UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 5
Spring 2007

Issued: Friday, February 23, 2007 **Due:** Thursday, March 8, 2007

Problem 5.1

Joe wishes to estimate the true fraction f of smokers in a large population without asking each and every person. He plans to select n people at random and then employ the estimator $F = S/n$, where S denotes the number of people in a size-n sample who are smokers. Joe would like to sample the minimum number of people, but also guarantee an upper bound p on the probability that the estimator F differs from the true value f by a value greater than or equal to d i.e., for a given accuracy d and given confidence p, Joe wishes to select the minimum n such that

$$\mathbf{P}(|F - f| \geq d) \leq p \quad .$$

For $p = 0.05$ and a particular value of d, Joe uses the Chebyshev inequality to conclude that n must be at least 50,000. Determine the new minimum value for n if:

1. the value of d is reduced to half of its original value.
2. the probability p is reduced to half of its original value, or $p = 0.025$.

Problem 5.2

For some random variable X, given that $E[X] = 7$ and $Var(X) = 6$, what can you say about $P(4 < X < 10)$? What if $Var(X) = 9$?

Problem 5.3

Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops. We are told that

$$p_{N|K}(n \mid k) = \frac{1}{k}, \qquad \text{for } n = 1, \ldots, k.$$

1. Find the joint PMF of K and N.
2. Find the marginal PMF of N.
3. Find the conditional PMF of K given that $N = 2$.
4. We are now told that he bought at least 2 but no more than 3 books. Find the conditional mean and variance of K, given this piece of information.

5. The cost of each book is a random variable with mean 3. What is the expected value of his total expenditure? *Hint:* Condition on events $N = 1, \ldots, N = 4$ and use the total expectation theorem.

Problem 5.4

Consider two random variables, X and Y, which take on only integer values, as indicated by their joint PMF:

$$p_{X,Y}(x, y) = \begin{cases} \frac{y}{20} & 1 \leq y \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

1. Find the marginal PMF $p_X(x)$ $\forall x$, and evaluate $\mathbf{E}[X]$.
2. Find the PMF $p_W(w)$, where $W = \max(X, 2Y)$.
3. For $R = X - Y$, evaluate $\mathbf{E}[R]$ and $\mathrm{var}(R)$.
4. With R defined as in part (c), let A denote the event $R \geq 2$, and evaluate the conditional standard deviation $\sigma_{R|A}$.

Problem 5.5

You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

Find the mean and the variance of the number of trials you will need to open the door, under the following alternative assumptions:

1. after an unsuccessful trial, you mark the corresponding key so that you never try it again, or
2. at each trial, you are equally likely to choose any key.

Problem 5.6

A coin has an a priori probability P of coming up heads, where P is a random variable with probability density:

$$f_P(p) = \begin{cases} p\,e^p, & \text{for } p \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

1. Find $P(\text{Heads})$.
2. Given that the last flip was heads, find the conditional density for P, i.e. find $f_{P|A}(p|A)$ where A denotes the event that the last flip came up heads.
3. Given A, find the conditional probability of heads on the next flip.

Problem 5.7

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of *independent* and *identically distributed* random variables. We define random variables $S_1, S_2, \ldots, S_n, \ldots$ in the following manner: $S_n = X_1 + X_2 + \cdots + X_n$. Find

$$E[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots]$$

Problem 5.8

A point X is uniformly distributed on a line segment S iff it has a density function:

$$f_X(x) = \begin{cases} \gamma & x \in S \\ 0 & \text{otherwise} \end{cases}$$

Similarly, a vector is uniformly distributed on a region R iff it has a density function:

$$f_{X,Y}(x,y) = \begin{cases} \gamma & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

1. Find γ.

2. Show that if R is the square of side 1, centered at (1/2,1/2) then X, Y are independent, and are both uniformly distributed from (0,1).

3. Show that for some R, X, Y need not be independent.

4. For the R in (b) find the probability that the point (X, Y) lies in the circle inscribed in the square R.