Noise: An Introduction

Adapted from a presentation in:
Transmission Systems for Communications,
Bell Telephone Laboratories, 1970, Chapter 7

Noise: An Introduction

- What is noise?
- Waveforms with incomplete information
  - Analysis: how?
  - What can we determine?
- Example: sine waves of unknown phase
  - Energy Spectral Density
  - Probability distribution function: P(v)
  - Probability density function: p(v)
  - Averages
- Common probability density functions
  - Gaussian
  - Exponential
- Noise in the real-world
- Noise Measurement
- Energy and Power Spectral densities

Background Material

- Probability
  – Discrete
  – Continuous
- The Frequency Domain
  – Fourier Series
  – Fourier Transform

Noise

- Definition

  Any undesired signal that interferes with the reproduction of a desired signal

- Categories
  - Deterministic: predictable, often periodic, noise often generated by machines
  - Random: unpredictable noise, generated by a “stochastic” process in nature or by machines

Random Noise

- Unpredictable
  - “Distribution” of values
  - Frequency spectrum: distribution of energy (as a function of frequency)
- We cannot know the details of the waveform only its “average” behavior

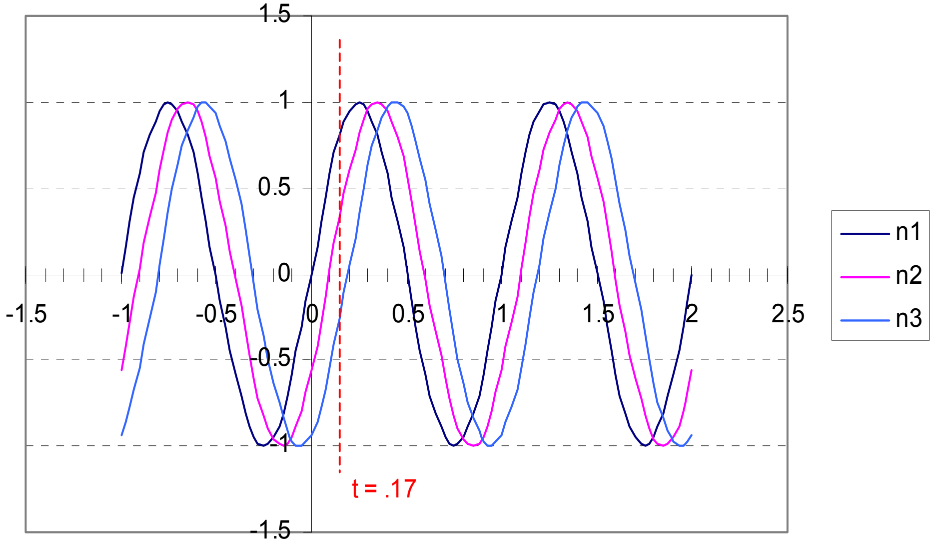
Noise analysis Introduction: a sine wave of unknown phase

- Single-frequency interference

  $n(t) = A \sin(\omega_n t + \phi)$

  A and ω_n are known, but ϕ is not known

- We cannot know its value at time "t"

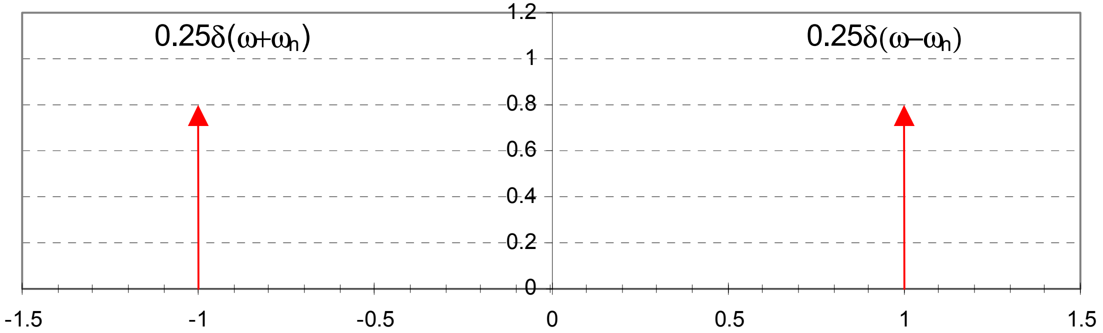


Energy Spectral Density

Here the "Energy Spectral Density" is just the magnitude squared of the Fourier transform of n(t)

