LECTURE 6

- **Readings:** Sections 2.1-2.3, start 2.4

Lecture outline

- Random variables
- Probability mass function (pmf)
  - Service facility example
- Expectation

Random variables

- An assignment of a value (number) to every possible outcome
- Mathematically: A function from the sample space to the real numbers
  - discrete or continuous
- Can have several random variables defined on the same sample space
- Notation:
  - random variable X
  - experimental value x

Probability mass function (pmf)

- ("probability law", "probability distribution")
- Notation:

$$p_X(x) = \mathbf{P}(X = x)$$

- **Example:** X=number of coin tosses until first head
  - assume independent tosses, $\mathbf{P}(H) = p > 0$

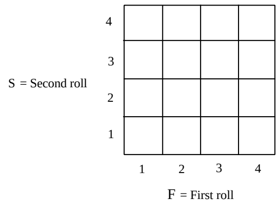
$$\begin{aligned} p_X(k) &= \mathbf{P}(X = k) \\ &= \mathbf{P}(TT \cdots TH) \\ &= (1-p)^{k-1}p, \qquad k = 1, 2, \ldots \end{aligned}$$

How to compute a pmf $p_X(x)$

- collect all possible outcomes for which X is equal to x
- add their probabilities
- repeat for all x

- **Example:** Two independent throws of a fair tetrahedral die

  F: outcome of first throw
  S: outcome of second throw
  $L = \min(F, S)$



$p_L(2) =$

Binomial pmf

- X: number of heads in n independent coin tosses
- $\mathbf{P}(H) = p$
- Let $n = 4$

$$\begin{aligned} p_X(2) &= \mathbf{P}(HHTT) + \mathbf{P}(HTHT) + \mathbf{P}(HTTH) \\ &\quad + \mathbf{P}(THHT) + \mathbf{P}(THTH) + \mathbf{P}(TTHH) \\ &= 6p^2(1-p)^2 \\ &= \binom{4}{2} p^2(1-p)^2 \end{aligned}$$

In general:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

Service facility design

- n: customers
- p: prob. customer requires service
- s: no. of service persons
- X: no. of service requests (r.v.)

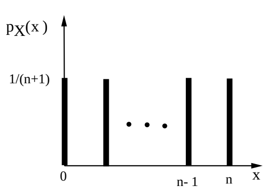
$$\begin{aligned} \mathbf{P}(X > s) &= \mathbf{P}(X = s+1) + \cdots + \mathbf{P}(X = n) \\ &= p_X(s+1) + \cdots + p_X(n) \\ &= \sum_{i=s+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \end{aligned}$$

Expectation

- Definition:

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

- Interpretations:
  - Center of gravity of pmf
  - Average in large number of repetitions of the experiment
    (to be substantiated later in this course)
- Example: Uniform on $0, 1, \ldots, n$



$$\mathbf{E}[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + \cdots + n \times \frac{1}{n+1} =$$

Properties of expectations

- Let X be a r.v. and let $Y = g(X)$
  - Hard: $\mathbf{E}[Y] = \sum_y y p_Y(y)$
  - Easy: $\mathbf{E}[Y] = \sum_x g(x) p_X(x)$
- "Second moment": $\mathbf{E}[X^2]$
- Caution: In general, $\mathbf{E}[g(X)] \neq g(\mathbf{E}[x])$
- Variance:

$$\text{var}(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \sum_x (x - \mathbf{E}[X])^2 p_X(x)$$

- If α is a constant: $\mathbf{E}[\alpha] =$
- $\mathbf{E}[\alpha X] =$
- $\mathbf{E}[\alpha X + \beta] =$
- How about second moments?

$$\mathbf{E}[(\alpha X)^2] = \mathbf{E}[\alpha^2 X^2]$$