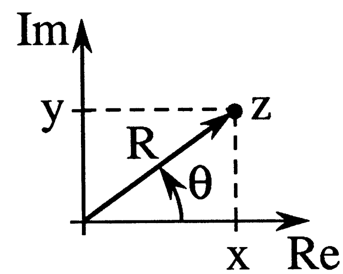
1.1.5 COMPLEX VARIABLES

Definition



Cartesian coordinates $z = (x, y)$

Polar coordinates $z = R\underline{/\theta}$

$R = \sqrt{x^2 + y^2}$ $x = R \cos \theta$

$\theta = \arctan(\frac{y}{x})$ $y = R \sin \theta$

Alternate representation

define $j = (0,1) = 1 \angle \pi 2$

then $z = x + jy$

Additional notation

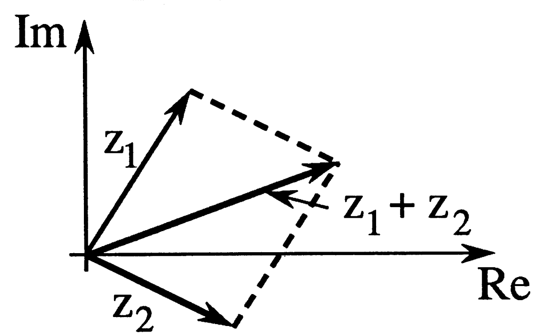
$Re\{z\} = x$

$Im\{z\} = y$

$|z| = R$

Algebraic Operations

Addition (Cartesian coordinates)



$$z_1 = x_1 + jy_1$$

$$z_2 = x_2 + jy_2$$

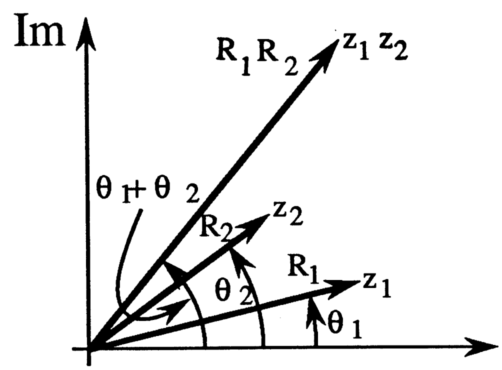
$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication (Polar coordinates)

$z_1 = R_1 \ \underline{/\theta_1}$

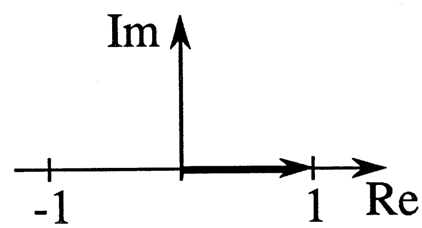
$z_2 = R_2 \ \underline{/\theta_2}$

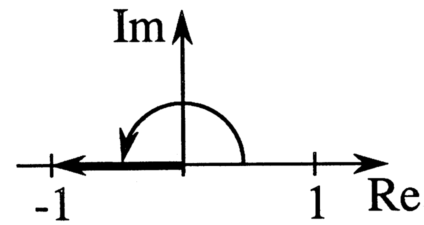
$z_1 z_2 = R_1 R_2 \ \underline{/\theta_1 + \theta_2}$



Special Examples

1. $1 = (1,0) = 1\ \underline{/0^\circ}$



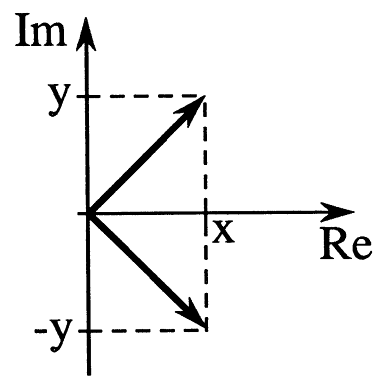


2. $-1 = (-1,0) = 1\ \underline{/180^\circ}$

3. $j^2 = (1\ \underline{/90^\circ})^2 = 1\ \underline{/180^\circ} = -1$

Complex Conjugate

$$z^* = x - jy$$
$$= R\,\underline{/-\theta}$$



Some useful identities:

$$\mathrm{Re}\{z\} = \frac{1}{2}\,[z + z^*]$$

$$\mathrm{Im}\{z\} = \frac{1}{j2}\,[z - z^*]$$

$$|z| = \sqrt{z\,z^*}$$

Complex Exponential

Taylor series for exponential function of real variable x

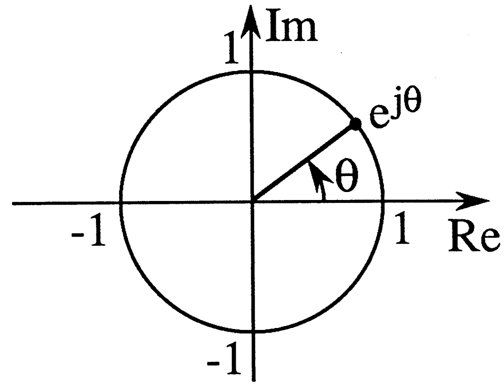
$$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Replace x by jθ:

$$\begin{aligned} e^{i\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + j(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots) \\ &= \cos\theta + j \sin \theta \end{aligned}$$

$$|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\underline{/e^{j\theta}} = \arctan\left(\frac{\sin\theta}{\cos\theta}\right) = \theta$$



