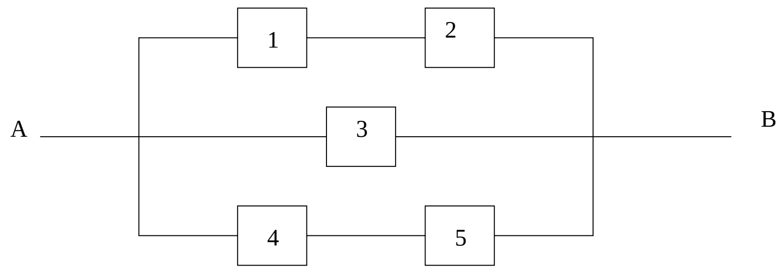
EE 3025 S2005 Exam 1 Solutions

Note: In each solution, I put the answer in simple enough form to be completely acceptable

1. The five relay switches in the circuit below operate independently:



p_i is the prob that switch i is closed ($i = 1, \cdots, 5$). $\{A \to B\}$ is the event that flow is possible from $A \to B$.

(a) Compute $P(\{A \to B\})$ in terms of the p_i's.

Solution. The circuit will not work if and only if all 3 parallel paths will not work, and these 3 probabilities of not working are

$$1 - p_1 p_2, 1 - p_3, 1 - p_4 p_5$$

Therefore

$$P(\{A \to B\}) = 1 - (1 - p_1 p_2)(1 - p_3)(1 - p_4 p_5)$$

In the remaining parts, assume all $p_i = 0.90$.

(b) $P[\textit{at least one switch is closed}] =?$

Solution. X, the number of switches closed, is Binomial(n, p) with $n = 5$ and $p = 0.90$. So the answer is

$$P[X \geq 1] = 1 - P[X = 0] = 1 - (0.10)^5.$$

(c) $P[\textit{exactly } 2 \textit{ switches closed}] =?$

Solution.

$$P[X = 2] = \binom{5}{2}(.90)^2(0.10)^3 = 10(.90)^2(0.10)^3.$$

(d) $P[\textit{exactly } 2 \textit{ closed} \mid \{A \to B\}] =?$

Solution. There are 10 ways exactly two switches can be closed. Out of those 10 ways, there are exactly 4 ways the circuit WILL NOT work:

- 3 will not work, exactly one of 1,2 will not work, and exactly one of 4,5 will not work. (4 ways)

These leaves 6 ways the circuit will work in which exactly two of the switches work. So the answer is

$$\frac{6(.90)^2(0.10)^3}{P[A \to B]}$$

2. A memory chip fabrication facility uses three lines to produce the chips: line A, line B, and line C. Line A produces 50% of the chips, line B produces 30% of the chips, and line C produces the rest. The percentage of good chips produced by lines A,B,C are 90, 80, and 70, respectively.

(a) What is the prob that a randomly sected chip from this facility is bad?

Solution. The matrix of forward cond probs:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} good & bad \end{array} \\ \begin{pmatrix} 0.90 & 0.10 \\ 0.80 & 0.20 \\ 0.70 & 0.30 \end{pmatrix} \end{array}$$

The matrix of joint probs:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} good & bad \end{array} \\ \begin{pmatrix} 0.50 * 0.90 & 0.50 * 0.10 \\ 0.30 * 0.80 & 0.30 * 0.20 \\ 0.20 * 0.70 & 0.20 * 0.30 \end{pmatrix} \end{array}$$

The prob of a bad chip is the sum of the right column:

$$0.50 * 0.10 + 0.30 * 0.20 + 0.20 * 0.30 = 0.17$$

(b) Given that a randomly selected chip is found to be bad, what is the cond prob that this chip came from line C?

Solution. Divide each column of the joint prob matrix by the col sum:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} good & bad \end{array} \\ \begin{pmatrix} 0.50 * 0.90/0.83 & 0.50 * 0.10.0.17 \\ 0.30 * 0.80/0.83 & 0.30 * 0.20/0.17 \\ 0.20 * 0.70/0.83 & 0.20 * 0.30/0.17 \end{pmatrix} \end{array}$$

The bottom entry of the right column is what we want:

$$0.20 * 0.30/0.17 = 6/17.$$

(c) Find the prob that a randomly selected chip is good and is selected from line A.

Solution. From the joint prob matrix, we want the entry in upper left corner:

$$0.50 * 0.90 = 0.45.$$

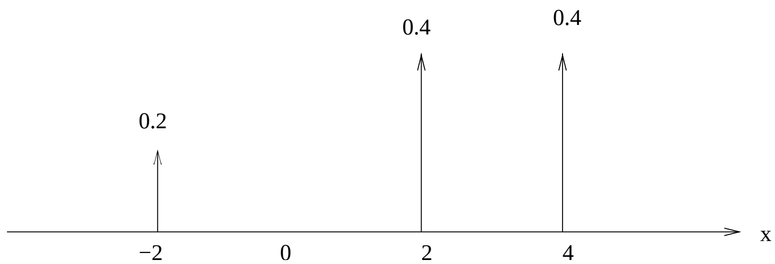
3. The CDF of RV X is:

$$F_X(x) = (0.2)u(x+2) + (0.4)u(x-2) + (0.4)u(x-4).$$

(a) Plot the density function $f_X(x)$.

