EE126: Probability and Random Processes **SP'07**

Problem Set 8 — Practice Problems for M2'

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We have provided the solutions for some of the problems but we strongly recommend you to try the exercises first and then compare your solutions to the one given in these notes.

Problem 8.1. *We have seen in class that a convenient way to write the error probability in Gaussian detection is the use of the Q-function defined as:*

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} exp\{-\frac{t^2}{2}\}dt$$

In this exercise we will study some properties of the Q-function:
(a) Show that $Q(-x) = 1 - Q(x)$
(b) Show that

$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}$$

Hint: rewrite the integrand (multiply and divide by t) and use integration by parts.

Solution:
(a)We will use the definition and a change of variable.

$$\begin{aligned}
Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{t^2}{2}} dt &= \frac{1}{\sqrt{2\pi}} \int_{-x}^{0} e^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{t^2}{2}} dt \\
\text{change of variable } x \leftrightarrow -x &= \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{t^2}{2}} dt + \frac{1}{2} \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} e^{-\frac{t^2}{2}} dt + \frac{1}{2} \\
&= 1 - Q(x)
\end{aligned}$$

(b)For the upper and lower bounds we will use integration by parts. We will drop the $\sqrt{2\pi}$ term in the calculations.

$$\begin{aligned}
\int_x^{\infty} e^{-\frac{t^2}{2}} &= \int_x^{\infty} \frac{t}{t} e^{-\frac{t^2}{2}} \\
(u = t, v' = te^{-\frac{t^2}{2}}) &= \left[-\frac{e^{-\frac{t^2}{2}}}{t}\right]_x^{\infty} - \int_x^{+\infty} \frac{1}{t^2} e^{-\frac{t^2}{2}} dt \qquad (8.1) \\
(\text{second term} \geq 0) &\geq \frac{e^{-\frac{x^2}{2}}}{x}
\end{aligned}$$

For the upper bound we will perform another integration by parts in the second term of equation 8.1.

$$\begin{aligned}
\int_x^{\infty} e^{-\frac{t^2}{2}} &= \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^{+\infty} \frac{t}{t \cdot t^2} e^{-\frac{t^2}{2}} dt \\
(u = \frac{1}{t^3}, v' = te^{-\frac{t^2}{2}}) &= \frac{e^{-\frac{x^2}{2}}}{x} - \left[-\frac{e^{-\frac{t^2}{2}}}{t^3} \right]_x^{\infty} + \int_x^{+\infty} \frac{1}{t^4} e^{-\frac{t^2}{2}} dt \\
(3^{rd} \text{ term} \geq 0) &\geq \frac{e^{-\frac{x^2}{2}}}{x} - \frac{e^{-\frac{x^2}{2}}}{x^3}
\end{aligned}$$

Problem 8.2. *The output of a system is normal* $\mathcal{N}(1.1, 1.1)$ *if the input* X *is equal to* 1. *When the input* X *is equal to* 2, *the output is normal* $\mathcal{N}(0.1, 0.1)$.
Observing the output of the system, we would like to compute our best estimate of X_i *(or estimate of* i*).*

1. *Find the best estimate of* $i \in 1, 2$ *given the output of the system (best in term of maximizing the probability of* X_i *given the system output). We assume that the input is* X_1 *with probability* 0.6.

2. *What would your answer in the previous part would be if the prior probabilities were equal?*

3. *Suppose now that we want to maximize the probability of guessing correctly when* $i = 1$ *and at the same time keeping the probability of guessing wrongly when* $x = 2$ *to be less than* 0.05. *What is the best decision rule?*

Problem 8.3. *In the hypothesis testing problem, we define the probability of Type I and Type II errors (resp. denoted* (α) *and* (β)*) by:*

$$\alpha = P(\hat{X} = 0 | H_1) \qquad \beta = P(\hat{X} = 1 | H_0)$$

They are also called false alarm (FA) *and* miss detection (MD). *In class, we have claimed that the probability of Type I and Type II errors (resp. denoted* (α) *and* (β)*) cannot be made both small at the same time. Mainly, reducing one type of error will result to increasing the other. We will verify that is this exercise. Also we will see that one way to reduce both type of errors is to increase the the number of observations.*
Let $(X_1, \ldots, X_n)$ *be* n *independent observations (or realizations) of a normal random variable with mean* μ *and variance* $\sigma^2 = 100$. *We would like to test the hypothesis* $H_0 : \mu = 50$ *against* $H_1 : \mu = \mu_1 > 50$ *for an observation size* $n = 25$.
A) Error calculation:
We decide to reject H_0 *is* $\bar{X} \geq 52$ *where* $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ *is the sample mean.*

1. *Find the probability of rejecting* H_0 *as a function of* μ.

