EE 3025 S2004 Dr. Kieffer

Exam 2 Solutions

In each problem below, determine the approximate value that the Matlab script will give. (Note to EE 3025 S2005 students: You will have no Matlab on your Exam 2, but the nonMatlab parts of this problem are still interesting.)

Problem 1:

```
x=randn(1,50000);
y=randn(1,50000);
z=randn(1,50000);
var(5*x-3*y-4*z)
```

Solution. You are estimating $Var(5X-3Y-4Z)$. Since each of X, Y, Z has variance 1, and they are independent, the answer must be

$$5^2 + (-3)^2 + (-4)^2 = 50.$$

Problem 2:

```
x=-log(rand(1,50000));
y=rand(1,50000);
u=x+2*y;
v=2*x-y;
mean(u.*v)
```

Solution.

$$E[UV] = E[(X+2Y)(2X-Y)] = 2E[X^2] + 3E[X]E[Y] - 2E[Y^2].$$

X is exponential with mean and variance 1, so $E[X^2] = 2$. Y is Uniform[0,1] with $E[Y] = 1/2$ and $E[Y^2] = 1/3$. So,

$$E[UV] = 29/6.$$

Let X, Y be RV's for which the following three statements all hold:

$$
\begin{aligned}
E[X|Y] &= (Y/3) + 2 \qquad (1) \\
E[Y|X] &= 2X - 1 \qquad (2) \\
Var[Y] &= 18 \qquad (3)
\end{aligned}
$$

Problem 3: Use the law of iterated expectation on equations (1) and (2) to compute $E[X]$, $E[Y]$.

Solution. You get the following system to solve:

$$
\begin{aligned}
E[X] &= E[Y]/3 + 2 \\
E[Y] &= 2E[X] - 1
\end{aligned}
$$

It follows that $E[Y] = 9$ and $E[X] = 5$.

Problem 4: Multiply equation (1) by Y, and then apply the law of iterated expectation along with equation (3) to compute the correlation $E[XY]$.

Solution. Following the hint, you get

$$E[XY] = E[Y^2]/3 + 2E[Y] = (81 + 18)/3 + 18 = 51.$$

Let R be the triangular region in the first quadrant of the xy-plane whose three vertices are $(0,0)$, $(0,4)$, and $(2,2)$. Let (X,Y) be uniformly distributed over R.

Problem 5: Compute μ_X, μ_Y.

Solution. Average up the three vertices. You get:

$$(\mu_X, \mu_Y) = (2/3, 2).$$

Problem 6: Compute $E[X|Y=3]$.

Solution. When $Y = 3$, X goes uniformly from 0 to 1. So, $E[X|Y=3] = (0+1)/2 = 0.5$.

Problem 7: Compute $P[Y > 1.75|X = 1]$.

Solution. When $X = 1$, Y goes uniformly from 1 to 3. So, the desired conditional probability is

$$\frac{3-1.75}{3-1} = 0.625.$$

Problem 8: Compute $P[X + Y < 1]$.

Solution. The area of the part of R below the line x+y=1 is 1/4. The area of R is 4. So, the answer is $(1/4)/4 = 1/16$.

Discrete RV's X, Y each take the values $-1, 0, 1$. The following is the joint PMF table:

0	1/8	1/4
1/8	0	1/8
1/8	1/4	0

Problem 9: Compute μ_X, σ_X.

Solution. The PMF values for X are 3/8, 1/4, 3/8, from which it is easily worked out that

$$\mu_X = 0, \quad \sigma_X = \sqrt{3}/2.$$

Problem 10: Compute μ_Y, σ_Y.

Solution. The PMF values for Y are $1/4, 3/8, 3/8$, from which one obtains

$$\mu_Y = 1/8, \quad \sigma_Y = \sqrt{39}/8.$$

Problem 11: Compute $E[XY]$, $Cov[X, Y]$, $\rho_{X,Y}$.