Alternate form for polar coordinate representation of complex number:

$$z = R\underline{/\theta} = Re^{j\theta}$$

Multiplication of two complex numbers can be done via usual rules for multiplication of exponentials:

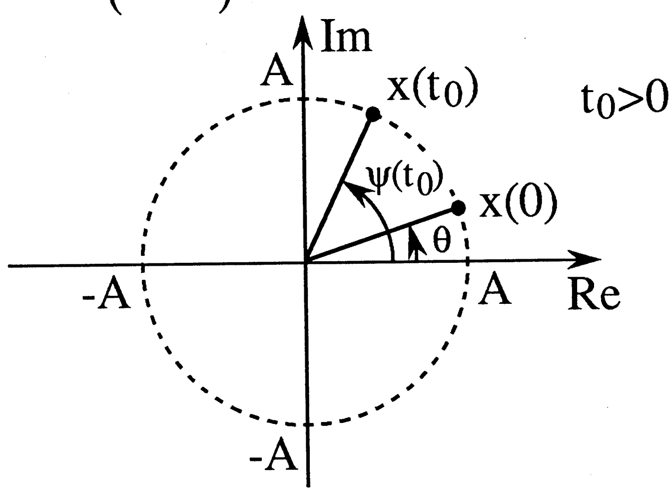
$$z_1 z_2 = (R_1 e^{j\theta_1})(R_2 e^{j\theta_2}) = R_1 R_2 e^{j(\theta_1+\theta_2)}$$

