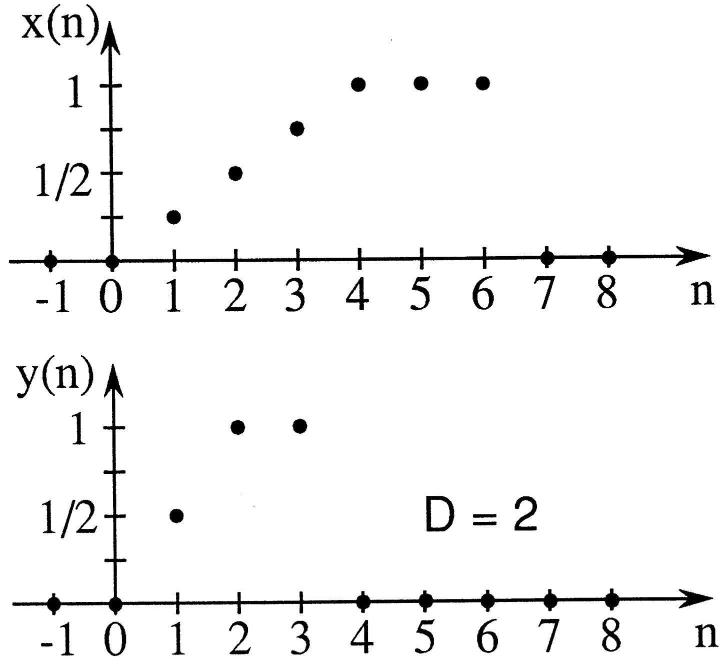
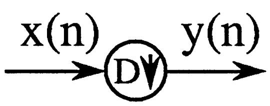
1.4.3 SAMPLING RATE CONVERSION (SCALING IN DT)

