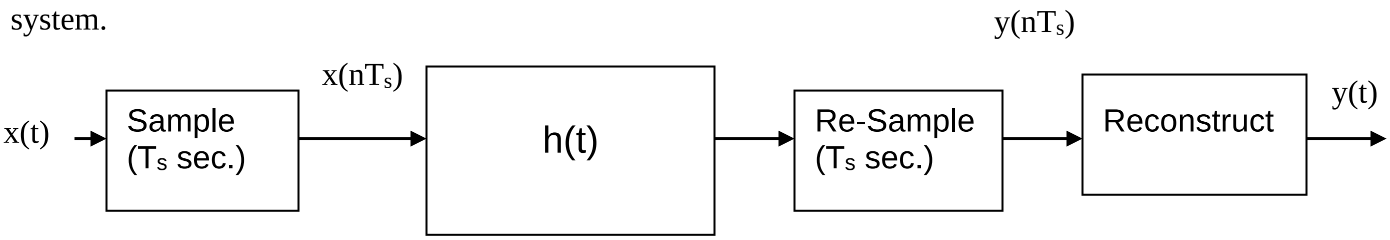
Discrete Time Systems

Introduction

In our discussion of Sampling we introduced an analog system which was equivalent to a discrete-time system.



If we simplify our notation,

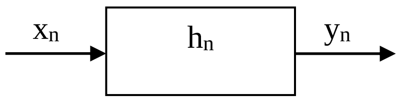
$$x_n = x(n * T_s) = x(t)\Big|_{t=n*T_s}$$

$$h_n = h(n * T_s) = h(t)\Big|_{t=n*T_s}$$

and

$$y_n = y(n * T_s) = y(t)\Big|_{t=n*T_s}$$

We get the discrete-time system.



Where the input, output and "pulse response" are each sequences of sampled values of the corresponding analog function.

Macroscopic Time Domain Analysis

In our Linear Systems discussion, we found that the output, y(t), of a linear system is related to its input, x(t), by the Convolution Integral,

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) * h(t-\tau)\,d\tau$$

Note: the limits normally are from 0 to t since the input starts at t=0 and the impulse response is 0 for values less than 0

But, in this case,

$$x(\tau) = \sum_{n=0}^{\infty} x_n * \delta(\tau - n * T_s)$$

For x(t) = 0 when t < 0 and assuming ideal sampling

Therefore

$$y(t) = \int_{\tau=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} x_k * \delta(\tau - k * T_s) \right] * h(t-\tau)\,d\tau$$

Now interchanging the order of integration and summation and using the sifting property of δ-functions

$$y(t) = \sum_{k=0}^{\infty} x_k \int_{\tau=-\infty}^{\infty} \delta(\tau - k*T_s) h(t-\tau) d\tau$$

And using the sifting property of the delta function

$$y(t) = \sum_{k=0}^{\infty} x_k * h(t - k*T_s)$$

Now we resample at t = n*T_s:

$$y(nT_s) = \sum_{k=0}^{\infty} x_k * h[(n-k)*T_s]$$

Again using our simplified notation, we obtain the “Convolution Sum”

$$y_n = \sum_{k=0}^{\infty} x_k * h_{(n-k)}$$

Where h_n is the sequence of samples of the original impulse response at the t = n*T_s

If we let the input sequence, x_n, be the unit pulse at n = 0

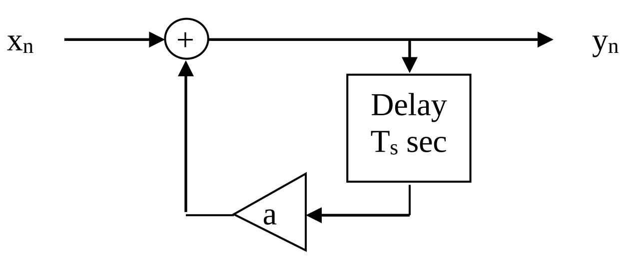
$$\delta_n = \begin{matrix} 0 & if\ n \neq 0 \\ 1 & if\ n = 0 \end{matrix}$$

$$y_n = \sum_{k=0}^{\infty} \delta_k * h_{(n-k)} = h_n$$

so h_n is the “Unit Pulse Response” of the discrete-time system

In general h(n,k) is the response at sample time n due to a pulse that occurred at sample time k

An example discrete-time system



Let $x_n = \delta_n$ Therefore y_n = { 1, a, a^2, a^3, …..) = a^n and $h_n = a^n$ for $n \geq 0$

Note: $y_n = 0$ for n < 0 and for stability $|a| \leq 1$

since $a^n = \varepsilon^{n*\ln(a)}$ this discrete-time system is equivalent (within a gain factor) to the simple RC low-pass filter (“leaky bucket”) which has been used often as an analog system example in these notes. The time Constant, RC, is:

$$\tau = \frac{1}{\ln(a)}$$

We can now solve for the output of this system given any input through the use of the convolution sum.