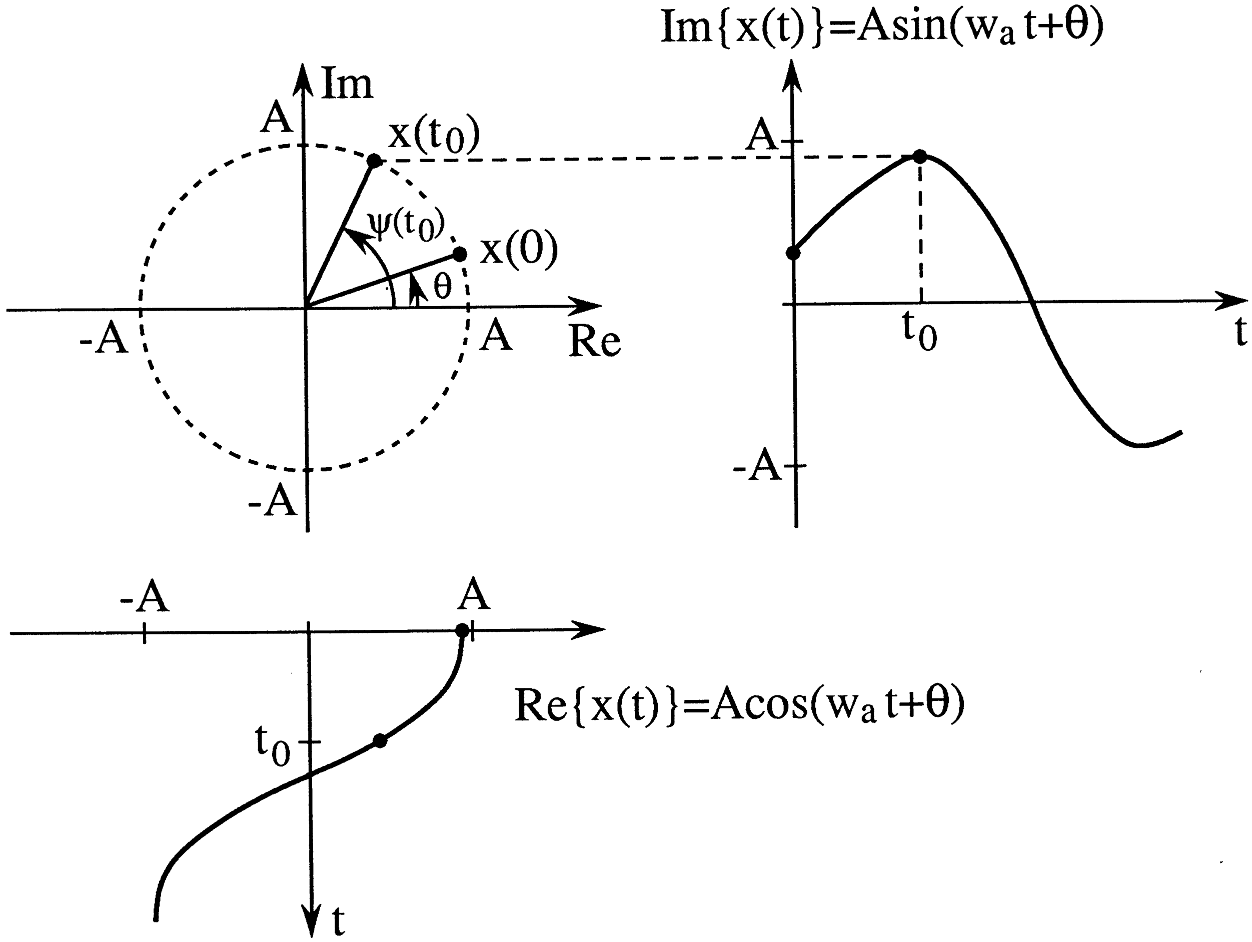
Complex Exponential Signal

CT $x(t) = Ae^{j\psi(t)}$

$\psi(t) = \omega_a t + \theta$ instantaneous phase

$\frac{d\psi(t)}{dt} = \omega_a \left(\frac{\text{rad.}}{\text{sec.}}\right)$ instantaneous phasor velocity (frequency)



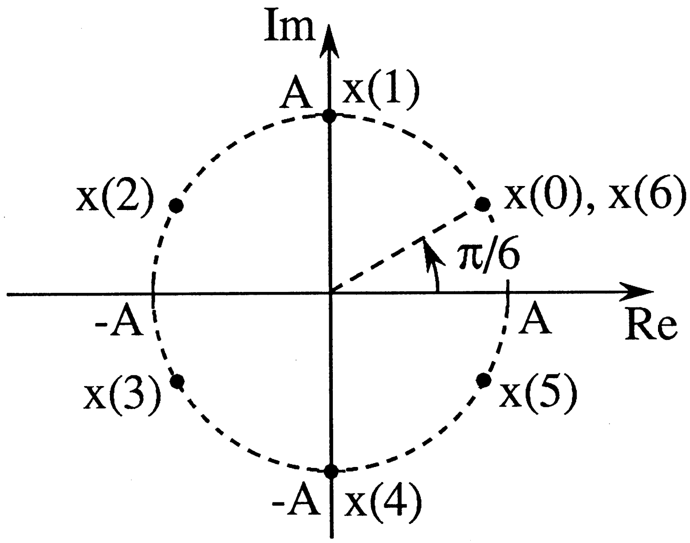

Im{x(t)}=Asin(wa t+θ)
Im
A
x(t0)
ψ(t0)
x(0)
θ
-A
A
Re
-A
A
-A
t0
t
-A
A
Re{x(t)}=Acos(wa t+θ)
t0
t

DT $\quad x(n) = Ae^{j(\omega_d n + \theta)}$ $\quad \omega_d$ is phasor velocity in radians/sample

Example:

$\omega_d = \pi/3$

$\theta = \pi/6$



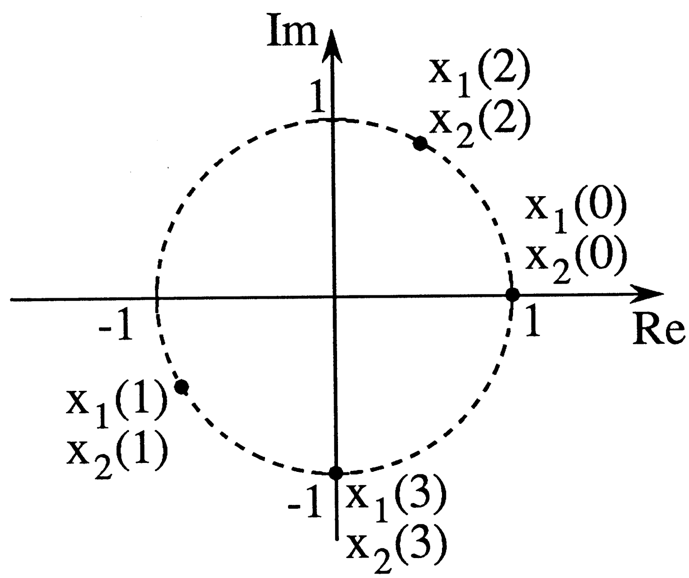
Aliasing

- The ability of one frequency to mimic another.
- For CT case, $e^{j\omega_1 t} = e^{j\omega_2 t}$ for all t $\Leftrightarrow \omega_1 = \omega_2$.
- In DT, $e^{j\omega_1 n} = e^{j\omega_2 n}$ if $\omega_2 = \omega_1 + 2\pi k$.
- Let's examine a specific case of this relation.

Consider

$$\omega_1 = 7\pi/6 \qquad \omega_2 = -\,5\pi/6$$

$$x_1(n) = e^{j\omega_1 n} \qquad x_2(n) = e^{j\omega_2 n}$$



In general, if $\omega_1 = \pi + \Delta$ and $\omega_2 = -(\pi - \Delta)$, then

$$
\begin{aligned}
x_1(n) &= e^{j\omega_1 n} \\
&= e^{j(\pi + \Delta)n} \\
&= e^{j[(\pi + \Delta)n - 2\pi n]} \\
&= e^{j(-\pi + \Delta)n} \\
&= e^{-j(\pi - \Delta)n} \\
&= x_2(n)
\end{aligned}
$$

We will see this relation again when we analyze the effect of sampling.