2. *Compute the probability of Type I error (α).*
3. *Find the probability of Type II error for (i)$\mu_1 = 53$ and for (ii) $\mu_1 = 55$.*

B) Now we consider that the decision rule is such that we reject H_0 if $\bar{X} \geq c$.

1. *Find the value of c such that the probability of Type I error is $\alpha = 0.05$.*
2. *Find the probability of Type II error β with the new decision rule when $\mu_1 = 55$.*
3. *Compare the values you get for α and β to the ones obtained in part A.*

C) Repeat part B) with a sample size $n = 100$ and conclude.

Solution:
A) Error calculation:
(1) Note the the sample mean $\bar{X}$ is normal $\mathcal{N}(\mu, \frac{\sigma^2}{n})$. Thus the probability of rejecting H_0 is given by

$$\begin{aligned} P(\mathcal{N}(\mu, \frac{\sigma^2}{n}) \geq 52) &= P(\mathcal{N}(0,1) \geq \frac{52-\mu}{\sqrt{\sigma^2/n}}) \\ &= Q(\frac{52-\mu}{\sqrt{\sigma^2/n}}), \quad \mu \geq 50 \end{aligned}$$

(2) Type I error is the probability of rejecting H_0 given $\mu = \mu_0 = 50$. Replacing μ by 50 in the result in part (1) gives:

$$\alpha = Q(\frac{52-50}{\sqrt{100/25}}) =\approx 0.1587$$

(3) Type II error is the probability of accepting H_0 given that $\mu = \mu_1 > \mu_0$. It is given by $P(\bar{X} \leq 52)$. Similar to part (1) we can express it as a function of μ and it is:

$$\begin{aligned} P(\mathcal{N}(\mu_1, \frac{\sigma^2}{n}) \leq 52) &= P(\mathcal{N}(0,1) \leq \frac{52-\mu_1}{\sqrt{\sigma^2/n}}) \\ &= Q(\frac{\mu_1-52}{\sqrt{\sigma^2/n}}), \quad \text{using properties of Q-function} \end{aligned}$$

For $\mu_1 = 53$ we obtain $\beta = 0.3085$ and for $\beta = \mu_1 = 55$ we obtain 0.0668.
Notice that clearly the probability of Type II error depends on μ_1.
B)-(1) Using the result in (2) of part A, we are looking for c such that

$$\alpha = P(\bar{X} \geq c | \mu = \mu_0) = 0.05$$

Again given that $\mu = \mu_0 = 50$, $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$. Thus we want c such that

$$Q(\frac{c-50}{\sqrt{\sigma^2/n}}) = c$$
$$\Leftrightarrow c = \sqrt{\sigma^2/n}Q^{-1}(0.05) + 50$$

Looking at the table we have $c = 53.29$
(2) Using part (A)-(3) with the modified decision rule, we obtain

$$\beta = Q(\frac{\mu_1 - 53.29}{\sqrt{\sigma^2/n}})$$

With $\mu_1 = 55$, we obtain $\beta = 0.1963$.
(3) Comparing with the results of part (A), we notice that with the change of the decision rule, α is reduced from 0.1587 to 0.05, but β is increased from 0.0668 to 0.1963.
(C) We will just use the results in part (B) with $n = 100$.
(1)

$$c' = \sqrt{\sigma^2/n}Q^{-1}(0.05) + 50, \quad \sigma^2 = 100, n = 100$$

Using tables, we obtain $c' = 51.645$.
(2) Setting $\mu_1 = 55$, we obtain

$$\beta' = Q(\frac{\mu_1 - 51.645}{\sqrt{\sigma^2/n}}) = 0.0004$$

(3)Notice that with sample size $n = 100$, both α and β have decreased from their respective original values of 0.1587 and 0.0668 when $n = 25$.

Problem 8.4. : *Bayes' Test*
In the Bayes' Test, each event (D_i, H_j) has an associated cost C_{ij}, where (D_i, H_j) is the event that hypothesis H_i is accepted while H_j is true (for $i, j = 0, 1$). The average cost or Bayes' risk *is defined as*

$$\bar{C} = C_{00}P(D_0; H_0) + C_{01}P(D_0; H_1) + C_{10}P(D_1; H_0) + C_{11}P(D_1; H_1)$$

The test that minimizes the average cost is called the Bayes' test and can be expressed in terms of the likelihood ratio

$$\Lambda(X) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{(C_{10} - C_{00})P(H_0)}{(C_{01} - C_{11})P(H_1)}$$

Now consider a binary decision problem with the following conditional pdf's:

$$f(x|H_0) = \frac{1}{2}\exp\{-|x|\}$$
$$f(x|H_1) = \exp\{-2|x|\}$$

The costs are given as follows:

$$C_{00} = C_{11} = 0 \quad C_{01} = 2 \quad C_{10} = 1$$

(a) Determine the Bayes' test if $P(H_0) = \frac{2}{3}$ and the associated Bayes' risk.
(b) Repeat part (a) for $P(H_0) = \frac{1}{2}$.

