LECTURE 11

- **Readings:** Section 3.6

More on continuous r.v.s, and derived distributions

Review

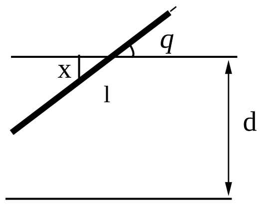
$$
\begin{array}{rl}
p_X(x) & f_X(x) \\
p_{X,Y}(x,y) & f_{X,Y}(x,y) \\
p_{X|Y}(x \mid y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)} & f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} \\
p_X(x) = \sum_y p_{X,Y}(x,y) & f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy
\end{array}
$$

$$F_X(x) = \mathbf{P}(X \le x)$$

$$\mathbf{E}[X], \quad \text{var}(X)$$

Buffon's needle

- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines)



- $X \in [0, d/2]$: distance of needle midpoint to nearest line
- Model: X, Θ uniform, independent

$$f_{X,\Theta}(x,\theta) = \qquad 0 \le x \le d/2, \ 0 \le \theta \le \pi/2$$

- Intersect if $X \le \dfrac{\ell}{2}\sin\Theta$

$$\mathbf{P}\left(X \le \frac{\ell}{2}\sin\Theta\right) = \frac{4}{\pi d}\int_0^{\pi/2} \frac{\ell}{2}\sin\theta\,d\theta = \frac{2\ell}{\pi d}$$

What is a derived distribution

- It is a pmf or pdf of a function of a random variable with known probability law.
  - Obtaining the PDF for

    $$g(X,Y) = Y/X$$

    involves deriving a distribution.
    Note: $g(X,Y)$ is a random variable

When not to find them

- Don't need PDF for $g(X,Y)$ if only want to compute expected value:

$$\mathbf{E}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

How to find them

- **Discrete case**
  - Obtain probability mass for each possible value of $Y = g(X)$

$$
\begin{aligned}
p_Y(y) &= \mathbf{P}(g(X) = y) \\
&= \sum_{x:\, g(x)=y} p_X(x)
\end{aligned}
$$

- **Two-step procedure for the continuous case:**
  - Get CDF of Y: $F_Y(y) = \mathbf{P}(Y \leq y)$
  - Differentiate to get

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

Example

- X: uniform on [0,2]
- Find PDF of $Y = X^3$
- **Solution:**

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) = \mathbf{P}(X^3 \leq y) \\ &= \mathbf{P}(X \leq y^{1/3}) = \frac{1}{2} y^{1/3} \end{aligned}$$

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

Example

- Joan is driving from Boston to New York. Her speed is uniformly distributed between 30 and 60 mph. What is the distribution of the duration of the trip?
- Let $T(V) = \frac{200}{V}$.
- Find $f_T(t)$

The pdf of Y=aX+b

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.
- General normal $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$