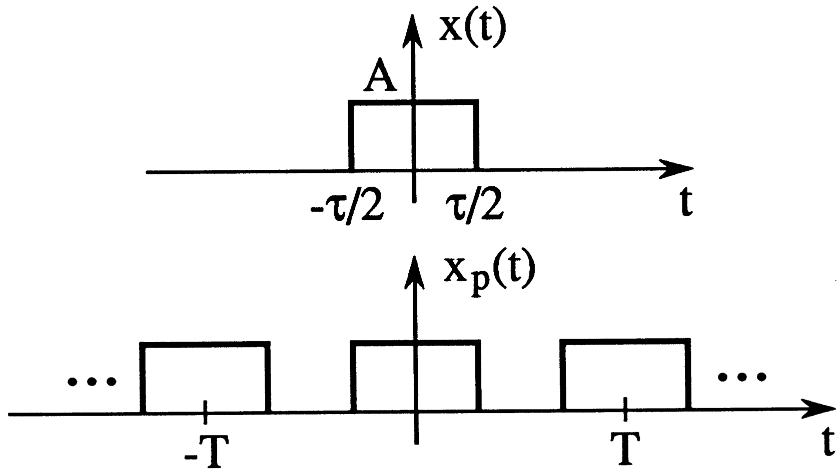
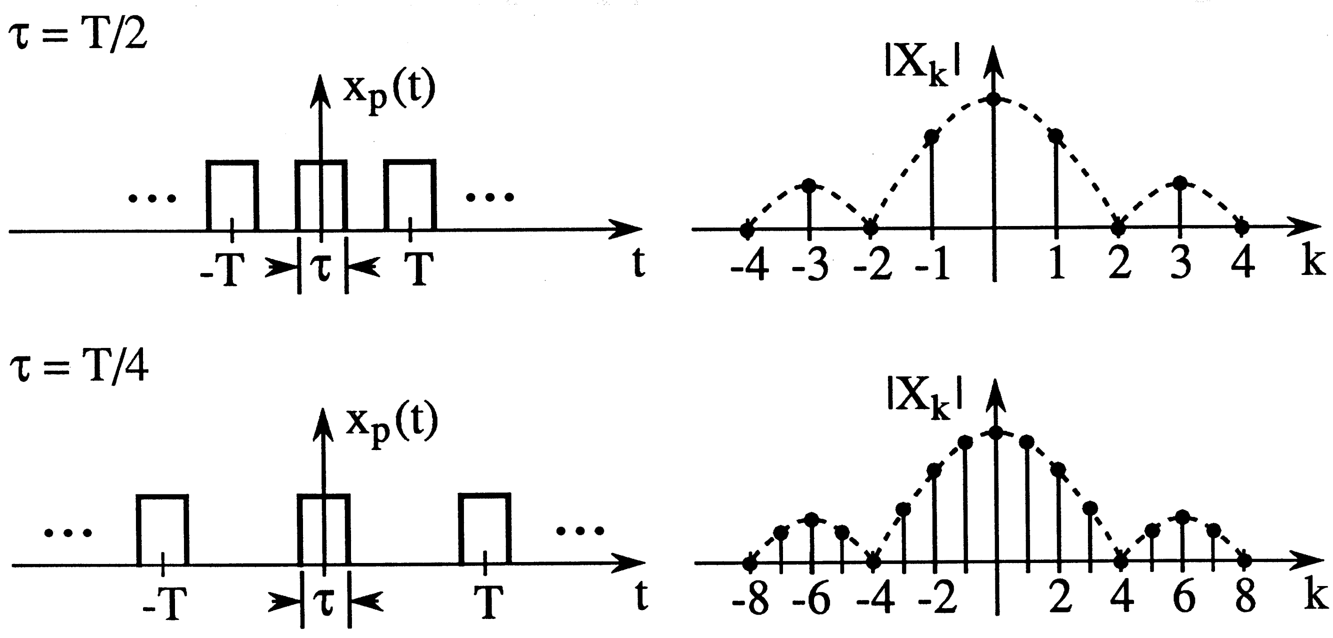
1.3.2 CONTINUOUS-TIME FOURIER TRANSFORM (CTFT)

Spectral representation for *aperiodic* CT signals

Consider a fixed signal x(t) and let $x_p(t) = \text{rep}_T[x(t)]$



What happens to Fourier series as T increases?



