LECTURE 20

Markov Processes – I

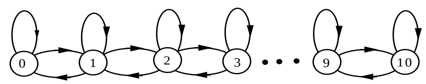
- **Readings:** Sections 6.1–6.2

Lecture outline

- Checkout counter example
- Markov process definition
- n-step transition probabilities
- Classification of states

Checkout counter model

- Discrete time $n = 0, 1, \ldots$
- Customer arrivals: Bernoulli(p)
  - geometric interarrival times
- Customer service times: geometric(q)
- "State" X_n: number of customers at time n



Finite State Markov models

- X_n: state after n transitions
  - belongs to a finite set, e.g., $\{1, \ldots, m\}$
  - X_0 is either given or random
- **Markov property/assumption:** (given current state, the past does not matter)

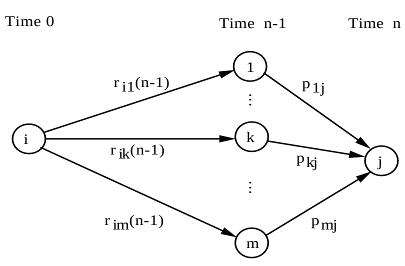
$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{n+1} = j \mid X_n = i) \\ &= \mathbf{P}(X_{n+1} = j \mid X_n = i,\ X_{n-1}, \ldots, X_0) \end{aligned}$$

- Modeling steps:
  - identify the possible states
  - mark the possible transitions
  - record the transition probabilities

n-step transition probabilities

- State occupancy probabilities, given initial state i:

$$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$$



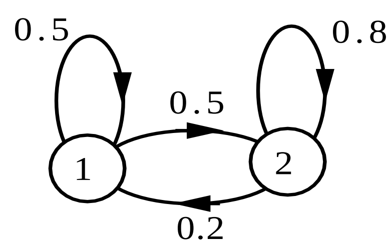
- Key recursion:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

- With random initial state:

$$\mathbf{P}(X_n = j) = \sum_{i=1}^{m} \mathbf{P}(X_0 = i) r_{ij}(n)$$

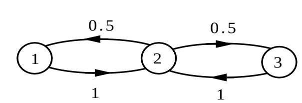
Example



	n=0	n=1	n=2	n=2563	n=2564
$r_{11}(n)$					
$r_{12}(n)$					

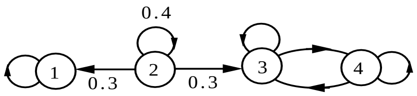
Generic question:

- Does r_{ij} converge to something?



n odd: $r_{22}(n)=$ n even: $r_{22}(n)=$

- Does the limit depend on initial state?



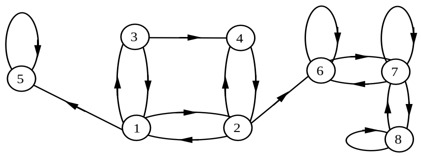
$r_{11}(n)=$

$r_{31}(n)=$

$r_{21}(n)=$

Recurrent and transient states

- State i is **recurrent** if:
  starting from i,
  and from wherever you can go,
  there is a way of returning to i
- If not recurrent, called **transient**

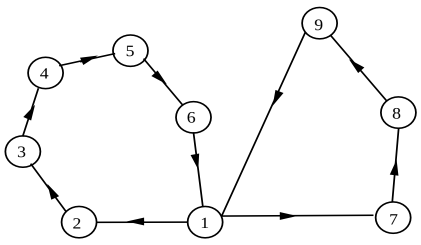


  - i transient:
    $\mathbf{P}(X_n = i) \to 0$,
    i visited finite number of times

- **Recurrent class**:
  collection of recurrent states that
  "communicate" to each other
  and to no other state

Periodic states

- A recurrent state is **periodic** if:
  there is an integer $d > 1$
  such that $p_{ii}(k) = 0$
  when k is not an integer multiple of d



- Then, $p_{ii}(n)$ cannot converge.