## LECTURE 25
## Outline

- Approximating binomial distributions
- Strong law of large numbers

## CLT review

- $X_i$: i.i.d., finite variance $\sigma^2$
- $S_n = X_1 + \cdots + X_n$
- $Z_n = \dfrac{(X_1 + \cdots + X_n) - n\mathbf{E}[X]}{\sigma\sqrt{n}}$
- $Z$: standard normal
  (zero mean, unit variance)
- **CLT:** $\mathbf{P}(Z_n \le c) \to \mathbf{P}(Z \le c) = \Phi(c)$
- Approximation: treat $S_n$ as if normal

## Apply to binomial

- Fix $p$, where $0 < p < 1$
- $X_i$: Bernoulli($p$)
- $S_n = X_1 + \cdots + X_n$: Binomial($n, p$)
  - mean $np$, variance $np(1-p)$
- $\dfrac{S_n - np}{\sqrt{np(1-p)}} \longrightarrow$ standard normal CDF

## Example

- $n = 36$, $p = 0.5$; find $\mathbf{P}(S_n \le 21)$
- Exact answer:

$$\sum_{k=0}^{21} \binom{36}{k} \left(\frac{1}{2}\right)^{36} = 0.8785$$

## The 1/2 correction for binomial approximation

- $\mathbf{P}(S_n \le 21) = \mathbf{P}(S_n < 22)$,
  because $S_n$ is integer
- Compromise: consider $\mathbf{P}(S_n \le 21.5)$

## De Moivre–Laplace CLT (for binomial)

- When the 1/2 correction is used, CLT can also approximate the binomial p.m.f. (not just the binomial CDF)

$$\mathbf{P}(S_n = 19) = \mathbf{P}(18.5 \le S_n \le 19.5)$$

$$18.5 \le S_n \le 19.5 \quad \Longleftrightarrow$$

$$\frac{18.5 - 18}{3} \le \frac{S_n - 18}{3} \le \frac{19.5 - 18}{3} \quad \Longleftrightarrow$$

$$0.17 \le Z_n \le 0.5$$

$$\begin{aligned}\mathbf{P}(S_n = 19) &\approx \mathbf{P}(0.17 \le Z \le 0.5) \\ &= \mathbf{P}(Z \le 0.5) - \mathbf{P}(Z \le 0.17) \\ &= 0.6915 - 0.5675 \\ &= 0.124\end{aligned}$$

- Exact answer:

$$\binom{36}{19}\left(\frac{1}{2}\right)^{36} = 0.1251$$

## Poisson vs. normal approximations of the binomial

- Poisson arrivals during unit interval equals: sum of $n$ (independent) Poisson arrivals during $n$ intervals of length $1/n$
  - Let $n \to \infty$, apply CLT (??)
  - Poisson=normal (????)
- Binomial$(n, p)$
  - $p$ fixed, $n \to \infty$: normal
  - $np$ fixed, $n \to \infty$, $p \to 0$: Poisson
- $p = 1/100$, $n = 100$: Poisson
- $p = 1/10$, $n = 500$: normal

## The strong law of large numbers

- **Theorem: (SLLN)** If $X_i$ are i.i.d., and $\mathbf{E}[|X|] < \infty$, then
$$M_n = \frac{X_1 + \cdots + X_n}{n} \to \mathbf{E}[X]$$
with probability 1.
- **One experiment:** Generate an infinite sequence $X_1, X_2, \ldots$ and the associated infinite sequence $M_1, M_2, \ldots$.
  - Sample space: set of all infinite sequences
- Sequences that do not converge to $\mathbf{E}[X]$ are also possible, but collectively their probability is zero
- Example: $X_i$ are Bernoulli$(p)$
$M_n =$ fraction of successes in $n$ trials
$M_n \to p$, with probability 1

## Convergence with probability one

- Let **one** experiment generate an **infinite** sequence of experimental values of $Y_1, Y_2, \ldots$
  - sample space: set of all sequences
- **Def:** $Y_n \to c$, with probability 1, if the set of all sequences that converge to $c$ has probability 1.
- It turns out that convergence w.p.1 always implies convergence in probability
- converse is not always true
  - Example: let $Y_n = 1$ if customer arrives at time slot $n$
    Model of customer arrivals: during each interval $2^k, \ldots 2^{k+1} - 1$, exactly 1 customer arrives, each slot being equally likely
    Check that $Y_n \to 0$ in probability, but not with probability 1