Let x_n be the discrete-time version of the unit step function

$$U_n = \begin{matrix} 0 & if\ n<0 \\ 1 & if\ n\geq 0 \end{matrix}$$

And let the system be initially at rest ($y_n = 0$).

The output is given by the convolution sum as

$$y_n = \sum_{k=0}^{\infty} x_k * h_{(n-k)} = \sum_{k=0}^{\infty} h_k * x_{(n-k)}$$

The second form is the commutative version and is easier to use for this calculation.

We have two cases:

For n < 0 all of the input values are zero and the sum is zero

For n ≥ 0 the sum becomes

$$y_n = \sum_{k=0}^{n} a^k * 1 = \sum_{k=0}^{n} a^k$$

This is a simple geometric series. To get a closed form without resorting to formulas pull out the first term and add/subtract an extra term

$$y_n = 1 + \sum_{k=1}^{n+1} a^k - a^{(n+1)}$$

Now let p = k - 1

$$y_n = 1 + \sum_{p=0}^{n} a^{p+1} - a^{(n+1)}$$

$$y_n = 1 + a y_n - a^{(n+1)}$$

$$(1-a) * y_n = 1 - a^{(n+1)}$$ or

$$y_n = \frac{1-a^{(n+1)}}{(1-a)}$$ for n ≥ 0

We therefore have our macroscopic approach to solving for the output of discrete-time systems.

Microscopic Time Domain analysis

For a microscopic analysis note how we determined h_n for the example system. We actually calculated each output at time n from a knowledge of the input at time n and the “state” of the memory (or delay elements) in the system. For the example system:

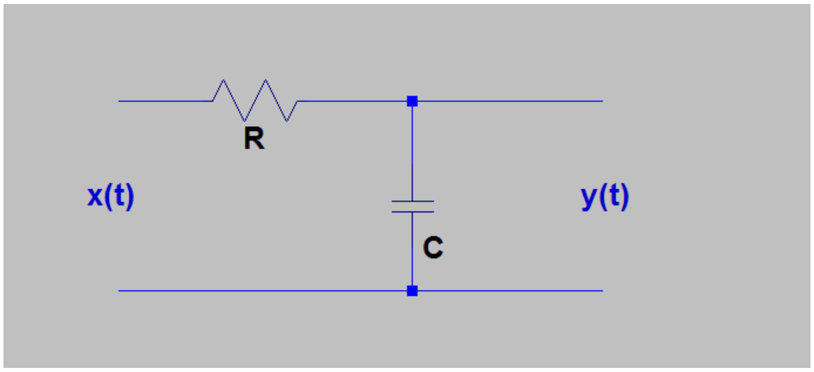
$$y_n = x_n + a * y_{n-1}$$ or

$$y_n - a * y_{n-1} = x_n$$

This is a finite difference equation which describes the example system at a moment in time. It is analogous to the differential equation:

$$\frac{dy(t)}{dt} - \frac{1}{RC} * y(t) = \frac{1}{RC} * x(t)$$

which describes the RC low pass example in the discussion of Linear Systems.



These difference equations can be solved directly using the same techniques used to solve differential equations. As an example we will solve the same problem used in the section on discrete convolution.

$y_n - a * y_{n-1} = x_n$; $0 < a < 1$; x_n is the unit step and the system is initially at rest with $y_n = 0$.

Step 1: Find the homogeneous (AKA natural or transient) solution

$$y_n - a * y_{n-1} = 0$$

Assume an exponential solution $y_n = C^n$

$C^n - aC^{(n-1)} = 0$

$C^n = aC^{(n-1)}$ or

C = a

The homogeneous solution is therefore

$y_n = K_H * a^n$

Step 2: Find the particular (AKA forced or steady-state) solution – we shall again use the method of undetermined coefficients

The particular solution for this problem must be of the form

$y_n = K_0 + K_1*n$ since the input is a constant. Therefore

$y_{(n-1)} = (K_0 - K_1) + K_1*n$

Now substituting into the difference equation for $n \geq 0$

$K_0 + K_1*n - a[(K_0 - K_1) + K_1*n = 1$

Or

$[K_0*(1-a) + a* K_1] + [K_1(1-a)]*n = 1$

Therefore

$K_1 = 0$ and $K_0 = 1/(1-a)$

The total solution is the sum of the homogeneous and particular solutions so

$y_n = K_H * a^n + 1/(1-a)$

but $y_0 = 1$ so

$1 = K_H * a^0 + 1/(1-a)$

Or $K_H = 1 – 1/(1-a) = -a/(1-a)$

This makes the total solution

$y_n = -a^{(n+1)}/(1-a) + 1/(1-a)$

Or

$y_n = [1-a^{(n+1)}]/(1-a)$ for $n \geq 0$

Which agrees with our previous solution.