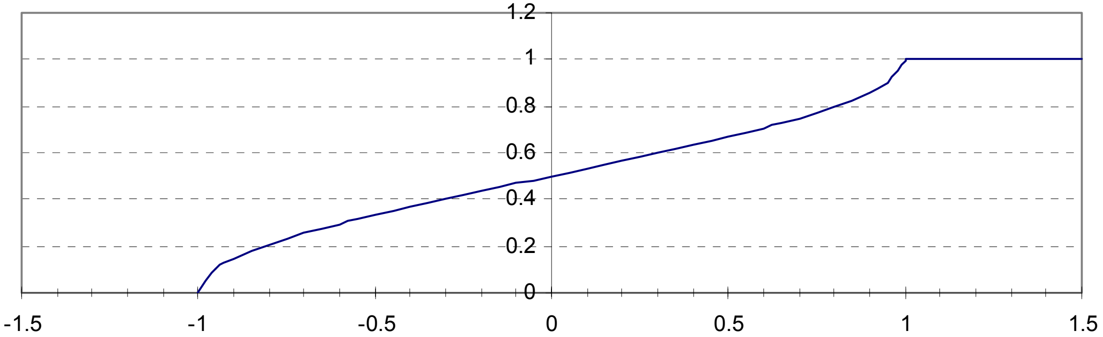
$$|N(\omega)|^2 = \frac{A^2}{4}\left[\delta(\omega-\omega_n)+\delta(\omega+\omega_n)\right]$$

since all of the energy is concentrated at ω_n and each half of the energy is at ± ω since the Fourier transform is based on the complex exponential not sine and cosine.



Probability Distribution

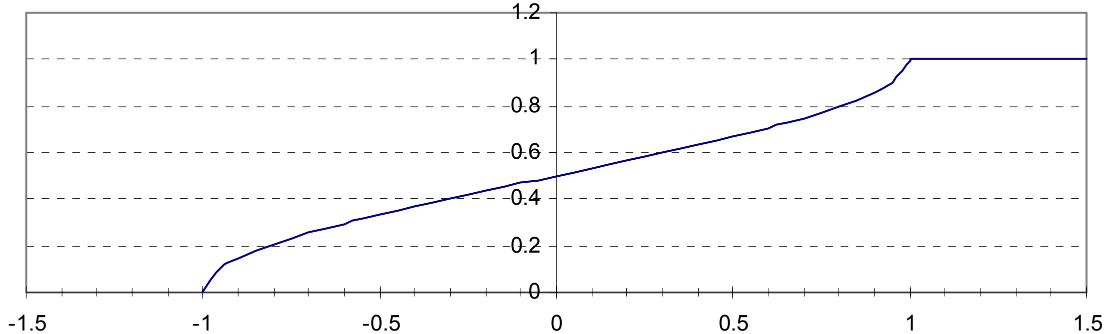
- The "distribution" of the 'noise" values
  - Consider the probability that at any time t the voltage is less than or equal to a particular value "v" $P(v_n) \equiv P[n(t) \leq v]$
- The probabilities at some values are easy
  - P(-A) = 0
  - P(A) = 1
  - P(0) = ½
- The actual equation is: $P(v_n) = \frac{1}{2} + (1/\pi)\arcsin(v_n/A)$



Shown for A=1

Probability Distribution
continued

- The actual equation is: $P(v_n) = ½ + (1/\pi)\arcsin(v/A)$



Shown for A=1

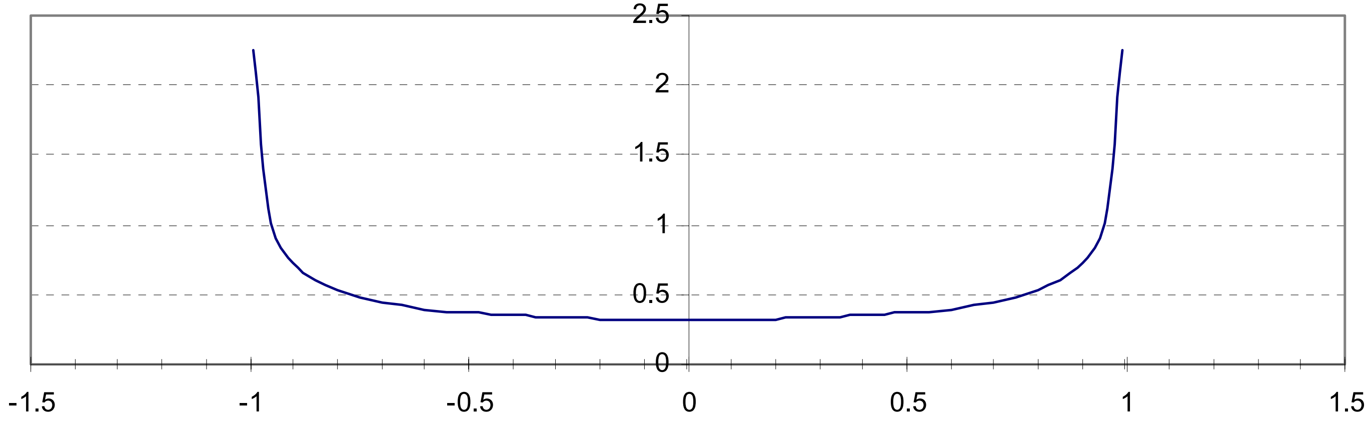
- Note that the noise spends more time near the extremes and less time near zero. Think of a pendulum:
  - It stops at the extremes and is moving slowly near them
  - It move fastest at the bottom and therefore spends less time there.
- Another useful function is the derivative of $P(v_n)$: the "Probability Density Function", $p(v_n)$ (note the lower case p)

