LECTURE 23
Limit theorems – I

- **Readings:** Sections 7.1-7.3
- $X_1, \ldots, X_n$ i.i.d.
$$M_n = \frac{X_1 + \cdots + X_n}{n}$$
What happens as $n \to \infty$?
- Why bother?
- A tool: Chebyshev's inequality
- Convergence "in probability"
- Convergence of M_n

Chebyshev's inequality

- Random variable X

$$\sigma^2 = \int (x - \mathbf{E}[X])^2 f_X(x)\, dx$$

$$\sigma^2 \;\geq\; c^2 \mathbf{P}(|X - \mathbf{E}[X]| \geq c)$$

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq c) \;\leq\; \frac{\sigma^2}{c^2}$$

$$\mathbf{P}(|X - \mathbf{E}[X]| \geq k\sigma) \;\leq\; \frac{1}{k^2}$$

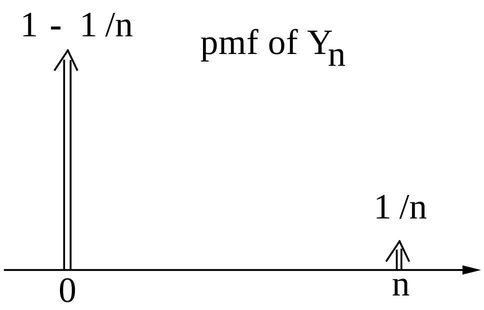
Deterministic limits (review)

- Sequence a_n
Number a
- a_n converges to a
$$\lim_{n\to\infty} a_n = a$$
"a_n eventually gets and stays (arbitrarily) close to a"
- For every $\epsilon > 0$,
there exists n_0,
such that for all $n \geq n_0$,
we have $|a_n - a| \leq \epsilon$.

Convergence "in probability"

- Sequence of random variables Y_n
- converges in probability to a number a:
"(almost all) of the PMF/PDF of Y_n , eventually gets concentrated (arbitrarily) close to a"
- For every $\epsilon > 0$,
$$\lim_{n\to\infty} \mathbf{P}(|Y_n - a| \geq \epsilon) = 0$$

Examples

- 

Does Y_n converge?

- Flip fair coin n times
  Y_n = (number of heads) minus $n/2$
  Converges?

Convergence of the sample mean

(Weak law of large numbers)

- $X_1, X_2, \ldots$ i.i.d.
  finite mean μ and variance σ^2

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

- $\mathbf{E}[M_n] =$

- $\mathrm{Var}(M_n) =$

$$\mathbf{P}(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

The pollster's problem

- f: fraction of population that do XYZ
- ith person polled:

$$X_i = \begin{cases} 1, & \text{if yes,} \\ 0, & \text{if no.} \end{cases}$$

- $M_n = (X_1 + \cdots + X_n)/n$
  fraction of "yes" in our sample
- Suppose we want:

$$\mathbf{P}(|M_n - f| \geq .01) \leq .05$$

- Use Chebyshev's inequality:

$$\begin{aligned} \mathbf{P}(|M_n - f| \geq .01) &\leq \frac{\sigma^2_{M_n}}{(0.01)^2} \\ &= \frac{\sigma^2_x}{n(0.01)^2} \leq \frac{1}{4n(0.01)^2} \end{aligned}$$

- If $n = 50{,}000$,
  then $\mathbf{P}(|M_n - f| \geq .01) \leq .05$
  (conservative)