EEL 5544 Noise in Linear Systems Lecture 5

STATISTICAL INDEPENDENCE AND CONDITIONAL PROBABILITY

- In the previous lecture, I claimed that if two events are statistically independent, then knowledge of whether one event occurred would not affect the probability of another event occurring
- Let's evaluate that using conditional probability:
- If $P(B) > 0$ and A and B are s.i., then

(When A and B are s.i., knowledge of B does not change the prob. of A (and vice versa))

EXAMPLE: BINARY COMMUNICATION SYSTEM

- Refer to Lecture 4 notes

Design of optimal receiver to minimize error probability

- Let H_{ij} be the hypothesis (i.e., we decide) that A_i was transmitted when B_j is received
- The error probability of our system is then the probability that we decide A_0 when A_1 is transmitted plus the probability that we decided A_1 when A_0 is transmitted:

$$P(E) = P(H_{10} \cap A_0) + P(H_{11} \cap A_0) + P(H_{01} \cap A_1) + P(H_{00} \cap A_1)$$

- Note that at this point our decision rule may be randomized. I.e., it may result in rules of the form: When B_0 is received, decide A_0 with probability η and decide A_1 with probability $1 - \eta$
- Since we only apply hypothesis H_{ij} when B_j is received, we can write

$$\begin{aligned} P(E) &= P(H_{10} \cap A_0 \cap B_0) + P(H_{11} \cap A_0 \cap B_1) \\ &\quad + P(H_{01} \cap A_1 \cap B_1) + P(H_{00} \cap A_1 \cap B_0) \end{aligned}$$

- We apply the chain rule to write this as

$$\begin{aligned} P(E) &= P(H_{10}|A_0 \cap B_0)P(A_0 \cap B_0) + P(H_{11}|A_0 \cap B_1)P(A_0 \cap B_1) \\ &\quad + P(H_{01}|A_1 \cap B_1)P(\cap A_1 \cap B_1) + P(H_{00}|A_1 \cap B_0)P(A_1 \cap B_0) \end{aligned}$$

- The receiver can only decide H_{ij} based on B_j; it has no direct knowledge of A_k. So $P(H_{ij}|A_k \cap B_j) = P(H_{ij}|B_j)$ for all i, j, k
- Furthermore, let $P(H_{00}|B_0) = q_0$ and $P(H_{11}|B_1) = q_1$.
- Then

$$\begin{aligned} P(E) &= (1-q_0)P(A_0 \cap B_0) + q_1 P(A_0 \cap B_1) \\ &\quad (1-q_1)P(A_1 \cap B_1) + q_0 P(A_1 \cap B_0) \end{aligned}$$

- The choices of q_0 and q_1 can be made independently, so $P(E)$ is minimized by finding q_0 and q_1 that minimize

$$\begin{aligned} &(1-q_0)P(A_0 \cap B_0) + q_0 P(A_1 \cap B_0) \text{ and} \\ &q_1 P(A_0 \cap B_1) + (1-q_1)P(A_1 \cap B_1) \end{aligned}$$

- We can further expand these using the chain rule as

$$\begin{aligned} &(1-q_0)P(A_0|B_0)P(B_0) + q_0 P(A_1|B_0)P(B_0) \text{ and} \\ &q_1 P(A_0|B_1)P(B_1) + (1-q_1)P(A_1|B_1)P(B_1) \end{aligned}$$

- $P(B_0)$ and $P(B_1)$ are constants that do not depend on our decision rule, so finally we wish to find q_0 and q_1 to minimize

$$(1-q_0)P(A_0|B_0) + q_0 P(A_1|B_0) \text{ and} \tag{1}$$

$$q_1 P(A_0|B_1) + (1-q_1)P(A_1|B_1) \tag{2}$$

- Both of these are linear functions of q_o and q_1, so the minimizing values must be at the endpoints.
  I.e., either $q_i = 0$ or $q_i = 1$

  $\Rightarrow$ Our decision rule is not randomized

- By inspection (1) is minimized by setting $q_0 = 1$ if $P(A_0|B_0) > P(A_1|B_0)$ and setting $q_0 = 0$ otherwise
- Similarly, (2) is minimized by setting $q_1 = 1$ if $P(A_1|B_1) > P(A_0|B_1$ and setting $q = 0$ otherwise
- These rules can be summarized as follows:

Given that B_j is received, decide A_i was transmitted if A_i maximizes $P(A_i|B_j)$

- In words: Choose the signal that was most probably transmitted given what you received <u>and</u> the *a priori* probabilities of what was transmitted

- The probabilities $P(A_i|B_j)$ are called *a posteriori* probabilities, meaning that they are the probabilities after the output of the system is measured
- This decision rule we have just derived is the **maximum *a posteriori* (MAP) decision rule**
- Problem: we don't know $P(A_i|B_j)$
- Need to express $P(A_i|B_j)$ in terms of $P(A_i)$, $P(B_j|A_i)$
- **General technique:**

  If $\{A_i\}$ is a partition of S and we know $P(B_j|A_i)$ and $P(A_i)$, then

  where the last step follows from the Law of Total Probability.

  The expression in the box is _________________________.

  Example Calculate the *a posteriori* probabilities and determine the MAP decision rules for the binary communication system with $P(A_0) = 2/5$, $p_{01} = 1/8$, and $p_{10} = 1/6$

  Example finished on separate sheets.

- Often, we may not know the *a posteriori* probabilities $P(A_0)$ and $P(A_1)$.
- In this case, we typically treat the input symbols as equally likely, $P(A_0) = P(A_1) = 0.5$
- Under this assumption, $P(A_0|B_0) = cP(B_0|A_0)$ and $P(A_1|B_0) = cP(A_1|B_0)$, so the decision rule becomes:

Given that B_j is received, decide A_i was transmitted if A_i maximizes $P(B_j|A_i)$

- This is known as the **maximum likelihood (ML) decision rule**