EE 3025 S2005 Homework Set #11

(due 10:10 AM Friday, April 29, 2005)

Directions: Work all 5 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading.

1. A server processes one message packet at each of the times $n = 1, 2, 3, \ldots$. The number of message packets that arrive at the server between time $n-1$ and time n is a random variable V_n with the geometric distribution

$$P[V_n = k] = p^k(1-p), \quad k = 0, 1, 2, \ldots$$

(where p is a parameter satisfying $0 < p < 1$). The V_n's are assumed independent. Let Q_n be the number of message packets queued up at the server at time n. It is not hard to see that for each $n = 1, 2, 3 \ldots$

$$Q_n = \begin{cases} Q_{n-1} + V_n - 1, & Q_{n-1} > 0 \\ V_n, & Q_{n-1} = 0 \end{cases}$$

(we take $Q_0 = 0$).

(a) Let U be a random variable uniformly distributed between 0 and 1. Let X be the random variable

$$X = \left\lfloor \frac{\log(U)}{\log(p)} \right\rfloor$$

For each of the values $p = 1/3$ and $p = 2/3$, simulate 100000 values of X using Matlab and show that the empirical estimates of the probabilities $P[X = x]$ fit the geometric PMF closely for $x = 0, 1, \ldots, 9$.

(b) For each of the values

$$p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$$

simulate the first 50000 values of Q_n and compute the average of these simulated values. Put your results in a table.

(c) Based on the evidence compiled in (b), for what values of p would you guess that the queueing system is stable, and for what values of p would you guess that the queueing system is unstable?

2. Let $X(t)$ be a WSS Gaussian random process with mean -4 and autocorrelation function $R_X(\tau) = 16 + 5\exp(-|\tau|)$. Let $Y(t)$ be a WSS Gaussian random process independent of $X(t)$ with mean 5 and autocorrelation function $R_Y(\tau) = 25 + 20\exp(-|\tau|)$.

(a) The process

$$Z_1(t) = X(t)Y(t)$$

is WSS. Find the autocorrelation function $R_{Z_1}(\tau)$. Find the power generated by $Z_1(t)$.

(b) The process

$$Z_2(t) = X(t) + Y(t)$$

is WSS. Find the autocorrelation function $R_{Z_2}(\tau)$. Find the power generated by $Z_2(t)$.

(c) The process

$$Z_3(t) = X(t)^2 Y(t)^2$$

is WSS. Find the power generated by $Z_3(t)$. (Note: This is the only part of the problem where you use the fact that $X(t)$ and $Y(t)$ are both Gaussian processes.)

3. Consider the following two step experiment which generates a WSS discrete-time random process

$$X_n, \quad n = 1, 2, 3, \cdots.$$

Step 1: Choose a real number at random from the interval $[0, 1]$ according to the Uniform(0,1) distribution.

Step 2: Choose a coin whose prob of Heads is equal to the number chosen on Step 1. Toss this coin infinitely many times. On Toss n ($n = 1, 2, 3, \cdots$), define X_n to be 1 if the coin comes up heads and define X_n to be -1 if the coin comes up tails.

(a) By examining time averages along different realizations of the X_n process, explain why the X_n process fails to satisfy the definition of ergodic process.

(b) Compute μ_X, σ_X^2, P_X, and $R_X(\tau)$.

(c) By examining the quantities you computed in (b), exploit a result stated in class to give another reason why the X_n process fails to be ergodic.

(d) Explain how you could use the X_n process to estimate the random real number that was used on Step 1. (Hint: The averages

$$(X_1 + X_2 + X_3 + \cdots + X_N)/N$$

converge stochastically as $N \to \infty$ to a random variable limit. How is this limit related to the random real number that was used on Step 1?)

4. A WSS process $X(t)$ has power spectral density function

$$S_X(f) = \begin{cases} 1, & |f| \leq B \\ 0, & \text{elsewhere} \end{cases}$$

It is the input to an RC filter with frequency response function

$$H(f) = \frac{1}{1 + jRC(2\pi f)}$$

(a) Given that $B = 2000\pi$, determine RC so that the filter output power will be 75% of the filter input power. (You will need Matlab to get the final answer.)

(b) You are told that $RC = 10^{-2}$, and that the filter output power is 87.5% of the input power. Determine B. (As in part(a), Matlab will be necessary toward the end of your solution to (b).)

5. Let $(X(t) : -\infty < t < \infty)$ be a CT zero-mean WSS process with autocorrelation function

$$R_X(\tau) = 3\exp(-0.5|\tau|), \quad -\infty < \tau < \infty$$

Let $(Y(t) : -\infty < t < \infty)$ be the process

$$Y(t) = 4X(t) - 3X(t-1) + 2X(t-4), \quad -\infty < t < \infty$$

(a) Find the impulse response $h(t)$ and the frequency response $H(f)$ of the linear time-invariant filter which carries the X process into the Y process.

(b) Find $R_Y(\tau)$ using the formula

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

(c) Find $S_Y(f)$ using the formula

$$S_Y(f) = |H(f)|^2 S_X(f)$$

(d) Compute P_Y two different ways (one way using $R_Y(\tau)$ and one way using $S_Y(f)$).

Supplementary Problems: (not to hand in) From the textbook, you can try Problems 11.1.3, 11.2.6, 11.3.4, 11.8.1, 11.8.4