ECE 541
Stochastic Signals and Systems
Problem Set 6 Solution

Problem Solutions : Yates and Goodman, 6.1.3 6.2.2 6.2.6 6.3.4 6.4.3 6.4.4 6.5.4 6.6.3 6.8.1 and 6.8.5

Problem 6.1.3 Solution

(a) The PMF of N_1, the number of phone calls needed to obtain the correct answer, can be determined by observing that if the correct answer is given on the nth call, then the previous $n-1$ calls must have given wrong answers so that

$$P_{N_1}(n) = \begin{cases} (3/4)^{n-1}(1/4) & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) N_1 is a geometric random variable with parameter $p = 1/4$. In Theorem 2.5, the mean of a geometric random variable is found to be $1/p$. For our case, $E[N_1] = 4$.

(c) Using the same logic as in part (a) we recognize that in order for n to be the fourth correct answer, that the previous $n-1$ calls must have contained exactly 3 correct answers and that the fourth correct answer arrived on the n-th call. This is described by a Pascal random variable.

$$P_{N_4}(n_4) = \begin{cases} \binom{n-1}{3}(3/4)^{n-4}(1/4)^4 & n = 4, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(d) Using the hint given in the problem statement we can find the mean of N_4 by summing up the means of the 4 identically distributed geometric random variables each with mean 4. This gives $E[N_4] = 4E[N_1] = 16$.

Problem 6.2.2 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Proceeding as in Problem 6.2.1, we must first find $F_W(w)$ by integrating over the square defined by $0 \le x, y \le 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 6.2.1 resulting in the following integrals for their appropriate regions. For $0 \le w \le 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx\, dy = w^2/2 \tag{2}$$

For $1 \le w \le 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx\, dy + \int_{w-1}^1 \int_0^{w-y} dx\, dy = 2w - 1 - w^2/2 \tag{3}$$

The complete CDF $F_W(w)$ is shown below along with the corresponding PDF $f_W(w) = dF_W(w)/dw$.

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \le w \le 1 \\ 2w - 1 - w^2/2 & 1 \le w \le 2 \\ 1 & \text{otherwise} \end{cases} \qquad f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 6.2.6 Solution

The random variables K and J have PMFs

$$P_J(j) = \begin{cases} \frac{\alpha^j e^{-\alpha}}{j!} & j = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad P_K(k) = \begin{cases} \frac{\beta^k e^{-\beta}}{k!} & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $n \ge 0$, we can find the PMF of $N = J + K$ via

$$P[N = n] = \sum_{k=-\infty}^{\infty} P[J = n - k, K = k] \tag{2}$$

Since J and K are independent, non-negative random variables,

$$\begin{aligned} P[N = n] &= \sum_{k=0}^{n} P_J(n-k)\, P_K(k) && (3) \\ &= \sum_{k=0}^{n} \frac{\alpha^{n-k} e^{-\alpha}}{(n-k)!} \frac{\beta^k e^{-\beta}}{k!} && (4) \\ &= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{\alpha}{\alpha+\beta}\right)^{n-k} \left(\frac{\beta}{\alpha+\beta}\right)^{k}}_{1} && (5) \end{aligned}$$

The marked sum above equals 1 because it is the sum of a binomial PMF over all possible values. The PMF of N is the Poisson PMF

$$P_N(n) = \begin{cases} \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 6.3.4 Solution

Using the moment generating function of X, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the nth moment of X, $E[X^n]$ by taking the nth derivative of $\phi_X(s)$ and setting $s = 0$.

$$\begin{aligned} E[X] &= \sigma^2 s e^{\sigma^2 s^2/2} \Big|_{s=0} = 0 && (1) \\ E\left[X^2\right] &= \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2} \Big|_{s=0} = \sigma^2. && (2) \end{aligned}$$

Continuing in this manner we find that

$$E\left[X^3\right] = \left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0 \tag{3}$$

$$E\left[X^4\right] = \left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4. \tag{4}$$

To calculate the moments of Y, we define $Y = X + \mu$ so that Y is Gaussian (μ, σ). In this case the second moment of Y is

$$E\left[Y^2\right] = E\left[(X+\mu)^2\right] = E\left[X^2 + 2\mu X + \mu^2\right] = \sigma^2 + \mu^2. \tag{5}$$

Similarly, the third moment of Y is

$$E\left[Y^3\right] = E\left[(X+\mu)^3\right] \tag{6}$$

$$= E\left[X^3 + 3\mu X^2 + 3\mu^2 X + \mu^3\right] = 3\mu\sigma^2 + \mu^3. \tag{7}$$

Finally, the fourth moment of Y is

$$E\left[Y^4\right] = E\left[(X+\mu)^4\right] \tag{8}$$

$$= E\left[X^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X + \mu^4\right] \tag{9}$$

$$= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4. \tag{10}$$

Problem 6.4.3 Solution

In the iid random sequence $K_1, K_2, \ldots$, each K_i has PMF

$$P_K(k) = \begin{cases} 1-p & k = 0, \\ p & k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The MGF of K is $\phi_K(s) = E[e^{sK}] = 1 - p + pe^s$.

