EE 3025 S2005 Homework Set #12

(due 10:10 AM Friday, May 6, 2005)

Directions: Work all 5 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading.

1. This Matlab problem concerns power spectrum estimation. Refer to Recitation 14 to help you with this problem.

A discrete-time ergodic Gaussian zero mean WSS process X_n has the autocorrelation function

$$R_X(\tau) = \begin{cases} 8, & \tau = 0 \\ -4, & \tau = \pm 1 \\ 1, & \tau = \pm 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the exact expression for $S_X(f)$ and then plot $S_X(f)$ using Matlab for $0 \leq f \leq 2$. Turn in printout of your plot. (You should see two periods of the $S_X(f)$ function in your plot.)

(b) Using the solution to Problem 1 of Homework Set 10, write a Matlab script that will simulate N consecutive samples of the process X_n. (N will be a parameter of your script.)

(c) Using your script for N simulated samples from (b), write and run a Matlab script to compute a periodogram estimate of $S_X(f)$ for $0 \leq f \leq 2$. Use as many samples N as your platform will allow you to use. Turn in your script and a plot of your estimated power spectral density function.

(d) Using your script for N simulated samples from (b), write and run a Matlab script to compute a Bartlett estimate of $S_X(f)$ for $0 \leq f \leq 2$. Use as many samples N as your platform will allow you to use. Turn in your script and a plot of your estimated power spectral density function.

2. Refer back to the WSS process $X(t)$ in Homework 10, Problem 4(a). This was a process whose realizations are all periodic signals with period 1. The autocorrelation function $R_X(\tau)$ was computed and you can find this function in the Solutions to Homework 10 and use it in the present problem. Suppose we pass random signal $X(t)$ through the ideal low pass filter with frequency response function $H(f)$ as follows:

$$H(f) = \begin{cases} 1, & -B \leq f \leq B \\ 0, & elsewhere \end{cases}$$

Let $Y(t)$ denote the output random signal coming out of this filter in response to filter input signal $X(t)$.

(a) Express $S_X(f)$ as a linear combination of the infinite collection of delta functions

$$\delta(f - i), \quad i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots$$

(b) For each of the five filter bandwidths

$$B = 1.5, 2.5, 3.5, 4.5, 5.5$$

compute the power ratio P_Y/P_X and then convert it to a percentage. Put your results in a two column table: the left column heading will be "bandwidth" and the right column heading will be "power ratio percentage".

(c) Among all possible filter bandwidths

$$B = N + 0.5, \quad N = 0, 1, 2, 3, \cdots,$$

find the smallest such bandwidth so that the power ratio P_Y/P_X, converted to a percentage, will be $\geq 90\%$.

3. In the block diagram below, the channel is an additive white noise channel in which the additive noise is modeled as a WSS process Z_n with $\mu_Z = 0$ and $R_Z(\tau) = \delta[\tau]$.

$$X_n \rightarrow \boxed{\text{channel}} \rightarrow Y_n = X_n + Z_n \rightarrow \boxed{\begin{array}{c}\text{optimal filter}\\ \text{(impulse response } h[n])\end{array}} \rightarrow \hat{X}_n$$

The channel input is a WSS process X_n satisfying $\mu_X = 0$ and

$$R_X(\tau) = 2\delta[\tau] + \delta[\tau - 1] + \delta[\tau + 1].$$

As usual, we assume that the random variables comprising the process X_n are independent of the random variables comprising the channel noise process Z_n. An optimal linear time-invariant filter with impulse response $h[n]$ is to be designed which filters the channel output random signal $Y_n = X_n + Z_n$ into a signal $\hat{X}_n$ so that the mean square estimation error

$$E[(X_n - \hat{X}_n)^2]$$

is minimized for all n. We suppose that the optimal filter is required to be a three-tap causal filter. This means that

$$h[n] = \begin{cases} 0, & n < 0 \\ 0, & n \geq 3 \end{cases}$$

The estimate $\hat{X}_n$ is therefore of the form

$$\hat{X}_n = h[0]Y_n + h[1]Y_{n-1} + h[2]Y_{n-2}.$$

(a) Use the orthogonality principle to set up three linear equations in the three unknowns $h[0], h[1], h[2]$. Solve for $h[0], h[1], h[2]$.