Probability Density Function

- The area under a portion of this curve is the probability that the voltage lies in that region.
- This PDF is zero for $|v_n| > A$

$$p(v_n) = \frac{d}{dv}\left[P(v_n)\right]$$

$$p(v_n) = \frac{1}{\pi\sqrt{A^2 - v_n^2}}$$



Averages

- Time Average of signals

$$\overline{n} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} n(t)dt$$

- "Ensemble" Average
  - Assemble a large number of examples of the noise signal. (the set of all examples is the "ensemble")
  - At any particular time (t_0) average the set of values of $v_n(t_0)$

$$\overline{v_n} = E(v) = \frac{1}{K}\sum_{l=1}^{K} v_l \qquad \text{or } E(v_n) = \int_{-\infty}^{\infty} p(v)dv \text{ for the infinite set}$$

to get the "Expected Value" of v_n

- When the time and ensemble averages give the same value (they usually do), the noise process is said to be "Ergodic"

Averages (2)

- Now calculate the ensemble average of our sinusoidal “noise”

$$E(v_n) = \int_{-\infty}^{\infty} v * p(v) dv$$

$$E(v_n) = \int_{-\infty}^{\infty} \frac{v}{\pi \left(A^2 - v^2\right)^{0.5}} dv$$

- Which is obviously zero

(odd symmetry, balance point, etc.
as it should since this noise the has no DC component.)

Averages (3)

- $E[v_n]$ is also known as the "First Moment" of $p(v_n)$

$$E(v_n) = \int_{-\infty}^{\infty} v * p(v)dv$$

- We can also calculate other important moments of $p(v_n)$. The "Second Central Moment" or "Variance" (σ^2) is:

$$\sigma^2 = E\left[\left(v - \overline{v_n}\right)^2\right] = \int_{-\infty}^{\infty} \left(v - \overline{v_n}\right)^2 * p(v)dv$$

Which for our sinusoidal noise is:

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{v^2}{\pi\left(A^2 - v^2\right)^{0.5}} dv$$

Averages (4)

Integrating this requires “Integration by parts

$$\int U * dV = U * V - \int V dU$$

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{v^2}{\pi\left(A^2 - v^2\right)^{0.5}} dv$$

let $\quad U = v$ and $dV = \dfrac{v}{\pi\left(A^2 - v^2\right)^{0.5}}$

Then $dU = dv$ and $V = -\dfrac{1}{\pi}\left(A^2 - v^2\right)^{0.5}$ and

0

$$\sigma^2 = -\frac{v}{\pi}\left(A^2 - v^2\right)^{0.5}\Bigg|_{-A}^{A} + \int_{-A}^{A} \frac{1}{\pi}\left(A^2 - v^2\right)^{0.5} dv$$

Averages (5)

Continuing $$\sigma^2 = \int_{-A}^{A} \frac{1}{\pi}\left(A^2 - v^2\right)^{0.5} dv$$

$$= \frac{A^2}{2\pi} \sin^{-1}\left(\frac{v}{A}\right)\bigg|_{-A}^{A}$$

$$= \frac{A^2}{2\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

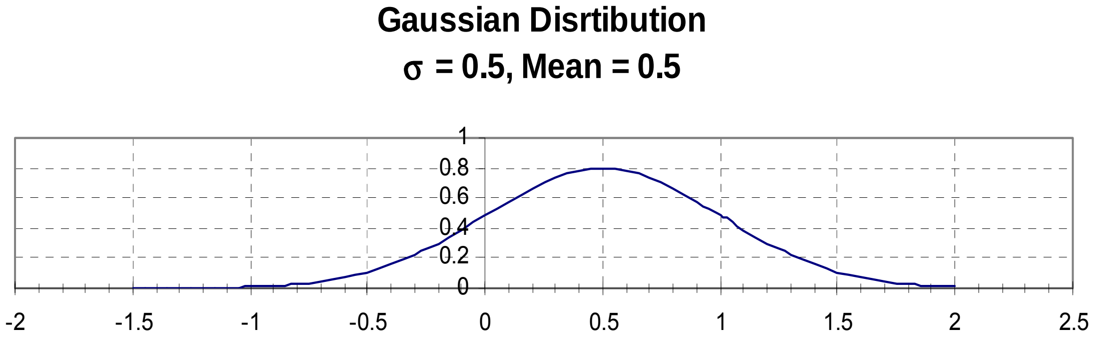
$$= \frac{A^2}{2}$$

Which corresponds to the power of our sine wave noise

Note: σ (without the "squared") is called the "Standard Deviation" of the noise and corresponds to the RMS value of the noise

