EE 3025 S2003 Dr. Kieffer

Hour Exam 2 — April 9, 2003

Problem 1

Given that $\mu_X = 1$ and $\sigma_X = 1$; also, given $X = x$, Y is conditionally uniformly distributed in the interval $[x, x+4]$. Compute the following:

(a) $E(Y)$

Solution. $E(Y|X = x)$ is the midpoint of the interval $[x, x+4]$. Therefore,

$$E(Y|X = x) = x + 2.$$

By the law of iterated expectation,

$$E(Y) = E(X + 2) = 3.$$

(b) $E(XY)$

Solution.

$$E(XY|X = x) = xE(Y|X = x) = x(x + 2) = x^2 + 2x$$

Applying the law of iterated expectation,

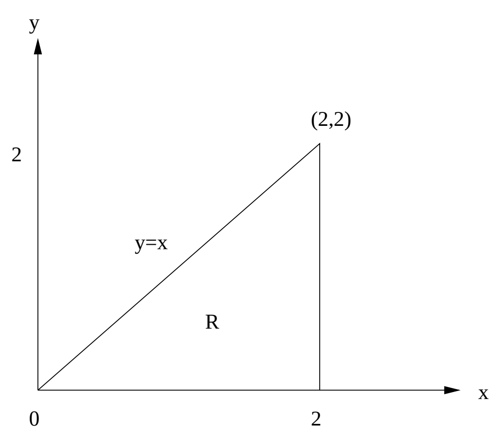
$$E(XY) = E(X^2 + 2X) = E(X^2) + 2E(X) = 4.$$

(c) $Cov(X, Y)$

Solution.

$$Cov(X, Y) = E(XY) - \mu_X \mu_Y = 4 - 3 = 1.$$

Problem 2



Given:

$$f_{X,Y}(x,y) = \begin{cases} (0.75)y, & (x,y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} E(Y|X=x) &= Ax + B, \quad 0 \le x \le 2 \\ E(X|Y=y) &= Cy + D, \quad 0 \le y \le 2 \end{aligned}$$

(a) What are the constants A, B?

Solution.

$$E(Y|X=x) = \frac{\int_0^x y^2 dy}{\int_0^x y dy} = 2x/3.$$

Therefore, $A = 2/3$ and $B = 0$.

(b) What are the constants C, D?

Solution. The conditional distribution for X given $Y = y$ is uniform. Therefore, $x = Cy + D$ is the straight line halfway between the straight lines forming the left and right boundaries of R, namely, the straight lines $x = y$ and $x = 2$. Therefore,

$$E(X|Y=y) = (y+2)/2,$$

and $C = 1/2$, $D = 1$.

Problem 3

$\bar{X}$ is the sample mean computed from 225 samples from a Gaussian distribution with unknown mean μ and variance $\sigma^2 = 900$.

(a) In the 90% confidence interval $[\bar{X} - C, \bar{X} + C]$ for μ, what is C?

Solution.

$$C = \frac{z_{0.05}\sigma}{\sqrt{n}} = \frac{1.645 * 30}{15} = 3.29.$$

(b) In the 95% confidence interval $[\bar{X} - D, \bar{X} + D]$ for μ, what is D?

Solution.

$$D = \frac{z_{0.025}\sigma}{\sqrt{n}} = \frac{1.96 * 30}{15} = 3.92.$$

Problem 4

A joint Gaussian density factors as

$$Ce^{-y^2/4}e^{-(x+3y)^2/4}.$$

Find:

(a) σ_Y^2

Solution. Use the "template"

$$Ce^{-y^2/2\sigma_y^2}e^{-(x-\mu_{x|y})^2/2\sigma_{y|x}^2}.$$

This tells us $\sigma_Y^2 = 2$, by inspection.

(b) $Var(X|Y = 0)$

Solution. Using the template again, we see that

$$Var(X|Y = 0) = \sigma_{x|y}^2 = 2.$$

(c) $Var(Y|X = 0)$

Solution. Plug $x = 0$ into the joint density, and simplify, getting

$$Ce^{-5y^2/2},$$

from which it is apparent that

$$Var(Y|X = 0) = 1/5$$

(d) $\rho_{X,Y}$

Solution. From the equation

$$\sigma_{y|x}^2 = \sigma_Y^2(1 - \rho^2),$$

you get

$$\rho^2 = 9/10.$$

Since $\mu_{x|y} = -3y$, ρ is negative. Therefore,

$$\rho = -\sqrt{9/10}.$$

Problem 5

Independent RV's X, Y have PMF's:

$$p_X(x) = \begin{cases} 1/3, & x = 1 \\ 1/3, & x = 2 \\ 1/3, & x = 4 \end{cases}$$

$$p_Y(y) = \begin{cases} 1/3, & y = 1 \\ 2/3, & y = 2 \end{cases}$$

Let $Z = X + Y$.

(a) $P(Z = 2) = ?$

Solution.

$$P(Z = 2) = P(X = 1, \;\; Y = 1) = p_X(1)p_Y(1) = 1/9.$$

(b) $P(Z = 3) = ?$

Solution.

$$\begin{aligned} P(Z = 3) &= P(X = 1, \;\; Y = 2) + P(X = 2, \;\; Y = 1) \\ &= p_X(1)p_Y(2) + p_X(2)p_Y(1) = 1/3 \end{aligned}$$

(c) $P(Z = 4) = ?$

Solution.

$$P(Z = 4) = P(X = 2, \;\; Y = 2) = p_X(2)p_Y(2) = 2/9.$$

(d) $P(Z = 5) = ?$

Solution.

$$P(Z = 5) = P(X = 4, \;\; Y = 1) = p_X(4)p_Y(1) = 1/9.$$

Problem 6

$X_1, X_2, X_3, X_4, X_5, X_6$ are independent, Gaussian RV's each having mean 0 and variance 1.