Problem 8.5. *Minimum Probability of Error*
Consider a Bayes' test with the following costs

$$C_{00} = C_{11} = 0 \quad C_{01} = 1 \quad C_{10} = 1$$

What is the average cost? (you should recognize this cost).
What is the Bayes' test? (this is a test that you already know)

Problem 8.6. *Let X and Y be two independent random variables with the same distribution $U(0,1)$.*
Find the MMSE of X^3 given $X+Y$.

Solution:
We know that the MMSE is given by $E[X^3|X+Y]$. So we only need to compute the distribution of X given $X+Y=z$.
Considering the cases where $0 \leq z < 1$ and $2 \geq z \geq 1$ we obtain

$$f_{X|X+Y}(x|z) = \begin{cases} \frac{1}{z}1_{[0\leq x\leq 1]} & \text{if } 0 \leq z < 1 \\ \frac{1}{2-z}1_{[z-1\leq x\leq 1]} & \text{if } 2 \geq z \geq 1 \end{cases}$$

Now we only need to compute $E[X^3|X+Y]$ and see that it is given by:

$$E[X^3|X+Y] = \begin{cases} \frac{1}{4}(x+y)^3 & \text{if } 0 \leq z < 1 \\ \frac{1}{4(2-x-y)}[1-(x+y)^4] & \text{if } 2 \geq z \geq 1 \end{cases}$$

Problem 8.7. *Let $Y = H \times X + Z$ where X is some Gaussian random vector in R^n with zero mean and covariance matrix K_X, H is a non-singular $n \times n$ known matrix, and Z is a gaussian random noise with zero mean and non-singular covariance matrix K_Z uncorrelated to X.*
(a) Find the MMSE estimator of X given Y.
(b) Explain what happens in the case where $|K_Z| = 0$.

Problem 8.8. *The values of a random sample, $2.9, 0.5, -0.1, 1.2, 3.5,$ and 0, are obtained from a random variable X uniformly distributed over the interval $[a,b]$. Find the maximum-likelihood estimates of a and b (assume that the samples are independent).*

Solution
The MLE is the value of (a, b) that maximizes the likelihood of observing the given sequence. In other terms, we are looking for $(\hat{a}, \hat{b})$ such that:

$$(\hat{a}, \hat{b}) = argmax_{(a,b)} f(x_1, \ldots, x_n; (a, b))$$

where x_i's are the observations.
The pdf $f(x_1, \ldots, x_n; (a, b))$ is equal to

$$f(x_1, \ldots, x_n; (a, b)) = \begin{cases} \frac{1}{b-a} & a \leq x_i \leq b, \forall i \\ 0 & \text{otherwise} \end{cases}$$

The choice of (a, b) that maximizes this expression is clearly $a = x_{min} = \min\{x_1, \ldots, x_n\}$ and $b = x_{max} = \max\{x_1, \ldots, x_n\}$. Thus the MLE of (a, b) is $(\hat{a}, \hat{b}) = (-0.1, 3.5)$.

Problem 8.9. *Consider a system consisting of two sensors, each making a simple measurement of an unknown constant x. Each measurement is noisy and may be modeled as follows*

$$\begin{aligned} y(1) &= x + v(1) \\ y(2) &= x + v(2) \end{aligned}$$

where $v_i, i = 1, 2$'s are zero mean, uncorrelated random variables with variances $\sigma_i^2, i = 1, 2$.
(a) We want to compute the best linear estimate of x of the form

$$\hat{x} = k_1 y(1) + k_2 y(2).$$

Find the values of k_1 and k_2 that produce an unbiased estimate of x that minimizes the mean-square error $E[(x - \hat{x})^2]$.

Problem 8.10. *The random variables X and Z are independent. X is uniformly distributed in $[0, 1]$ and Y is exponentially distributed with mean 1. Let $Y = X + Z$.*

1. *What is the $LLSE(X|Y)$?*
2. *What is the $MMSE(X|Y)$?*