Common Probability Density Functions: The Gaussian Distribution

$$p(v)=\frac{1}{\sigma\sqrt{2\pi}}\varepsilon^{-\left[(v-\bar{v})^2 / 2\sigma^2\right]}$$

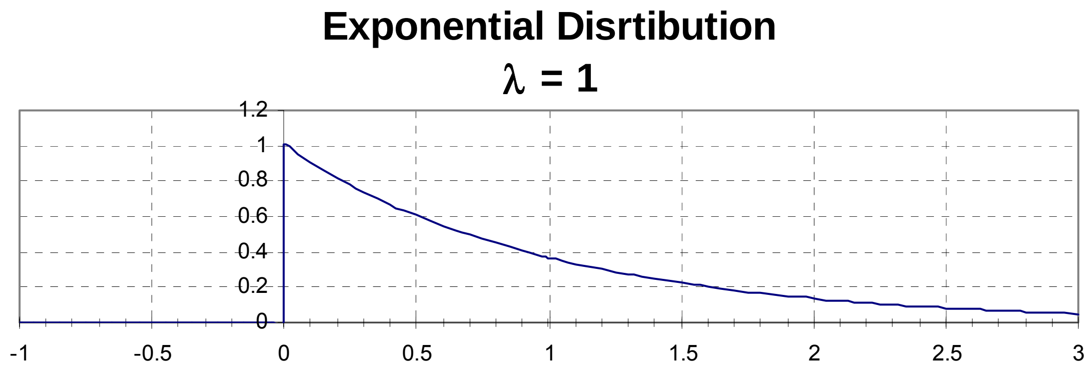


- Central Limit Theorem

  The probability density function for a random variable that is the result of adding the effects of many small contributors tends to be Gaussian as the number of contributors gets large.

Common Probability Density Functions: The Exponential Distribution

$$p(v) = \lambda * \varepsilon^{-\lambda v} \text{ for v} \geq 0 \text{ , 0 for v} < 0$$



- Occurs naturally in discrete "Poison Processes"
  - Time between occurrences
    - Telephone calls
    - Packets

Common Noise Signals

- Thermal Noise
- Shot Noise
- 1/f Noise
- Impulse Noise

Thermal Noise

- From the Brownian motion of electrons in a resistive material.

  $p_n(f) = kT$ is the power spectrum where:

  $k = 1.3805 * 10^{-23}$ (Boltzmann's constant) and

  T is the absolute temperature (°Kelvin)

- This is a "white" noise ("flat" spectrum)
  - From a color analogy
  - White light has all colors at equal energy
- The probability distribution is Gaussian

Thermal Noise (2)

- A more accurate model (Quantum Theory)

$$p_n(f) = \frac{h^*f}{\varepsilon^{\left(h^*f/_{kT}\right)} - 1}$$

Which corrects for the high frequency roll off (above 4000 GHz at room temperature)

- The power in the noise is simply

P_n = k*T*BW Watts or

P_n = -174 + 10*$\log_{10}$(BW) in dBm (decibels relative to a milliwatt)

Note: dB = 10*$\log_{10}$ (P/P_{ref}) = 20*$\log_{10}$ (V/V_{ref})

Shot Noise

- From the irregular flow of electrons

  $I_{rms} = 2*q*I*f$ where:

  $q = 1.6 * 10^{-19}$ the charge on an electron

- This noise is proportional to the signal level (not temperature)
- It is also white (flat spectrum) and Gaussian

1/f Noise

- Generated by:
  - irregularities in semiconductor doping
  - contact noise
  - Models many naturally occurring signals
    - “speech”
    - Textured silhouettes (Mountains, clouds, rocky walls, forests, etc.)
- $p_n(f) = A / f^{\alpha}$ $(0.8 < \alpha < 1.5)$