Solution. Differentiating the CDF, we get

$$f_X(x) = (0.2)\delta(x+2) + (0.4)\delta(x-2) + (0.4)\delta(x-4).$$



(b) Find $E[X]$

Solution.

$$E[X] = 0.2(-2) + 0.4(2) + 0.4(4) = 2.$$

(c) Find the standard deviation of X.

Solution.

$$E[X^2] = 0.2(-2)^2 + 0.4(2)^2 + 0.4(4)^2 = 8.8.$$

$$Var[X] = E[X^2] - \mu_X^2 = 8.8 - 2^2 = 4.8.$$

$$\sigma_X = \sqrt{4.8}.$$

4. The number of message packets arriving at a server in a 10 millisecond time interval is a Poisson RV. The expected number of packets in 10 ms is α (unknown).

(a) In terms of α,

$$P[exactly\ 2\ packets\ in\ 20\ ms] = ?$$

Solution. The Poisson parameter is 2α. So, the answer is:

$$\exp(-2\alpha)(2\alpha)^2/2 = 2\exp(-2\alpha)\alpha^2.$$

(b) $P[exactly\ 2\ packets\ in\ 30\ ms] = ?$

Solution. The Poisson parameter is now 3α. So, the answer is:

$$\exp(-3\alpha)(3\alpha)^2/2 = (4.5)\exp(-3\alpha)\alpha^2.$$

(c) Assume the probs in (a),(b) are equal. Compute α.

Solution.

$$\alpha = \log_e(9/4).$$

(d) $P[exactly\ 2\ packets\ in\ 10\ ms | at\ least\ one]$

Solution. The prob of at least one packet is

$$1 - P[no\ packet] = 1 - \exp(-\alpha) = 5/9.$$

So, the answer is

$$\frac{P[exactly\ 2]}{5/9} = \frac{(4/9)0.5 * \alpha^2}{5/9} = .4 * \alpha^2.$$

5. X is a Gaussian RV with mean 1, var 4.

(a) Use Table on page 123 to compute $P[-1 < X < 3]$.
Solution. Use change of variable

$$Z = \frac{X-1}{2}.$$

The answer is

$$P[-1 < Z < 1] = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2 * (0.8413) - 1 = 0.6826.$$

(b) Compute $E[(2X+1)^2]$.
Solution. Expand as

$$E[4X^2 + 4X + 1] = 4E[X^2] + 4E[X] + 1 = 4E[X^2] + 5.$$

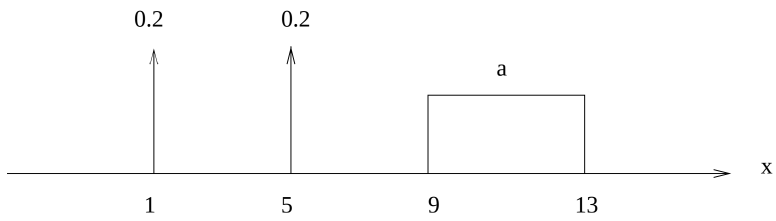
$$E[X^2] = Var[X] + \mu_X^2 = 5,$$

so the answer is 25.

(c) If $Y = aX + b$ is to be a Gaussian RV with mean 5, var 9, determine values of constants a, b.
Solution. Solve

$$\begin{aligned} \sigma_Y^2 &= a^2\sigma_X^2 \\ \mu_Y &= a\mu_X + b \end{aligned}$$

You get $a = 3/2$ and $b = 3.5$.



6. The PDF $f_X(x)$ of RV X is plotted above.

(a) Compute a.

Solution. The area under the rect pulse has to be 0.6. So its amplitude a must be $.6/4 = 0.15$.

(b) Compute μ_X

Solution. By inspection,

$$\mu_X = (0.2)1 + (0.2)5 + (0.6)11 = 7.8$$

(c) Compute $E[X^2|X \geq 7]$

Solution. The conditional variance is

$$\frac{(13-9)^2}{12} = 4/3$$

by App A. The conditional mean is 11. Therefore,

$$E[X^2|X \geq 7] = (4/3) + (11)^2 = 122.3333.$$

(d) Compute $E[X|X \geq 3]$

Solution.

$$\mu_X = 7.8 = (0.2)1 + (0.8)E[X|X \geq 3].$$

Solving,

$$E[X|X \geq 3] = 7.6/.8 = 9.5$$