(b) By Theorem 6.8, $M = K_1 + K_2 + \ldots + K_n$ has MGF

$$\phi_M(s) = [\phi_K(s)]^n = [1 - p + pe^s]^n \tag{2}$$

(c) Although we could just use the fact that the expectation of the sum equals the sum of the expectations, the problem asks us to find the moments using $\phi_M(s)$. In this case,

$$E[M] = \frac{d\phi_M(s)}{ds}\bigg|_{s=0} = n(1-p+pe^s)^{n-1}pe^s\big|_{s=0} = np \tag{3}$$

The second moment of M can be found via

$$E\left[M^2\right] = \frac{d\phi_M(s)}{ds}\bigg|_{s=0} \tag{4}$$

$$= np\left((n-1)(1-p+pe^s)pe^{2s} + (1-p+pe^s)^{n-1}e^s\right)\big|_{s=0} \tag{5}$$

$$= np[(n-1)p + 1] \tag{6}$$

The variance of M is

$$\text{Var}[M] = E\left[M^2\right] - (E[M])^2 = np(1-p) = n\,\text{Var}[K] \tag{7}$$

Problem 6.4.4 Solution

Based on the problem statement, the number of points X_i that you earn for game i has PMF

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0,1,2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The MGF of X_i is

$$\phi_{X_i}(s) = E\left[e^{sX_i}\right] = 1/3 + e^s/3 + e^{2s}/3 \tag{2}$$

Since $Y = X_1 + \cdots + X_n$, Theorem 6.8 implies

$$\phi_Y(s) = [\phi_{X_i}(s)]^n = [1 + e^s + e^{2s}]^n/3^n \tag{3}$$

(b) First we observe that first and second moments of X_i are

$$E[X_i] = \sum_x x P_{X_i}(x) = 1/3 + 2/3 = 1 \tag{4}$$

$$E\left[X_i^2\right] = \sum_x x^2 P_{X_i}(x) = 1^2/3 + 2^2/3 = 5/3 \tag{5}$$

Hence,

$$\text{Var}[X_i] = E\left[X_i^2\right] - (E[X_i])^2 = 2/3. \tag{6}$$

By Theorems 6.1 and 6.3, the mean and variance of Y are

$$E[Y] = nE[X] = n \tag{7}$$

$$\text{Var}[Y] = n\,\text{Var}[X] = 2n/3 \tag{8}$$

Another more complicated way to find the mean and variance is to evaluate derivatives of $\phi_Y(s)$ as $s = 0$.

Problem 6.5.4 Solution

Donovan McNabb's passing yardage is the random sum of random variables

$$V + Y_1 + \cdots + Y_K \tag{1}$$

where Y_i has the exponential PDF

$$f_{Y_i}(y) = \begin{cases} \frac{1}{15}e^{-y/15} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

From Table 6.1, the MGFs of Y and K are

$$\phi_Y(s) = \frac{1/15}{1/15 - s} = \frac{1}{1 - 15s} \qquad \phi_K(s) = e^{20(e^s - 1)} \tag{3}$$

From Theorem 6.12, V has MGF

$$\phi_V(s) = \phi_K(\ln \phi_Y(s)) = e^{20(\phi_Y(s) - s)} = e^{300s/(1-15s)} \tag{4}$$

The PDF of V cannot be found in a simple form. However, we can use the MGF to calculate the mean and variance. In particular,

$$E[V] = \left.\frac{d\phi_V(s)}{ds}\right|_{s=0} = e^{300s/(1-15s)} \left.\frac{300}{(1-15s)^2}\right|_{s=0} = 300 \tag{5}$$

$$E\left[V^2\right] = \left.\frac{d^2\phi_V(s)}{ds^2}\right|_{s=0} \tag{6}$$

$$= e^{300s/(1-15s)}\left(\frac{300}{(1-15s)^2}\right)^2 + e^{300s/(1-15s)} \left.\frac{9000}{(1-15s)^3}\right|_{s=0} = 99,000 \tag{7}$$

Thus, V has variance $\text{Var}[V] = E[V^2] - (E[V])^2 = 9,000$ and standard deviation $\sigma_V \approx 94.9$.