Impulse Noise

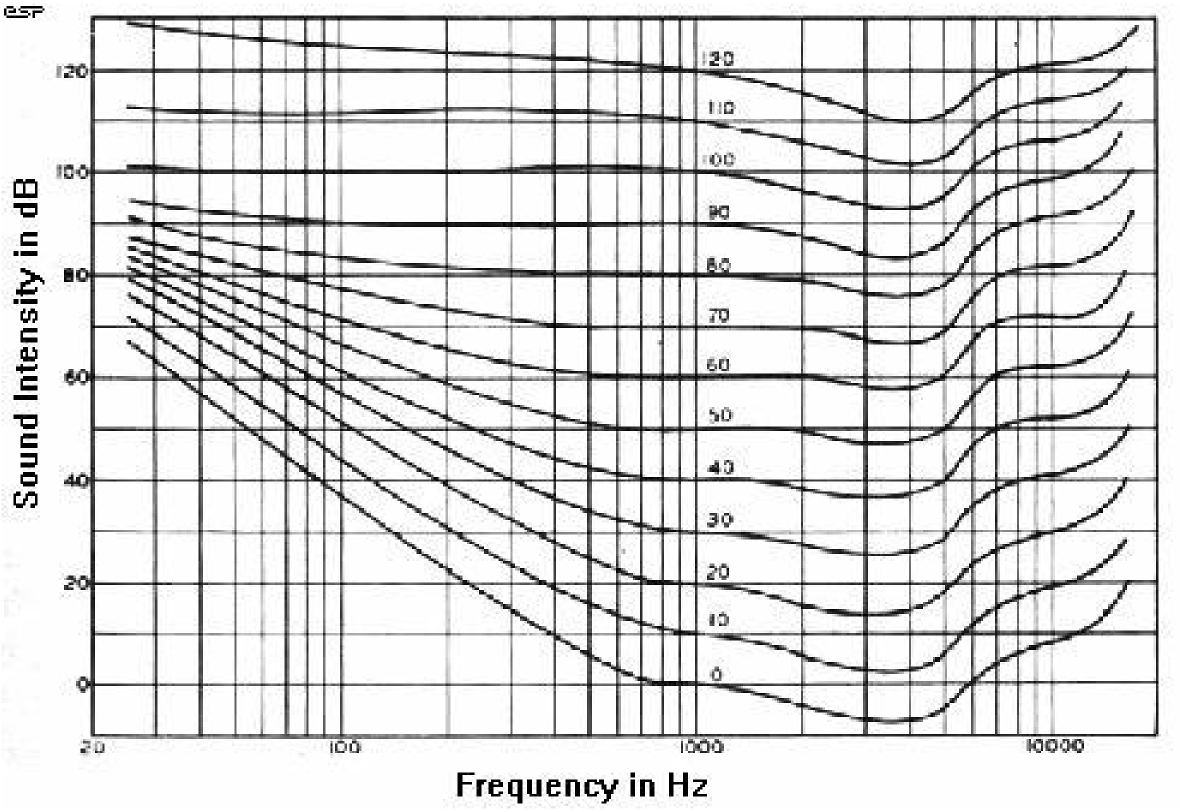
- Random energy spikes, clicks and pops
  - Common sources
    - Lightning
    - Vehicle ignition systems
  - This is a white noise, but NOT Gaussian
    - Adding multiple sources - more impulse noise
    - An exception to the “Central Limit Theorem”

Noise Measurement

- The Human Ear
  - Average Performance
  - The Cochlea
  - Hearing Loss
- Noise Level
  - A-Weighted
  - C-Weighted

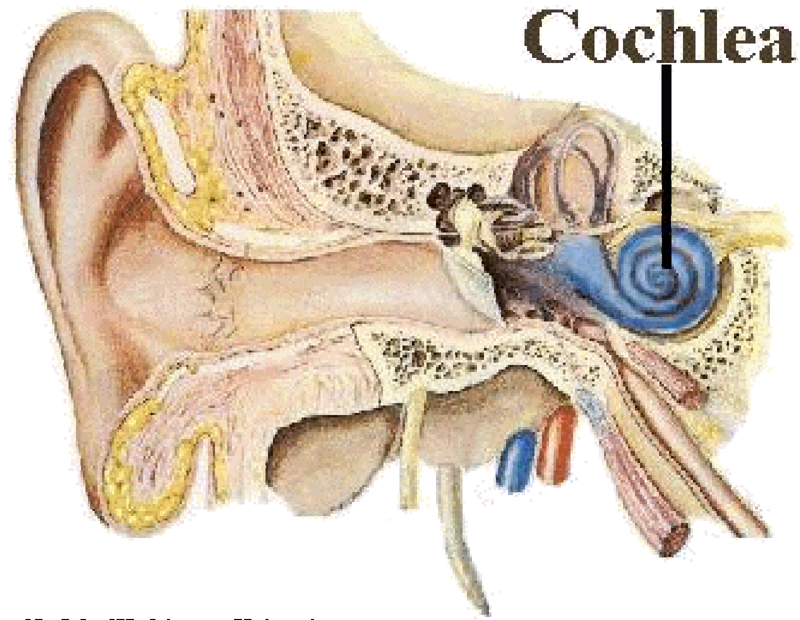
Hearing Performance
(an average, good, ear)

