MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2002)

6.041 Quiz 1 Solutions
Handed out: March 15, 2002

Problem 1: (2 points) Answers may vary.

Problem 2:

(a) (12 points) Let X be the number of medals won while in a good mood and Y be the number of medals won while in a bad mood. Because the outcomes of different races are independent when given the driver's mood, X is binomial with probability of success p and 5 trials, Y is binomial with probability of success q and 6 trials, and X and Y are independent. The mean and variance of the number of medals won, $X+Y$, is therefore:

$$E[X+Y] = E[X] + E[Y] = 5p + 6q$$

$$var(X+Y) = var(X) + var(Y) = 5p(1-p) + 6q(1-q)$$

(b) (12 points) The number of medals won is binomial with probability of success p and 11 trials. The probability they won at least 6 medals is therefore $\sum_{k=6}^{11} \binom{11}{k} p^k (1-p)^{11-k}$.

(c) (13 points) Because $p = q$, any outcome with exactly 5 medal races and 6 nonmedal races is equally likely. Finding the probability that all 4 good mood races were also medal races is thus a counting problem. The number of ways to choose 5 medal races from 11 total races is $\binom{11}{5}$. The number of ways to choose 5 medal races such that all 4 good mood races are medal races is 7 (all good mood races are medal races, then choose 1 medal race from 7 bad mood races). Therefore, the answer is $\frac{7}{\binom{11}{5}}$.

(d) (13 points) **6.431 ONLY** The probability of any outcome with a total of 3 gold, 3 silver, and 1 bronze is $(\frac{1}{5})^3(\frac{2}{5})^3(\frac{2}{5})$. The number of outcomes with a total of 3 gold, 3 silver, and 1 bronze is the number of ways to order 3 gold, 3 silver, and 1 bronze, $\frac{7!}{3!3!1!}$. Therefore, the probability of winning a total of 3 gold, 3 silver, and 1 bronze is $\frac{7!}{3!3!1!}(\frac{1}{5})^3(\frac{2}{5})^3(\frac{2}{5})$.

(e) (13 points) There are four different ways to win the third race: win win win (which occurs with probability p^3), win lose win ($p(1-p)q$), lose win win ($(1-p)qp$), and lose lose win ($(1-p)(1-q)q$). Of these 4 outcomes, only two involve winning the second race: win win win (p^3) and lose win win ($(1-p)qp$). The probability they won the second race given that they won the third is therefore $\frac{p^3+(1-p)qp}{p^3+p(1-p)q+(1-p)qp+(1-p)(1-q)q}$.

Problem 3:

(a) (12 points)

$$p_X(x) = \begin{cases} (1-p)^{x-1}p & x = 1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$p_{Y,Z|X}(y,z|x) = \begin{cases} 1 & y=0,\ z=0,\ x=2,4,6,\ldots \\ \frac{1}{4} & y=0,2,\ z=0,2,\ x=1,3,5,\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} p_{X,Y,Z}(x,y,z) &= p_{Y,Z|X}(y,z|x)p_X(x) \\ &= \begin{cases} (1-p)^{x-1}p & y=0,\ z=0,\ x=2,4,6,\ldots \\ \frac{1}{4}(1-p)^{x-1}p & y=0,2,\ z=0,2,\ x=1,3,5,\ldots \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) (i) (4 points) If $Y = 2$, then Z is equally likely to be either 0 or 2. If $Y = 0$, then Z is more likely to be 0 than 2. Because knowing the value of Y changes the probability distribution for Z, Y and Z are not independent.

(ii) (4 points) If $Y = 2$ and $X = x$, then Z is equally likely to be either 0 or 2 for all possible values of x. Because knowing the value of X does not change the conditional probability distribution for Z, X and Z are conditionally independent given that $Y = 2$.

(iii) (4 points) If $Y = 0$ and X is even, then Z must be 0. If $Y = 0$ and X is odd, then Z is equally likely to be either 0 or 2. Because knowing the value of X changes the conditional probability distribution for Z, X and Z are not conditionally independent given that $Y = 0$.

(c) (12 points)

$$\begin{aligned} E[X|Y=2] &= \sum_x x p_{X|Y}(x|2) = \sum_x \frac{x p_{X,Y}(x,2)}{p_Y(2)} \\ &= \sum_x \frac{x(p_{X,Y,Z}(x,2,0) + p_{X,Y,Z}(x,2,2))}{\sum_x (p_{X,Y,Z}(x,2,0) + p_{X,Y,Z}(x,2,2))} \\ &= \sum_{x=1,3,5,\ldots} \frac{x(\frac{1}{4}(1-p)^{x-1}p + \frac{1}{4}(1-p)^{x-1}p)}{\sum_{x=1,3,5,\ldots}(\frac{1}{4}(1-p)^{x-1}p + \frac{1}{4}(1-p)^{x-1}p)} \\ &= \frac{\sum_{x=1,3,5,\ldots} x(1-p)^{x-1}p}{\sum_{x=1,3,5,\ldots}(1-p)^{x-1}p} \end{aligned}$$

(d) (12 points) Given that X is odd, Y and Z are conditionally independent, and are each uniformly distributed on the set {0,2}.

$$\begin{aligned} var(Y+Z|X \text{ is odd}) &= var(Y|X \text{ is odd}) + var(Z|X \text{ is odd}) \\ &= 2var(Y|X \text{ is odd}) \\ &= 2(\frac{1}{2}(2-1)^2 + \frac{1}{2}(0-1)^2) = 2 \end{aligned}$$

Another way to do this part is to find the PMF for $Y + Z$,

$$p_{Y+Z}(k) = \begin{cases} \frac{1}{4} & k = 0,4 \\ \frac{1}{2} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

then calculate the variance of $Y + Z$ using its PMF, noticing that $E[Y + Z] = 2$ by symmetry:

$$\begin{aligned} var(Y+Z|X \text{ is odd}) &= E[(Y+Z-E[Y+Z])^2] \\ &= \frac{1}{4}(0-2)^2 + \frac{1}{2}(2-2)^2 + \frac{1}{4}(4-2)^2 = 2 \end{aligned}$$