# Response of LTI Systems Using Laplace Transforms

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt$$

Where *h*(*t*) is an impulse response, is called the system function or transfer function and it completely characterizes the input/output relationship of an LTI system. We can use it to determine time responses of LTI systems.

## Transfer Functions

We can use Laplace Transforms to solve differential equations for systems (assuming the system is initially at rest for one-sided systems) of the form:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t)$$

Taking the Laplace Transform of both sides of this equation and using the Differentiation Property, we get:

$$Y(s) \sum_{k=0}^{n} a_k s^k = X(s) \sum_{k=0}^{m} b_k s^k$$

From this, we can define the transfer function, ***H*(*s*)**, as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}$$

Which is the ratio of two polynomials in "s"

## Partial Fraction Expansion

Instead of taking contour integrals to invert Laplace Transforms, we will use Partial Fraction Expansion. We review it here. Given a Laplace Transform,

$$F(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}, \quad m < n$$

If m isn't less than n, perform **polynomial division** and then the remainder can be analyzed by Partial Fraction Expansion.
We write its Partial Fraction Expansion as:

$$F(s) = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$
$$= \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}$$

where

Assuming that all of the poles have unique values!

$$k_j = (s-p_j)\ F(s)\Big|_{s=p_j}$$

is the *residue* of the pole at *pj*.

Thus

$$f(t) = \sum_{j=1}^{n} k_j e^{p_j t} u(t)$$

because the Inverse Laplace Transform of

$$\frac{k_j}{(s+p_j)} \quad \text{is} \quad k_j e^{p_j t}$$

## Convolution

An important property of Laplace Transforms is that the Laplace transform of the convolution of two signals is the product of their Laplace transforms:

$$x(t)*h(t) \leftrightarrow X(s)H(s)$$

This is useful for studying LTI systems. In fact, we can completely characterize an LTI system from:

1. The system differential equation or
2. the system transfer function *H*(*s*) or
3. the system impulse response *h*(*t*).

**Example 1** Find *y*(*t*) where the transfer function *H*(*s*) and the input *x*(*t*) are given. Use Partial Fraction Expansion to find the output *y*(*t*):

$$H(s) = \frac{3s+1}{s^2+6s+5}, \quad x(t) = e^{-3t}u(t)$$

First find the Laplace transform of x(t) from a table of Laplace transforms: X(s) = 1/(s+3)

Now find the Laplace Transform of the output by multiplying H(s) by X(s)

Y(s) = (3s+1) / [ (s+3)*(s+2)*(s+3) ]

To do partial fractions we need to separate out the poles, but one of the poles is repeated so we need to find

Y(s) = A/(s+2) + B/(s+3) + C/$(s+3)^2$

The third term is to account for the repeated root.

Multiplying both sides of the equation by (s+2) yields:

$(s+2)^* Y(s) = (3s+1) / [ (s+3)^*(s+3) ] = A + (s+2) * [ B/(s+3) + C/(s+3)^2 ]$

Letting s ---> -2 now yields

-5 / [ (1)*(1) ] = A so **A = -5**

If we now repeat the process but multiply both sides by $(s+3)^2$ we get:

$(s+3)^{2*} Y(s) = (3s+1) / [ (s+2) ] = A^*(s+3)^2 + B^*(s+3) + C$

Now take the limit as s ---> -3 and

(-9 + 1) / (-1) = C so **C = 8**

there are two methods to find the remaining constant, B

1. Take [2]the derivative of both sides of the original equation and then B can be isolated by multiplying both sides by (s+3) and taking the limit as s ---> -3. This is the method shown in most textbooks.
2. go back to the original equation setting A = 1 and C = 8. Then put the three terms over a common denominator and the extra terms should cancel out to leave the "B" term.

Using the second method here:

Y(s) = -5/(s+2) + 8/(s+3) + B/(s+3)

$Y(s) = [-5(s+3)^2 + 8(s+2) + B(s+2)(s+3)] / [(s+2)(s+3)^2]$

$Y(s) = [-5s^2 - 30s - 45- +8s +16 + B(s^2 + 5s + 6)] / [(s+2)(s+3)^2]$

$Y(s) = (B-5)s^2 +(5B - 22)s + (6B-45) / [(s+2)(s+3)^2]$

But Y(s) = (3s+1) / [ (s+3)*(s+2)*(s+3) ] so

B-5 = 0 or **B = 5** from the $s^2$ term but as a check from the $s^1$ term:

5B-22 = 3 **OK** and from the $s^0$ term

6B-29 = 1 **OK**

We Have the partial fraction expansion of:

**$Y(s) = -5/(s+2) + 5/(s+3) + 8/(s+3)^2$**

And using the Table of Laplace Transforms

**y(t) = [-5*exp(-2t) +5*exp(-3t) +8t*exp(-3t)]*U(t)**

## Stability

We saw that a condition for bounded-input bounded-output stability was:

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Let's look at stability from a system function standpoint.

Given a Laplace Transform *H*(*s*), we expand *H*(*s*) with Partial Fraction Expansion:

$$H(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}$$

The corresponding impulse response is:

$$h(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots + k_n e^{p_n t} u(t)$$

What happens to *h*(*t*) as *t* → ∞? For a system to be stable, its impulse response must not blow up as *t* → ∞.

If $Re\{p_i\}$, $\forall i$ , then $h(t)$ decays to 0 as $t \to \infty$ and the system is stable **(just evaluate** $\int_{-\infty}^{\infty} |h(t)|\,dt$**).**

Therefore, the system is BIBO stable if and only if all poles of $H(s)$ are in the left half plane of the $s$-plane.