

Problem Set 6
Spring 2007

Issued: Thursday, March 8, 2007

Due: Thursday, March 15, 2007

Problem 6.1

1. Since X and Y are zero-mean jointly Gaussian random variables, their joint pdf is of the form

$$f_{X,Y}(x,y) = ce^{-q(x,y)},$$

where c is a normalization constant

$$c = \frac{1}{2\pi\sqrt{E[X^2]E[Y^2] - \text{cov}(X,Y)^2}},$$

and

$$\begin{aligned} q(x,y) &= \left[\frac{1}{E[X^2]}x^2 - 2\frac{\text{cov}(X,Y)}{E[X^2]E[Y^2]}xy + \frac{1}{E[Y^2]}y^2 \right] \frac{1}{2\left(1 - \frac{\text{cov}(X,Y)^2}{E[X^2]E[Y^2]}\right)} \\ &= \left[\frac{1}{E[X^2]}x^2 - 2\frac{\text{cov}(X,Y)}{E[X^2]E[Y^2]}xy + \frac{1}{E[Y^2]}y^2 \right] \frac{E[X^2]E[Y^2]}{2(E[X^2]E[Y^2] - \text{cov}(X,Y)^2)} \\ &= \left[E[Y^2]x^2 - 2\text{cov}(X,Y)xy + E[X^2]y^2 \right] \frac{1}{2(E[X^2]E[Y^2] - \text{cov}(X,Y)^2)}. \end{aligned}$$

Note that

$$\begin{aligned} K^{-1} &= \frac{1}{|K|} \begin{pmatrix} E[Y^2] & -E[XY] \\ -E[XY] & E[X^2] \end{pmatrix} \\ &= \frac{1}{E[X^2]E[Y^2] - \text{cov}(X,Y)^2} \begin{pmatrix} E[Y^2] & -E[XY] \\ -E[XY] & E[X^2] \end{pmatrix}, \end{aligned}$$

and by straight comparison,

$$q(x,y) = -\underline{v}^t K^{-1} \underline{v} / 2,$$

which implies that

$$f_{X,Y}(x,y) = ce^{-\underline{v}^t K^{-1} \underline{v} / 2}.$$

Finally,

$$c = \frac{1}{2\pi\sqrt{E[X^2]E[Y^2] - \text{cov}(X,Y)^2}} = \frac{1}{2\pi\sqrt{|K|}},$$

which yields the desired result.

2. (a) Since X and Y are jointly Gaussian, there exist two independent normal random variables U and V , such that

$$\begin{aligned} X &= aU + bV \\ Y &= cU + dV \end{aligned}$$

and therefore,

$$\begin{aligned} Z &= (2a + c)U + (2b + d)V \\ W &= (a - 2c)U + (b - 2d)V \end{aligned}$$

and Z and W are jointly Gaussian.

Note: Since X and Y are jointly Gaussian, any linear combination of these two variables is a Gaussian random variable. With this derivation we have established that any *two* such linear combinations are actually jointly Gaussian.

- (b) Given that Z and W are jointly Gaussian, their joint pdf is of the form given in (a) and therefore it suffices to find the covariance matrix M of Z and W .

Using a vector notation where the expected value of a matrix is the matrix of the expected value of each of its elements

$$\begin{aligned} M &= E \left[\begin{pmatrix} Z \\ W \end{pmatrix} \begin{pmatrix} Z & W \end{pmatrix} \right] \\ &= E \left[\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} E \left[\begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} \right] \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} K \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \end{aligned}$$

Problem 6.2

Three independent and identically distributed continuous random variables will always satisfy

$$\begin{aligned} P(X_1 < X_2 < X_3) &= P(X_1 < X_3 < X_2) = P(X_2 < X_1 < X_3) \\ &= P(X_2 < X_3 < X_1) = P(X_3 < X_1 < X_2) = P(X_3 < X_2 < X_1) = \frac{1}{6}. \end{aligned}$$