Fourier Coefficients

$$X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T} dt$$

Let $T \rightarrow \infty$

$k/T \rightarrow f$

$X_k \rightarrow X(f)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Fourier Series Expansion

$$x_p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_k \, e^{j2\pi kt/T}$$

Let $T \rightarrow \infty$

$$x_p(t) \rightarrow x(t)$$

$$k/T \rightarrow f \qquad X_k \rightarrow X(f)$$

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty} df$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \, e^{j2\pi ft} df$$

Fourier Transform Pair

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \qquad (1)$$

Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \qquad (2)$$

Sufficient Conditions for Existence of CTFT

1. x(t) has finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

2. x(t) is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

and it satisfies the Dirichlet conditions

Transform Relations

1. linearity

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$$

2. scaling and shifting

$$x\left(\frac{t - t_0}{a}\right) \overset{\text{CTFT}}{\leftrightarrow} |a| \; X(af) \; e^{-j2\pi f t_0}$$

3. modulation

$$x(t) \; e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} X(f - f_0)$$

4. reciprocity

$$X(t) \overset{\text{CTFT}}{\leftrightarrow} x(-f)$$

5. Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

6. Initial value

$$\int_{-\infty}^{\infty} x(t)dt = X(0)$$

Comments

1. Reflection is a special case of scaling and shifting with a $= -1$ and $t_0 = 0$, *i.e.*

$$x(-t) \overset{\text{CTFT}}{\leftrightarrow} X(-f)$$

2. The scaling relation exhibits reciprocal spreading.

3. Uniqueness of the CTFT follows from Parseval's relation.

CTFT for Real Signals

If x(t) is real, $X(f) = [X(-f)]^*$

$$\Rightarrow |X(f)| = |X(-f)| \quad \text{and} \quad \underline{/X(f)} = -\underline{/X(-f)}$$

In this case, the inverse transform may be written as

$$x(t) = 2\int_0^\infty |X(f)| \cos[2\pi ft + \underline{/X(f)}]df$$

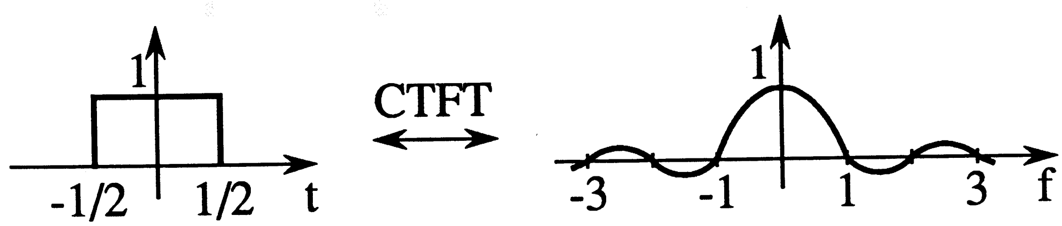
Additional symmetry relations:

x(t) is real and even $\Leftrightarrow$ X(f) is real and even

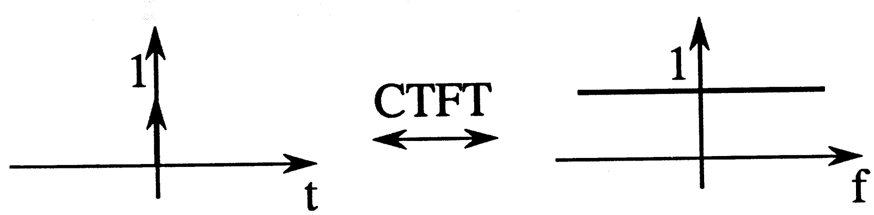
x(t) is real and odd $\Leftrightarrow$ X(f) is imaginary and odd