- Frequency response is a function of sound level
- 0 dB here is the threshold of hearing
- Higher intensities yield flatter response



The Cochlea

- A fluid-filled spiral vibration sensor
  - Spatial filter:
    - Low frequencies travel the full length
    - High frequencies only affect the near end
  - Cillia: hairs put out signals when moved
    - Hearing damage occurs when these are injured
    - Those at the near end are easily damaged (high frequency hearing loss)



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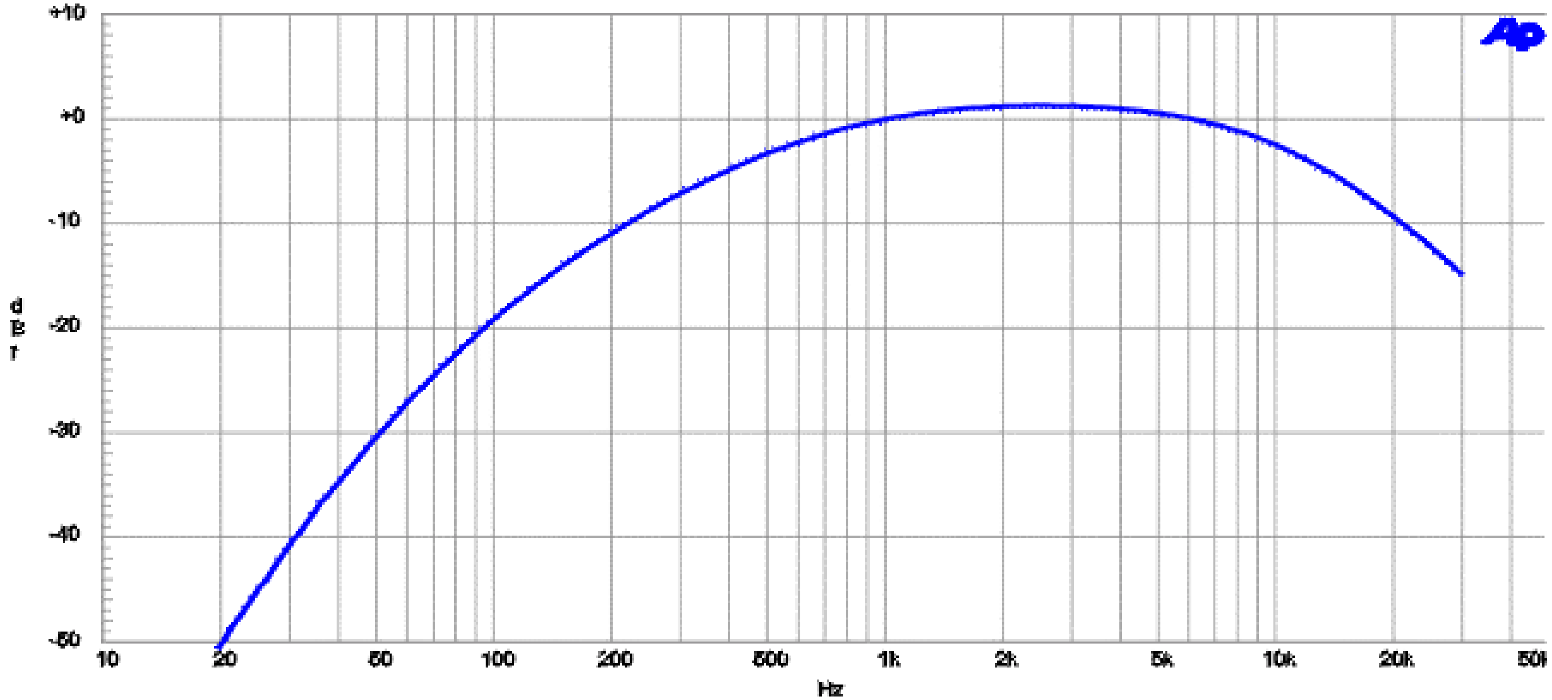
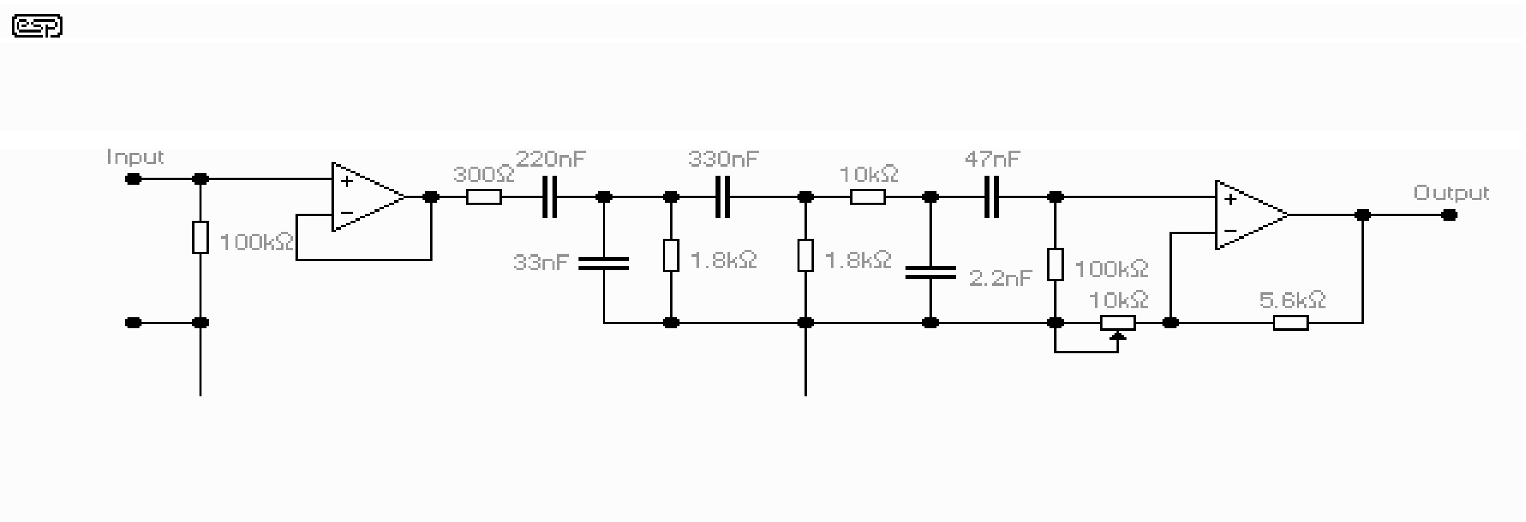
Noise Intensity Levels:
The A- Weighted Filter



