0.2905)z^{-1}} + \frac{j0.4303}{1 + (0.3750 + \mathrm{j}0.2905)z^{-1}}.$$

The output data generated by Program 6_4 is as follows:

```
Numerator polynomial coefficients
   -4.5000   -6.8750   -3.6375   -0.8438

Denominator polynomial coefficients
    1.0000    1.5000    0.7875    0.1688
```

Hence, $X_4(z) = -\dfrac{4.5 + 6.875z^{-1} + 3.6375z^{-2} + 0.84375z^{-3}}{1 + 1.5z^{-1} + 0.7875z^{-2} + 0.16875z^{-3}}$.

M6.4 **(a)** The inverse $\mathcal{Z}$–transform of $X_a(z)$ from it partial-fraction expansion form is thus $x_a[n] = 3\delta[n] - \frac{4}{5}(-1/5)^n \mu[n] - \frac{7}{6}(-1/6)^n \mu[n]$. The first 10 samples of $x_a[n]$ obtained by evaluating this expression in MATAB are given by

```
Columns 1 through 4
1.0333333333   0.3544444444  -0.0644074074   0.0118012345

Columns 5 through 8
-0.0021802057   0.0004060343  -0.0000762057   0.0000144076

Columns 9 through 10

-0.0000027426   0.0000005254
```

The first 10 samples of the inverse z–transform of the rational form of $X_a(z)$ obtaining using the M-file `impz` are identical to the samples given above.

(b) The inverse $\mathcal{Z}$–transform of $X_b(z)$ from it partial-fraction expansion form is thus

$$x_b[n] = -2.5\delta[n] + 3(-0.4)^n \mu[n] - (0.7 - j0.6454972243679)(j0.774596669241)^n \mu[n]$$
$$- (0.7 + j0.6454972243679)(-j0.774596669241)^n \mu[n].$$

The first 10 samples of $x_b[n]$ obtained by evaluating this expression in MATAB are given by

```
Columns 1 through 4
-0.900000000  -2.200000000   1.320000000   0.408000000
```

```
Columns 5 through 8
-0.427200000  -0.390720000   0.314688000   0.211084800

Columns 9 through 10
-0.179473920  -0.130386432
```

The first 10 samples of the inverse z–transform of the rational form of $X_b(z)$ obtaining using the M-file `impz` are identical to the samples given above.

(c) The inverse $\mathcal{Z}$–transform of $X_c(z)$ from it partial-fraction expansion form is thus

$$x_c[n] = 5(-0.64)^n \mu[n] + 1.5(-0.5)^n \mu[n] - 0.25(n+1)(-0.5)^n \mu[n].$$

The first 10 samples of $x_c[n]$ obtained by evaluating this expression in MATAB are given by

```
Columns 1 through 5
6.250000000  -3.700000000   2.235500000  -1.373220000
0.854485800

Columns 6 through 10
-0.536870912   0.3396911337  -0.215996076   0.137807801
-0.0881188675
```

The first 10 samples of the inverse z–transform of the rational form of $X_c(z)$ obtaining using the M-file `impz` are identical to the samples given above.

(d) The inverse $\mathcal{Z}$–transform of $X_d(z)$ from it partial-fraction expansion form is thus

$$x_4[n] = -5 + 0.5(-0.75)^n \mu[n] - j0.430331483(-0.375 + j0.29047375)^n \mu[n]$$
$$+ j0.430331483(-0.375 - j0.29047375)^n \mu[n].$$

The first 10 samples of $x_d[n]$ obtained by evaluating this expression in MATAB are given by

```
Columns 1 through 5
4.50000000000000  -0.12499999991919   0.09374999993940  -
0.12656249997773   0.13710937500069

Columns 6 through 10
-0.12181640625721   0.09610839844318  -0.07136938476820
0.05192523193410  -0.03790274963350
```

The first 10 samples of the inverse z–transform of the rational form of $X_d(z)$ obtaining using the M-file `impz` are identical to the samples given above.

M6.5 To verify using MATLAB that $H_2(z) = \dfrac{1 + 0.3z^{-1} - 0.1z^{-2}}{1.1 - 0.04z^{-1} - 0.12z^{-2}}$ is the inverse of $H_1(z) = \dfrac{1.1 - 0.04z^{-1} - 0.12z^{-2}}{1 + 0.3z^{-1} - 0.1z^{-2}}$, we determine the first 20 samples of $h_1[n]$ and $h_2[n]$, and then form the convolution of these two sequences using the M-file `conv`. The first samples of the convolution result are as follows:

Columns 1 through 9

1.0000 0 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 -0.0000

Columns 10 through 18

0.0000 -0.0000 -0.0000 -0.0000 -0.0000 0.0000 -0.0000 0.0000 0.0000

Columns 19 through 21

-0.0000 -0.0000 -0.0000

M6.6

```
% As an example try a sequence x = 0:24;
% calculate the actual uniform dft
% and then use these uniform samples
% with this ndft program to get the
% original sequence back
% [X,w] = freqz(x,1,25,'whole');
% use freq = X and points = exp(j*w)

freq = input('The sample values = ');
points = input('Frequencies at which samples are taken = ');
L = 1;
len = length(points);
val = zeros(size(1,len));
L = poly(points);
for k = 1:len
   if  (freq(k) ~= 0)
      xx  = [1 -points(k)];
      [yy, rr] = deconv(L,xx);
      F(k,:) = yy;
      Down = polyval(yy,points(k))*(points(k))*(points(k)^(-len+1));
      F(k,:) = freq(k)/down*yy;
      val = val+F(k,:);
   end
end
coeff = val;
```