The equality of the above probabilities is due to the symmetry of the situation, and then the value of $\frac{1}{6}$ follows from the fact that there are six orderings. (Note that equality of random variables can be ignored because the probability of equality is zero with continuous random

variables.) Writing the definition of conditional probability,

$$\begin{aligned}
 P(X_1 < X_2 \mid X_2 < X_3) &= \frac{P(X_1 < X_2 \text{ and } X_2 < X_3)}{P(X_2 < X_3)} \\
 &= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3) + P(X_2 < X_3 < X_1)} \\
 &= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}} = \frac{1}{3}
 \end{aligned}$$

Problem 6.3

If $y \leq z$, then

$$\begin{aligned}
 F_{Y,Z}(y, z) &= P(Y \leq y, Z \leq z) \\
 &= P(\max(X_1, \dots, X_n) \leq y \& \min(X_1, \dots, X_n) \leq z) \\
 &= P(\max(X_1, \dots, X_n) \leq y) \\
 &= \prod_{i=1}^n P(X_i \leq y) \\
 &= F(y)^n,
 \end{aligned}$$

because the $\{X_i\}$'s are IID.

On the other hand, if $y > z$,

$$\begin{aligned}
 F_{Y,Z}(y, z) &= P(Y \leq y, Z \leq z) \\
 &= P(\max(X_1, \dots, X_n) \leq y \& \min(X_1, \dots, X_n) \leq z) \\
 &= P(\max(X_1, \dots, X_n) \leq y) - P(\max(X_1, \dots, X_n) \leq y \& \min(X_1, \dots, X_n) > z) \\
 &= F(y)^n - P(z < X_i \leq y, \forall i) \\
 &= F(y)^n - \prod_{i=1}^n (F(y) - F(z)) \\
 &= F(y)^n - (F(y) - F(z))^n
 \end{aligned}$$

where we have used the fact that the $\{X_i\}$'s are IID and the total probability theorem. We can see that Y and Z are not independent as the form of the joint distribution depends on whether $y \leq z$ or $y > z$. Below we derive the joint PDF of Y and Z and it is shown that the joint PDF can not be factored into the product of f_Y and f_Z .

Problem 6.4

1. An error will occur if
 - If the system concludes 0 or 1 sent, when -1 was sent.
 - If the system concludes -1 or 1 sent, when 0 was sent.

- If the system concludes -1 or 0 sent, when 1 was sent.

Therefore, the probability of error, $P(\varepsilon)$, is:

$$\begin{aligned} P(\varepsilon) &= P(X = -1)P(Y \geq -\frac{1}{2}|X = -1) + P(X = 0)P(Y < -\frac{1}{2} \text{ OR } Y > \frac{1}{2}|X = 0) \\ &\quad + P(X = 1)P(Y \leq \frac{1}{2}|X = 1) \\ &= \frac{1}{3}P(N \geq \frac{1}{2}) + \frac{1}{3}P(N < -\frac{1}{2} \text{ OR } N > \frac{1}{2}) + \frac{1}{3}P(N \leq -\frac{1}{2}) \end{aligned}$$

from Symmetry of Normal Distribution

$$P(N \geq \frac{1}{2}) = P(N \leq -\frac{1}{2})$$

Since N is a gaussian random variable, we can calculate $P(N \geq n)$ using the Φ function,

$$P(N \geq n) = \left(1 - \Phi\left(\frac{n - \mu}{\sigma}\right)\right)$$

Therefore

$$\begin{aligned} P(\varepsilon) &= \frac{4}{3}P(N \geq \frac{1}{2}) \\ &= \frac{4}{3}\left(1 - \Phi\left(\frac{1-1}{2-2}\right)\right) \\ &= \boxed{0.535} \end{aligned}$$

2. Similarly

$$\begin{aligned} P(\varepsilon) &= P(X = -2)P(Y \geq -1|X = -2) + P(X = 0)P(Y < -1 \text{ OR } Y > 1|X = 0) \\ &\quad + P(X = 2)P(Y \leq 1|X = 2) \\ &= \frac{1}{3}P(N \geq 1) + \frac{1}{3}P(N < -1 \text{ OR } N > 1) + \frac{1}{3}P(N \leq -1) \end{aligned}$$

Therefore

$$\begin{aligned} P(\varepsilon) &= \frac{4}{3}P(N \geq 1) \\ &= \frac{4}{3}\left(1 - \Phi\left(\frac{1}{2}\right)\right) \\ &= \boxed{0.41} \end{aligned}$$