Important Transform Pairs

1. $\text{rect}(t) \overset{\text{CTFT}}{\leftrightarrow} \text{sinc}(f)$



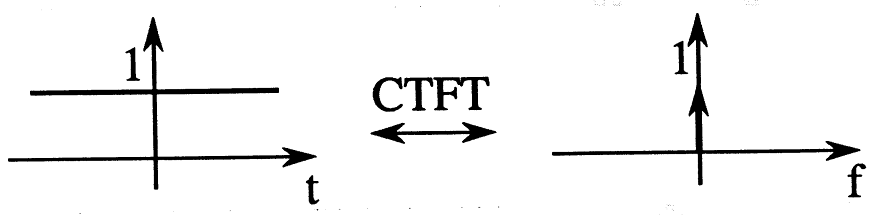
2. $\delta(t) \overset{\text{CTFT}}{\leftrightarrow} 1$ (by sifting property)



Proof:

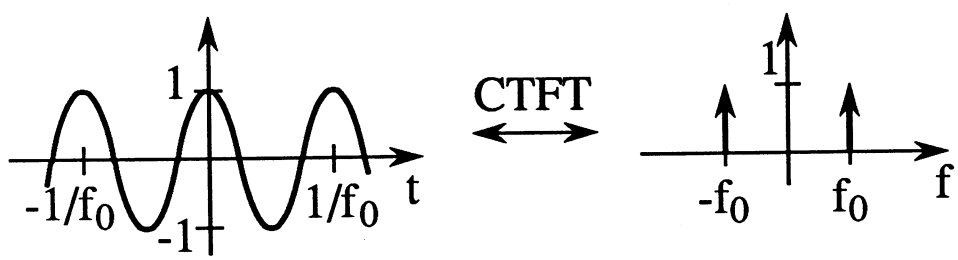
$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft} dt = 1$$

3. $1 \overset{\text{CTFT}}{\leftrightarrow} \delta(f)$ (by reciprocity)



4. $e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} \delta(f - f_0)$ (by modulation property)

5. $\cos(2\pi f_0 t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$



Generalized Fourier Transform

Note that $\delta(t)$ is absolutely integrable but not square integrable.

Consider $\delta_\Delta(t) = \frac{1}{\Delta}\,\mathrm{rect}\left(\frac{t}{\Delta}\right)$

$$\int_{-\infty}^{\infty} |\,\delta_\Delta(t)\,|\,dt = 1$$

$$\int_{-\infty}^{\infty} |\,\delta_\Delta(t)\,|^2 dt = \frac{1}{\Delta}$$

$$\therefore \lim_{\Delta \to 0} \int_{-\infty}^{\infty} |\,\delta_\Delta(t)\,|^2 dt = \infty$$

The function $x(t) \equiv 1$ is neither absolutely nor square integrable; and the integral

$$\int_{-\infty}^{\infty} 1\, e^{-j2\pi ft}\, dt$$

is undefined.

Even when neither condition for existence of the CTFT is satisfied, we may still be able to define a Fourier transform through a limiting process.

Let $x_n(t)$, $n = 0, 1, 2, \ldots$ denote a sequence of functions each of which has a valid CTFT $X_n(f)$

Suppose that $\lim_{n \to \infty} x_n(t) = x(t)$, a function that does not have a valid transform

If $X(f) = \lim_{n \to \infty} X_n(f)$ exists, we call it the *generalized* Fourier transform of x(t) *i.e.*

$$x_0(t) \overset{\text{CTFT}}{\leftrightarrow} X_0(f)$$

$$x_1(t) \overset{\text{CTFT}}{\leftrightarrow} X_1(f)$$

$$x_2(t) \overset{\text{CTFT}}{\leftrightarrow} X_2(f)$$

$$\vdots \qquad\qquad \vdots$$

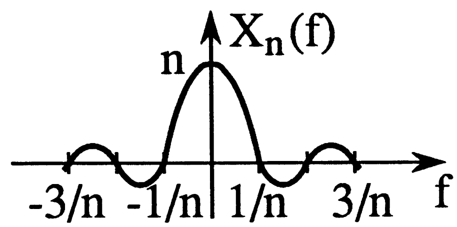
$$x(t) \overset{\text{GCTFT}}{\leftrightarrow} X(f)$$

Example

Let $x_n(t) = \text{rect}(t/n)$

$$X_n(f) = n\ \text{sinc}(nf) \qquad \text{(by scaling)}$$

$$\lim_{n\to\infty} x_n(t) = 1$$



What is $\lim_{n\to\infty} X_n(f)$?

$$X_n(f)\to 0, \qquad f \neq 0$$

$$X_n(0)\to\infty$$

What is $\int_{-\infty}^{\infty} X_n(f)df$?