A second way to calculate the mean and variance of V is to use Theorem 6.13 which says

$$E[V] = E[K]\,E[Y] = 20(15) = 200 \tag{8}$$

$$\text{Var}[V] = E[K]\,\text{Var}[Y] + \text{Var}[K](E[Y])^2 = (20)15^2 + (20)15^2 = 9000 \tag{9}$$

Problem 6.6.3 Solution

(a) Let $X_1, \ldots, X_{120}$ denote the set of call durations (measured in minutes) during the month. From the problem statement, each $X-I$ is an exponential (λ) random variable with $E[X_i] = 1/\lambda = 2.5$ min and $\text{Var}[X_i] = 1/\lambda^2 = 6.25$ min^2. The total number of minutes used during the month is $Y = X_1 + \cdots + X_{120}$. By Theorem 6.1 and Theorem 6.3,

$$E[Y] = 120E[X_i] = 300 \qquad \text{Var}[Y] = 120\,\text{Var}[X_i] = 750. \tag{1}$$

The subscriber's bill is $30 + 0.4(y-300)^+$ where $x^+ = x$ if $x \geq 0$ or $x^+ = 0$ if $x < 0$. the subscribers bill is exactly \$36 if $Y = 315$. The probability the subscribers bill exceeds \$36 equals

$$P[Y > 315] = P\left[\frac{Y-300}{\sigma_Y} > \frac{315-300}{\sigma_Y}\right] = Q\left(\frac{15}{\sqrt{750}}\right) = 0.2919. \tag{2}$$

(b) If the actual call duration is X_i, the subscriber is billed for $M_i = \lceil X_i \rceil$ minutes. Because each X_i is an exponential (λ) random variable, Theorem 3.9 says that M_i is a geometric (p) random variable with $p = 1 - e^{-\lambda} = 0.3297$. Since M_i is geometric,

$$E[M_i] = \frac{1}{p} = 3.033, \qquad \text{Var}[M_i] = \frac{1-p}{p^2} = 6.167. \tag{3}$$

The number of billed minutes in the month is $B = M_1 + \cdots + M_{120}$. Since $M_1, \ldots, M_{120}$ are iid random variables,

$$E[B] = 120E[M_i] = 364.0, \qquad \text{Var}[B] = 120\,\text{Var}[M_i] = 740.08. \tag{4}$$

Similar to part (a), the subscriber is billed \$36 if $B = 315$ minutes. The probability the subscriber is billed more than \$36 is

$$P[B > 315] = P\left[\frac{B - 364}{\sqrt{740.08}} > \frac{315 - 365}{\sqrt{740.08}}\right] = Q(-1.8) = \Phi(1.8) = 0.964. \qquad (5)$$

Problem 6.8.1 Solution

The $N[0,1]$ random variable Z has MGF $\phi_Z(s) = e^{s^2/2}$. Hence the Chernoff bound for Z is

$$P[Z \geq c] \leq \min_{s \geq 0} e^{-sc} e^{s^2/2} = \min_{s \geq 0} e^{s^2/2 - sc} \qquad (1)$$

We can minimize $e^{s^2/2-sc}$ by minimizing the exponent $s^2/2 - sc$. By setting

$$\frac{d}{ds}\left(s^2/2 - sc\right) = 2s - c = 0 \qquad (2)$$

we obtain $s = c$. At $s = c$, the upper bound is $P[Z \geq c] \leq e^{-c^2/2}$. The table below compares this upper bound to the true probability. Note that for $c = 1, 2$ we use Table 3.1 and the fact that $Q(c) = 1 - \Phi(c)$.

	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
Chernoff bound	0.606	0.135	0.011	3.35×10^{-4}	3.73×10^{-6}
$Q(c)$	0.1587	0.0228	0.0013	3.17×10^{-5}	2.87×10^{-7}

(3)

We see that in this case, the Chernoff bound typically overestimates the true probability by roughly a factor of 10.

Problem 6.8.5 Solution

Let $W_n = X_1 + \cdots + X_n$. Since $M_n(X) = W_n/n$, we can write

$$P[M_n(X) \geq c] = P[W_n \geq nc] \qquad (1)$$

Since $\phi_{W_n}(s) = (\phi_X(s))^n$, applying the Chernoff bound to W_n yields

$$P[W_n \geq nc] \leq \min_{s \geq 0} e^{-snc} \phi_{W_n}(s) = \min_{s \geq 0} \left(e^{-sc} \phi_X(s)\right)^n \qquad (2)$$

For $y \geq 0$, y^n is a nondecreasing function of y. This implies that the value of s that minimizes $e^{-sc}\phi_X(s)$ also minimizes $(e^{-sc}\phi_X(s))^n$. Hence

$$P[M_n(X) \geq c] = P[W_n \geq nc] \leq \left(\min_{s \geq 0} e^{-sc} \phi_X(s)\right)^n \qquad (3)$$