(b) The mean square estimation error $E[(X_n - \hat{X}_n)^2]$ for our optimal three-tap filter can be computed in terms of cross-correlations as follows:

$$\begin{aligned} E[(X_n - \hat{X}_n)^2] &= E[(X_n - \hat{X}_n)X_n] \\ &= = R_X(0) - h[0]E[X_nY_n] - h[1]E[X_nY_{n-1}] - h[2]E[X_nY_{n-2}] \end{aligned}$$

Compute the cross-correlations $E[X_nY_n]$, $E[X_nY_{n-1}]$, $E[X_nY_{n-2}]$. Then compute the estimation error in decibels as follows:

$$10\log_{10}\left[\frac{E[X_n^2]}{E[(X_n-\hat{X}_n)^2]}\right]. \tag{1}$$

(c) The best possible LTI estimation filter to estimate X_n is the so-called noncausal Wiener filter, which is an IIR LTI stable filter that possibly uses all the Y process samples (at all times). The mean square estimation error of the noncausal Wiener filter is known to be

$$E_{Wiener} = \int_0^1 \frac{S_X(f)S_Z(f)}{S_X(f)+S_Z(f)}df.$$

Compute what this is in decibels, that is, compute

$$10\log_{10}\left[\frac{E[X_n^2]}{E_{Wiener}}\right]. \tag{2}$$

(Meaning of this result: the difference of the decibel figures (2) and (1) tells you how much of an improvement can be made if you design a more sophisticated estimation filter.)

4. We assume the same channel model as Problem 3, with the same random channel input signal X_n, the same random channel noise signal Z_n, and the same channel output random signal $Y_n = X_n + Z_n$. In this problem, you are going to construct a 3-tap causal LTI filter with impulse response $h[n]$ to process the Y_n signal, but you are going to choose $h[n]$ in a different way than in Problem 3. We can write our block diagram as

$$X_n \rightarrow \boxed{\text{channel}} \rightarrow Y_n = X_n + Z_n \rightarrow \boxed{h[n]} \rightarrow X_n^0 + Z_n^0,$$

where X_n^0 is that component of the $h[n]$ filter output that is in response to X_n, and Z_n^0 is that component of the $h[n]$ filter output that is in response to Z_n. The so-called signal-to-noise ratio (SNR) at the $h[n]$ filter output is measured in decibels as

$$SNR(decibels) = 10\log_{10}\left[\frac{X_n^0\ power}{Z_n^0\ power}\right]. \tag{3}$$

For a three-tap filter $h[n]$, it is not hard to show that

$$\begin{aligned} X_n^0\ power &= \sum_{i=0}^{2}\sum_{j=0}^{2} h[i]h[j]R_X(i-j) \\ Z_n^0\ power &= h[0]^2 + h[1]^2 + h[2]^2 \end{aligned}$$

In Problem 3, you found the 3-tap filter $h[n]$ to minimize the mean square estimation error. In the current problem, you will instead design the 3-tap filter $h[n]$ to maximize the filter output SNR given by (3).

(a) Compute $SNR(decibels)$ in (3) for the case when $h[n] = \delta[n]$. (The filter $h[n]$ does nothing in this case, so this is the same thing as the SNR at the input to the $h[n]$ filter. This is the SNR figure we are trying to improve upon by properly designing the filter $h[n]$.)

(b) Compute $SNR(decibels)$ in (3) for the case when $h[n]$ is the 3-tap filter you found in Problem 3 (which minimized the MS estimation error). (The decibel figure you obtain here may or may not be bigger than the decibel figure you found in (a), where no filtering was done.)

(c) Use Matlab to find a choice for the 3 tap weights $h[0]$, $h[1]$, $h[2]$ which will make $SNR(decibels)$ in (3) a maximum. Give not only your choice for $h[0], h[1], h[2]$, but also give the $SNR(decibels)$ figure that this filter will give, which will be a higher decibel figure that for (a). (Hint: If you apply the same scaling factor to the tap weights $h[0], h[1], h[2]$, the $SNR(decibels)$ figure does not change. So, it is OK to assume that

$$h[0]^2 + h[1]^2 + h[2]^2 = 1.$$

Subject to this constraint, there might only be two possible solutions for $h[0]$, $h[1]$, $h[2]$ to maximize $SNR(decibels)$.)

5. Let $Z(t)$ be Gaussian white noise with $R_Z(\tau) = \delta(\tau)$. Let $X(t)$ be the Gaussian process

$$X(t) = \int_0^t sZ(s)ds, \quad t \geq 0.$$

(a) Compute the mean and variance of the random variable $X(4)$. Write down the PDF of this random variable.

(b) Compute $\mathrm{Cov}(X(4), X(7))$ and compute the correlation coefficient ρ for the random variables $X(4)$ and $X(7)$. Write down the joint PDF of the random variables $X(4)$ and $X(7)$.

Supplementary Problems: (not to hand in) From the textbook, you can try Problems 11.4.5, 11.8.6, 11.8.7, 11.8.10, 11.9.2