(a) $E[(X_1 + X_3 + X_5)(X_2 + X_4 + X_6)] = ?.$

Solution. The 2 RV's in parentheses are independent. Therefore, their correlation is the same as the product of their expected values which is $0 * 0 = 0$.

(b) $Cov(X_1 + X_2 + X_3 + X_4, X_3 + X_4 + X_5 + X_6) = ?.$

Solution. The common part of the two arguments of the covariance function in this case is $X_3 + X_4$. Therefore, the covariance reduces to

$$Cov(X_3 + X_4, X_3 + X_4) = Var(X_3 + X_4) = 2.$$

(c) $Var\left(\frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6}{6}\right) = ?.$

Solution. You square the $1/6$ when you pull it out of the variance operator, getting $1/36$. You are then left with the variance of the sum, which is 6. So the answer is $6/36 = 1/6$.

(d) Find constant A so that

$$\frac{X_1 + \sqrt{2}X_2 + \sqrt{3}X_3 + \sqrt{3}X_4 + \sqrt{2}X_5 + X_6}{A}$$

will be Gaussian(0,1).

Solution. The mean of the expression in the numerator is zero, so all we have to do is divide by the standard deviation. The variance of the expression is the sum of the squares of the coefficients, which is 12. Therefore, you should divide by $\sqrt{12}$.

Problem 7
Given $p_{X,Y}(x,y)$ table:

X \ Y	1	2	3
1	0	c/2	0
2	c	0	c
3	0	c/2	0

(a) Find c.

Solution. The numbers $c, c, c/2, c/2$ must add up to 1, which yields $c = 1/3$.

(b) Compute $P(X = 1)$, $P(X = 2)$, $P(X = 3)$.

Solution. These are the row sums, which are $1/6, 2/3, 1/6$, respectively.

(c) Compute $E(XY)$.

Solution. XY is equal to 2 half of the time and equal to 6 the other half of the time. Therefore, $E(XY)$ is equal to $(2 + 6)/2 = 4$.

(d) $P(X = Y + 1) =?$

Solution. For the given condition to be true, it is necessary and sufficient that (X, Y) belong to the set $\{(2, 1), (3, 2)\}$. The probability of this happending is $1/2$.

(e) $E(X|Y = 2)$, $Var(X|Y = 2)$ are?

Solution. Given $Y = 2$, X is equiprobably distributed over the two values $1, 3$. Therefore the conditional mean is $(1 + 3)/2 = 2$. The square of the deviation of each of the values $1, 3$ from the cond mean is 1. Therefore, the cond variance is 1.

Problem 8
Explain what number each Matlab script will approximately give.

(a)
```
x=-log(rand(1,50000));
y=-log(rand(1,50000));
mean((x+2*y).*(x-2*y))
```

Solution. This is an approximation to $E[(X + 2Y)(X - 2Y)]$, where X and Y are each exponentially distributed with mean 1 (and therefore variance 1). We have

$$E[(X + 2Y)(X - 2Y)] = E(X^2) - 4E(Y^2) = 2 - 4 * 2 = -6.$$

(b)
```
x=randn(1,50000);
y=randn(1,50000);
mean(x+y>sqrt(2))
```

Solution. This is an approximation to $P(U > \sqrt{2})$, where U is Gaussian with mean 0 and variance 2. Notice that $U/\sqrt{2}$ is a gaussian(0,1) RV Z. Therefore

$$P(U > \sqrt{2}) = P(Z > 1) = 1 - \Phi(1).$$

Problem 9

Let X be a continuous RV uniformly distributed in the interval $[0, 2]$. Let $Y = 4 - X^2$.

(a) Draw a plot of the curve $y = 4 - x^2$ in xy-plane for $0 \le x \le 2$.

Solution. Nearly everyone got this part of the problem.

(b) Compute $P(Y \le 2)$.

Solution.

$$P(Y \le 2) = P(X \ge \sqrt{2}) = (2 - \sqrt{2})/2$$

(c) Compute $P(Y \le y)$ as a function of y $(0 \le y \le 4)$.

Solution.

$$P(Y \le y) = P(X \ge \sqrt{4 - y}) = \frac{2 - \sqrt{4 - y}}{2}.$$

(d) Compute the density $f_Y(y)$ by differentiating $P(Y \le y)$.

Solution. The derivative is

$$\frac{1}{4\sqrt{4 - y}}$$

for $0 < y < 4$.

Problem 10

Let X, Y be independent discrete RV's each equidistributed over the set of values $\{1, 2, 3\}$.

(a) Find PMF of $U = \max(X, Y)$.

Solution. In the 3×3 (X, Y) array, we fill each position with the max of the x, y values for that position:

X \ Y	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

The answer is now clear:

$$P(U = 1) = 1/9, \quad P(U = 2) = 1/3, \quad P(U = 3) = 5/9.$$

(b) Find PMF of $V = \min(X, Y)$.

Solution. By symmetry, you can just turn the PMF obtained in part(a) around:

$$P(V = 1) = 5/9, \quad P(V = 2) = 1/3, \quad P(V = 3) = 1/9.$$