Solution.

$$E[XY] = (-1)(1)(1/4) + 1(-1)1/8 = -3/8.$$

The product of the means is 0, so this is the same thing as $\sigma_{X,Y}$.

$$\rho_{X,Y} = \frac{-3/8}{(\sqrt{3}/2)(\sqrt{39}/8)} = -0.555.$$

X, Y are jointly Gaussian with density

$$C * \exp\left[-0.5\left(\frac{x^2 - Axy + y^2}{5}\right)\right],$$

where C is a positive constant you don't have to know and A is a positive parameter that will be chosen in the problems below.

Problem 12: For $A = 0$, determine the common value of σ_X^2, σ_Y^2, and determine the value of $\rho_{X,Y}$.

Solution. The joint density factors as $f_X(x)f_Y(y)$ (no xy term in exponent). Therefore, $\rho = 0$. The common variances are 5.

Problem 13: For $A = 1$, determine the common value of σ_X^2, σ_Y^2, and determine the value of $\rho_{X,Y}$.

Solution. Let σ^2 be the common value of the variance. Then, from the xy term

$$\frac{2\rho}{(1-\rho^2)\sigma^2} = 1/5.$$

From the squared terms,

$$\frac{1}{(1-\rho^2)\sigma^2} = 1/5.$$

These two equations allow you to figure out that $\rho = 1/2$ and $\sigma^2 = 20/3$.

$\bar{X}$ is the sample mean of a sample of size n from a certain probability distribution with mean μ and known standard deviation σ.

Problem 14: Suppose the underlying distribution is Gaussian, and you want your confidence interval for μ to be

$$[\bar{X} - \frac{\sigma}{\sqrt{n}}, \ \ \bar{X} + \frac{\sigma}{\sqrt{n}}].$$

Use the table on the next page to compute the percentage of confidence for this confidence interval.

Solution. You compute the prob a gaussian RV lies within one standard deviation of its mean, which is 0.683. So, you have roughly a 68% confidence interval.

Problem 15: Suppose your confidence interval is now of the form

$$[\bar{X} - \frac{C\sigma}{\sqrt{n}}, \ \ \bar{X} + \frac{C\sigma}{\sqrt{n}}],$$

where the constant $C > 1$ is chosen so that you will achieve at least the percentage of confidence in Problem 14 for ANY underlying distribution and ANY n. Compute the value of C using Chebyshev's inequality.

Solution. Solve

$$1 - \frac{1}{C^2} = 0.683.$$

This gives $C = 1.775$.

Problem 16: As the result of Problems 14 and 15, you now have two confidence intervals. Let's say you use sample size $n_1 = 10000$ for the Problem 14 confidence interval, and then you choose sample size n_2 for the Problem 15 confidence interval to have the same width as the Problem 14 interval. Compute what n_2 would be.

Solution. Set

$$\frac{\sigma}{\sqrt{n_1}} = \frac{C\sigma}{\sqrt{n_2}}.$$

You get

$$n_2 = C^2 n_1 \approx 31500.$$

(X, Y) is continuously distributed over the square

$$S = \{(x, y) : 0 \le x \le 2, \ \ 0 \le y \le 2\}.$$

Its CDF function is defined over this square by

$$F(x, y) = Ax^2y^2 + \frac{xy}{16}, \quad (x, y) \in S.$$

Problem 17: Determine the value of the constant A.

Solution. Set $F(2, 2) = 1$. You get $A = 3/64$.

Problem 18: Compute the probability $P[0 < X < 1, \ \ 1 < Y < 2]$

Solution. The answer is $F(1, 2) - F(1, 1) = 13/64$.

Problem 19: By taking partial derivatives, find the expression for the joint density $f_{X,Y}(x, y)$ valid in the square S.

Solution. Take the partial derivative first with respect to x and then to y or vice-versa. You get

$$f_{X,Y}(x, y) = (3xy + 1)/16,$$

in the square S.