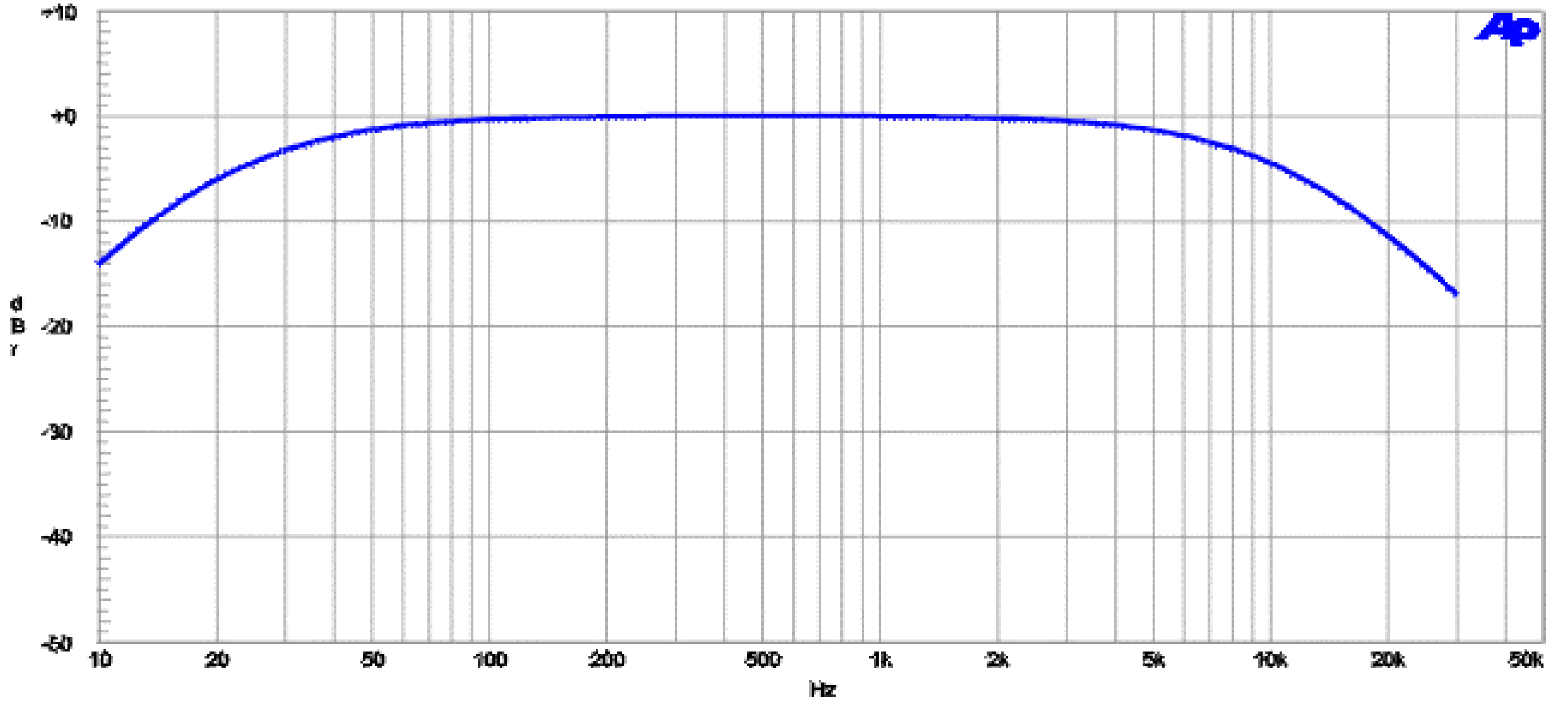
- Corresponds to the sensitivity of the ear at the threshold of hearing; used to specify OSHA safety levels (dBA)