By the initial value relation

$$\int_{-\infty}^{\infty} X_n(f)df = x_n(0) = 1$$

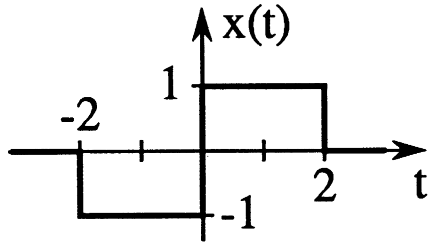
$$\therefore \quad \lim_{n \to \infty} X_n(f) = \delta(f)$$

and we have

$$1 \overset{\text{GCTFT}}{\leftrightarrow} \delta(f)$$

Efficient Calculation of Fourier Transforms

Suppose we wish to determine the CTFT of the following signal



Brute force approach:

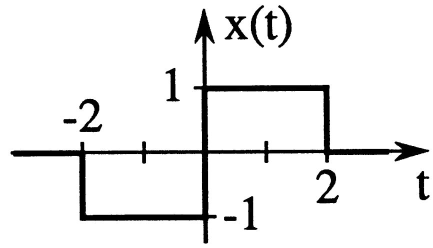
1. evaluate transform integral directly

$$X(f) = \int_{-2}^{0} (-1)\, e^{-j2\pi ft} dt + \int_{0}^{2} (1)\, e^{-j2\pi ft} dt$$

2. collect terms, simplify, etc...

Faster approach:

1. write x(t) in terms of functions whose transforms are known

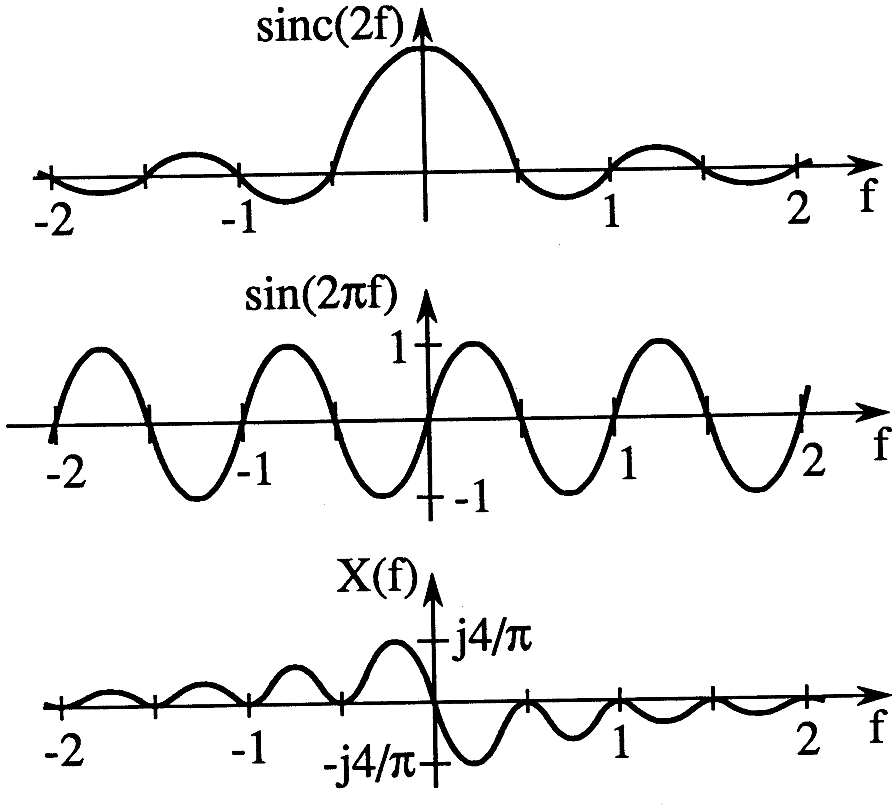


$$x(t) = -\,\text{rect}\left(\frac{t+1}{2}\right) + \text{rect}\left(\frac{t-1}{2}\right)$$