Problem 6.5

(a) We are given that \mathbf{X} is a zero-mean Gaussian random variable with variance σ^2 . The moment of X^n is expressed as follows

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^n \exp \frac{-x^2}{\sigma^2} dx \quad (1)$$

Using integration by parts we get the following recursion,

$$E[X^n] = \sigma^2(n-1)E[X^{n-2}] \quad (2)$$

We note that when n is odd, the base case for the recursion is zero. Therefore, any odd n will yield a moment of X^n that is 0. When n is even the recursion yields the following formulae,

$$E[X^{2n}] = \frac{(2n)!}{n!2^n} \sigma^{2n} \quad (3)$$

(b) From part a) we discern that the mean of X^2 is σ^2 . Therefore ,

$$\begin{aligned} E[Y] &= E[X_1^2] + E[X_2^2] + \dots + E[X_n^2] \\ &= n\sigma^2 \end{aligned} \quad (4)$$

The variance of any of the X_i^2 can be expressed as follows,

$$Var(X_i^2) = E[X_i^4] - E[X_i^2]^2 \quad (5)$$

$$= 3\sigma^4 - \sigma^4 \quad (6)$$

$$= 2\sigma^4 \quad (7)$$

Therefore the variance of Y is the sum of the variances of the random variables X_i^2 . Because the variables are identically distributed the variance of Y becomes $n2\sigma^4$.

Problem 6.6

1. We know that the total length of the edge for red interval is two times that for black interval. Since the ball is equally likely to fall in any position of the edge, probability of falling in a red interval is $\frac{2}{3}$.
2. Conditioned on the ball having fallen in a black interval, the ball is equally likely to fall anywhere in the interval. Thus, the PDF is

$$f_{Z|\text{black interval}}(z) = \begin{cases} \frac{15}{\pi r} & , \quad z \in [0, \frac{\pi r}{15}] \\ 0 & , \quad \text{otherwise} \end{cases}$$

3. Since the ball is equally likely to fall on any point of the edge, we can see it is twice as likely for $z \in [0, \frac{\pi r}{15})$ than $z \in [\frac{\pi r}{15}, \frac{2\pi r}{15}]$. Therefore, intuitively, let

$$f_Z(z) = \begin{cases} 2h & , \quad z \in [0, \frac{\pi r}{15}) \\ h & , \quad z \in [\frac{\pi r}{15}, \frac{2\pi r}{15}] \\ 0 & , \quad \text{otherwise} \end{cases}$$

Using the fact that $\int_{-\infty}^{\infty} f_Z(z) = 1$,

$$(2h)\left(\frac{\pi r}{15}\right) + (h)\left(\frac{\pi r}{15}\right) = 1 \Rightarrow h = \frac{5}{\pi r}$$

$$f_Z(z) = \begin{cases} \frac{10}{\pi r} & , z \in \left[0, \frac{\pi r}{15}\right) \\ \frac{5}{\pi r} & , z \in \left[\frac{\pi r}{15}, \frac{2\pi r}{15}\right] \\ 0 & , \text{otherwise} \end{cases}$$

4. The total gains (or losses), T , equals to the sum of all X_i , i.e. $T = X_1 + X_2 + \dots + X_n$. Since all the X_i 's are independent of each other, and they have the same Gaussian distribution, the sum will also be a Gaussian with

$$E[T] = E[X_1] + E[X_2] + \dots + E[X_n] = 0$$

$$\text{var}(T) = \text{var}(X_1) + \text{var}(X_2) + \dots + \text{var}(X_n) = n\sigma^2$$

Therefore, the standard deviation for T is $\sqrt{n}\sigma$.

5.

$$\begin{aligned} P(|T| > 2\sqrt{n}\sigma) &= P(T > 2\sqrt{n}\sigma) + P(T < -2\sqrt{n}\sigma) \\ &= 2P(T > 2\sqrt{n}\sigma) \\ &= 2\left(1 - \Phi\left(\frac{2\sqrt{n}\sigma - E[T]}{\sigma_T}\right)\right) \\ &= 2(1 - \Phi(2)) \simeq 0.0454. \end{aligned}$$