An A-Weighting Filter

- Below is an active filter that will accurately perform A-Weighting for sound measurements
  Thanks to: Rod Elliott at http://sound.westhost.com/project17.htm



Noise Intensity Levels:
The C- Weighted Filter



- Corresponds to the sensitivity of the ear at normal listening levels; used to specify noise in telephone systems (dBC)

Energy Spectral Density (ESD)

$E = \int_{t=-\infty}^{\infty} f^2(t)dt$ is the Energy in a time waveform,

but $f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\varepsilon^{-j\omega t}d\omega$ is the Inverse Fourier Transform

$E = \int_{t=-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\varepsilon^{-j\omega t}d\omega\right]dt$ Substituting for one $f(t)$

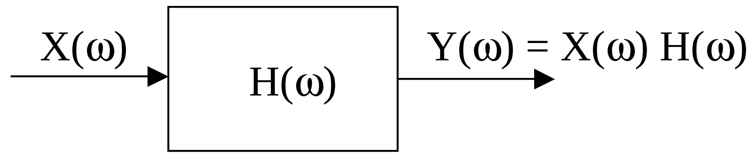
$E = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\left[\int_{t=-\infty}^{\infty} f(t)\varepsilon^{-j\omega t}dt\right]d\omega$ Interchanging the order of integration

but the inner integral is almost the Fourier Transform (except for the "-" $j\omega t$)

$E = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)F(-\omega)d\omega$ but $F(-\omega) = F^*(\omega)$ the complex conjugate

$E = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |F(\omega)|^2 d\omega$ so $|F(\omega)|^2$ is the "Energy Spectral Density"

Energy Spectral Density (ESD) and Linear Systems



$$E_y = \int_{f=-\infty}^{\infty} \left|Y(\omega)\right|^2 df \quad \text{changing to f eliminates the } \frac{1}{2\pi}$$

$$E_y = \int_{f=-\infty}^{\infty} \left|H(\omega) * X(\omega)\right|^2 df \quad \text{so if } H(\omega) \text{ and } X(\omega) \text{ are uncorrelated}$$

$$E_y = \int_{f=-\infty}^{\infty} \left|H(\omega)\right|^2 * \left|X(\omega)\right|^2 df$$

Therefore the ESD of the output of a linear system is obtained by multiplying the ESD of the input by **$|H(\omega)|^2$**

Power Spectral Density (PSD)

- Functions that exist for all time have an infinite energy so we define power as:

$$P = \lim_{t \to \infty} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt \right] \text{ this is energy/time}$$

$$\text{Define } f_T(t) \equiv f(t) \text{ for } -\frac{T}{2} < t < \frac{T}{2} \text{ and zero elsewhere}$$

$$E_T = \int_{-\infty}^{\infty} \left| F_T(\omega) \right|^2 df \quad \text{which does exist and the Power is:}$$

$$P = \lim_{t \to \infty} \left[\frac{1}{T} E_T \right] = \int_{-\infty}^{\infty} \left[\lim_{t \to \infty} \left(\frac{\left| F_T(\omega) \right|^2}{T} \right) \right] df$$

Power Spectral Density (PSD-2)

- As before, the function in the integral is a density. This time it's the PSD

$$PSD = \lim_{t \to \infty} \left(\frac{\left| F_T(\omega) \right|^2}{T} \right)$$

- Both the ESD and PSD functions are real and even functions