2. Use transform relations to determine X(f)

$$X(f) = 2\ \text{sinc}(2f)\ [e^{-j2\pi f} - e^{j2\pi f}]$$

$$X(f) = -\,j4\ \mathrm{sinc}(2f)\ \sin(2\pi f)$$



Comments

1. $A_x = 0$ and $X(0) = 0$

2. $x(t)$ is real and odd and $X(f)$ is imaginary and odd

CTFT and CT LTI Systems

The key factor that made it possible to express the response y(t) of an LTI system to an *arbitrary* input x(t) in terms of the impulse response h(t) was the fact that we could write x(t) as a superposition of impulses $\delta(t)$.

We can similarly express x(t) as a superposition of complex exponential signals:

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df$$

Let $\tilde{H}(f)$ denote the frequency response of the system, *i.e.* for a fixed frequency f

$$e^{j2\pi ft} \rightarrow \boxed{\text{System}} \rightarrow \tilde{H}(f)\, e^{j2\pi ft}$$

then by homogeneity

$$X(f)\, e^{j2\pi ft} \rightarrow \boxed{\text{System}} \rightarrow \tilde{H}(f)X(f)\, e^{j2\pi ft}$$

and by superposition

$$\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, dt \rightarrow \boxed{\text{System}} \rightarrow \int_{-\infty}^{\infty} \tilde{H}(f)X(f)\, e^{j2\pi ft}\, dt$$

Thus, the response to x(t) is

$$y(t) = \int_{-\infty}^{\infty} \tilde{H}(f)X(f)\, e^{j2\pi ft} dt$$

But also,

$$y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{j2\pi ft} dt$$

$$\therefore \quad Y(f) = \tilde{H}(f)\, X(f) \tag{1}$$

We also know that

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau$$

What is relation between h(t) and $\tilde{H}$(f)?

$$\begin{array}{llcll} \text{Let } x(t) & = \delta(t) & \Rightarrow & y(t) & = h(t) \\ \text{then } X(f) & = 1 & \text{and} & Y(f) & = H(f) \end{array}$$

From Eq. (1), conclude that $\tilde{H}(f) = H(f)$

Since the frequency response is the CTFT of the impulse response, we will drop the tilde.

Summarizing, we have two equivalent characterizations for CT LTI systems

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)\, x(\tau) d\tau$$

$$Y(f) = H(f)\, X(f)$$

Convolution Theorem

Since x(t) and h(t) are arbitrary signals, we also have the following Fourier transform relation

$$\int x_1(\tau)\, x_2(t-\tau) d\tau \overset{\text{CTFT}}{\leftrightarrow} X_1(f)\, X_2(f)$$

or

$$x_1(t) * x_2(t) \overset{\text{CTFT}}{\leftrightarrow} X_1(f)\, X_2(f)$$

Product Theorem

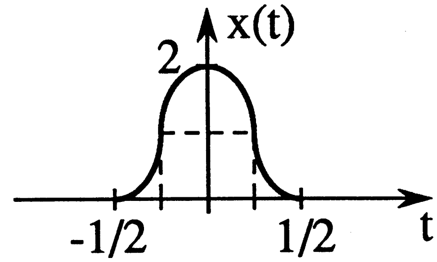
By reciprocity, we also have the following result

$$x_1(t)\, x_2(t) \overset{\text{CTFT}}{\leftrightarrow} X_1(f) * X_2(f)$$

This can be very useful for calculating transforms of certain functions.

Example

$$x(t) = \begin{cases} \frac{1}{2}\,[1 + \cos(2\pi t)] \;, & |t| \le 1/2 \\ 0 \;, & |t| > 1/2 \end{cases}$$



Find X(f)

$$x(t) = \frac{1}{2}\,[1 + \cos(2\pi t)]\ \mathrm{rect}(t)$$

$$\therefore\ X(f) = \frac{1}{2}\left\{\delta(f) + \frac{1}{2}[\delta(f-1) + \delta(f+1)]\right\} * \mathrm{sinc}(f)$$

Since convolution obeys linearity, we can write this as

$$X(f) = \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\}$$

All three convolutions here are of the same general form.

Identity

For any signal w(t),

$$w(t) * \delta(t - t_0) = w(t - t_0)$$

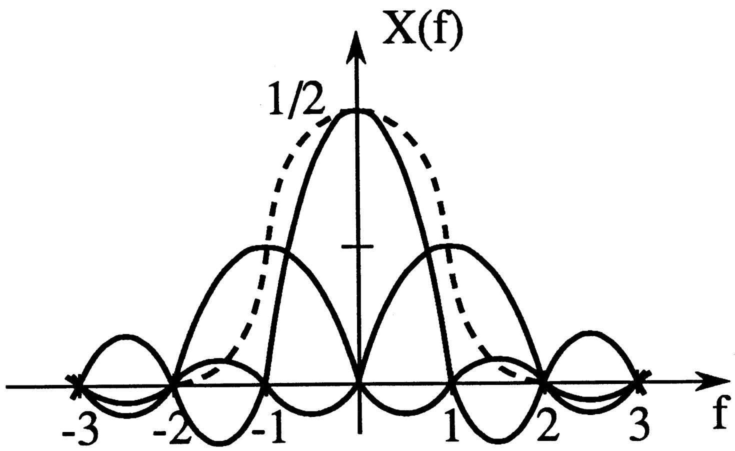
Proof:

$$\begin{aligned} w(t) * \delta(t - t_0) &= \int w(\tau)\, \delta(t - \tau - t_0) d\tau \\ &= w(t - t_0) \quad \text{(by sifting property)} \end{aligned}$$

Using the identity,

$$X(f) = \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\}$$

$$= \frac{1}{2}\left\{\text{sinc}(f) + \frac{1}{2}\,[\text{sinc}(f-1) + \text{sinc}(f+1)]\right\}$$



Fourier Transform of Periodic Signals

- We previously developed the Fourier series as a spectral representation for periodic CT signals.

- Such signals are neither square integrable nor absolutely integrable, and hence do not satisfy the conditions for existence of the CTFT.

- However, by applying the concept of the generalized Fourier transform, we can obtain a Fourier transform for periodic signals.

- This allows us to treat the spectral analysis of *all* CT signals within a single framework.

- We can also obtain the same result directly from the Fourier series.

Let $x_0(t)$ denote one period of a signal that is periodic with period T, *i.e.* $x_0(t) = 0$, $|t| > T/2$.

Define $x(t) = \text{rep}_T[x_0(t)]$

The Fourier series representation for x(t) is

$$x(t) = \frac{1}{T} \sum_k X_k \, e^{j2\pi kt/T}$$

Taking the CTFT directly, we obtain

$$X(f) = \mathscr{F}\left\{\frac{1}{T}\sum_{k} X_k\ e^{j2\pi kt/T}\right\}$$

$$= \frac{1}{T}\sum_{k} X_k\ \mathscr{F}\left\{e^{j2\pi kt/T}\right\} \quad \text{(by linearity)}$$

$$= \frac{1}{T}\sum_{k} X_k\ \delta(f - k/T)$$

Also

$$X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T} dt$$

$$= \int_{-\infty}^{\infty} x_0(t)\, e^{-j2\pi kt/T} dt$$

$$= X_0(k/T)$$

Thus

$$X(f) = \frac{1}{T} \sum_k X_0(k/T)\, \delta(f - k/T)$$

$$= \frac{1}{T} \text{comb}\, \frac{1}{T}\, [X_0(f)]$$

Dropping the subscript 0, we may state this result in the form of a transform relation:

$$\mathrm{rep}_T[x(t)] \overset{\mathrm{CTFT}}{\leftrightarrow} \frac{1}{T}\,\mathrm{comb}_{\frac{1}{T}}\,[X(f)]$$

For our derivation, we required that $x(t) = 0$, $|t| > T/2$. However, when the generalized transform is used to derive the result, this restriction is not needed.

Example