Recall definitions for DT scaling



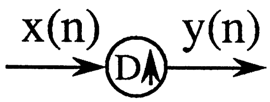
Downsampling

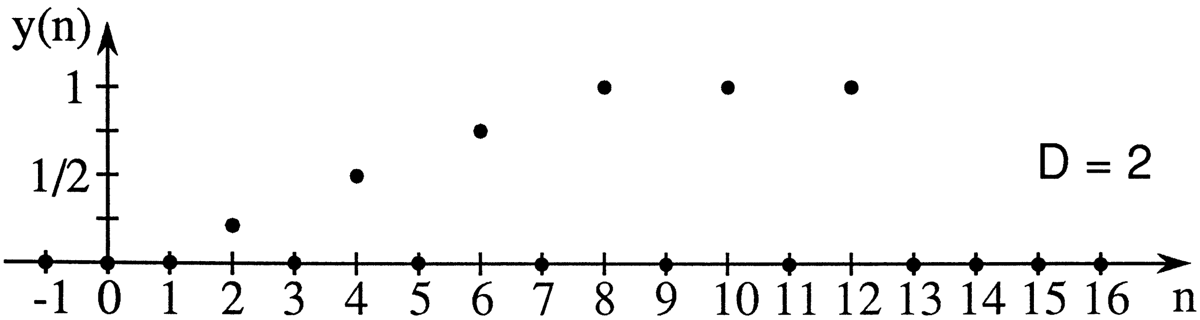
$y(n) = x(Dn)$



Upsampling

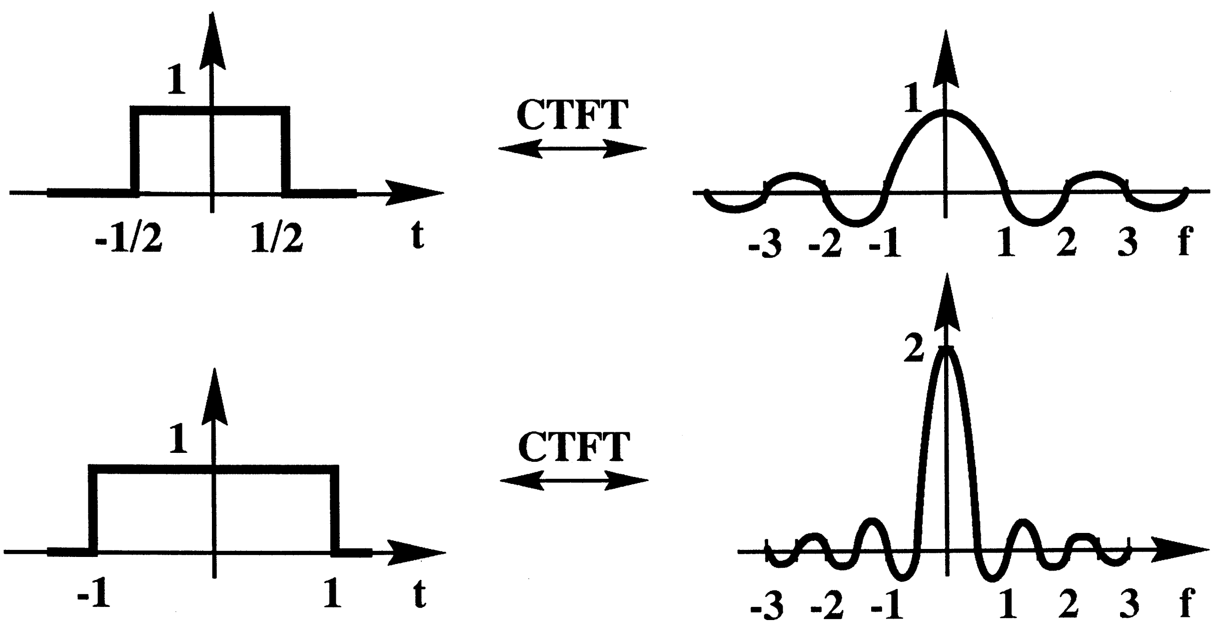
$$y(n) = \begin{cases} x(n/D) & , \quad \text{if } n/D \text{ is an integer} \\ 0 & , \quad \text{else} \end{cases}$$





In CT, we have a very simple transform relation for scaling:

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$



What is transform relation for DT?

- As in definitions for downsampling and upsampling, we must treat these cases separately.
- Relations will combine elements from both *scaling* and *sampling* transform relations for CT.

DOWNSAMPLING

$$y(n) = x(Dn)$$

$$Y(e^{j\omega}) = \sum_{n} x(Dn)\, e^{-j\omega n}$$

$$\text{let} \quad m = Dn \;\Rightarrow n = m/D$$

$$Y(e^{j\omega}) = \sum_{m} x(m)\, e^{-j\omega m/D}$$

(m/D is an integer)

To remove restriction on m, define a sequence

$$s_D(m) = \begin{cases} 1\,, & m/D \text{ is an integer} \\ 0\,, & \text{else} \end{cases}$$

then

$$Y(e^{j\omega}) = \sum_m s_D(m)\, x(m)\, e^{-j\omega m/D}$$

Alternate expression for $s_D(m)$

$D = 2$

$$s_2(m) = \frac{1}{2}[1 + (-1)^m]$$

$$= \frac{1}{2}[1 + e^{-j2\pi m/2}]$$

$$s_2(0) = 1$$

$$s_2(1) = 0$$

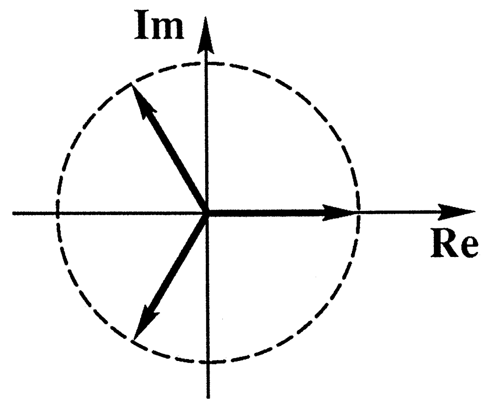
$$s_2(m + 2k) = s(m)$$

D = 3

$$s_3(m) = \frac{1}{3}[1 + e^{-j2\pi m/3} + e^{-j2\pi(2)m/3}]$$

$$s_3(0) = \frac{1}{3}[1 + 1 + 1] = 1$$

$$s_3(1) = \frac{1}{3}[1 + e^{-j2\pi/3} + e^{-j2\pi(2)/3}] = 0$$



$$s_3(2) = \frac{1}{3}[1 + e^{-j2\pi(2)/3} + e^{-j2\pi(4)/3}] = 0$$

$$s_3(m + 3k) = s_3(m)$$

In general,

$$s_D(m) = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi km/D}$$

$$= \frac{1}{D}\,\frac{1 - e^{-j2\pi m}}{1 - e^{-j2\pi m/D}}$$

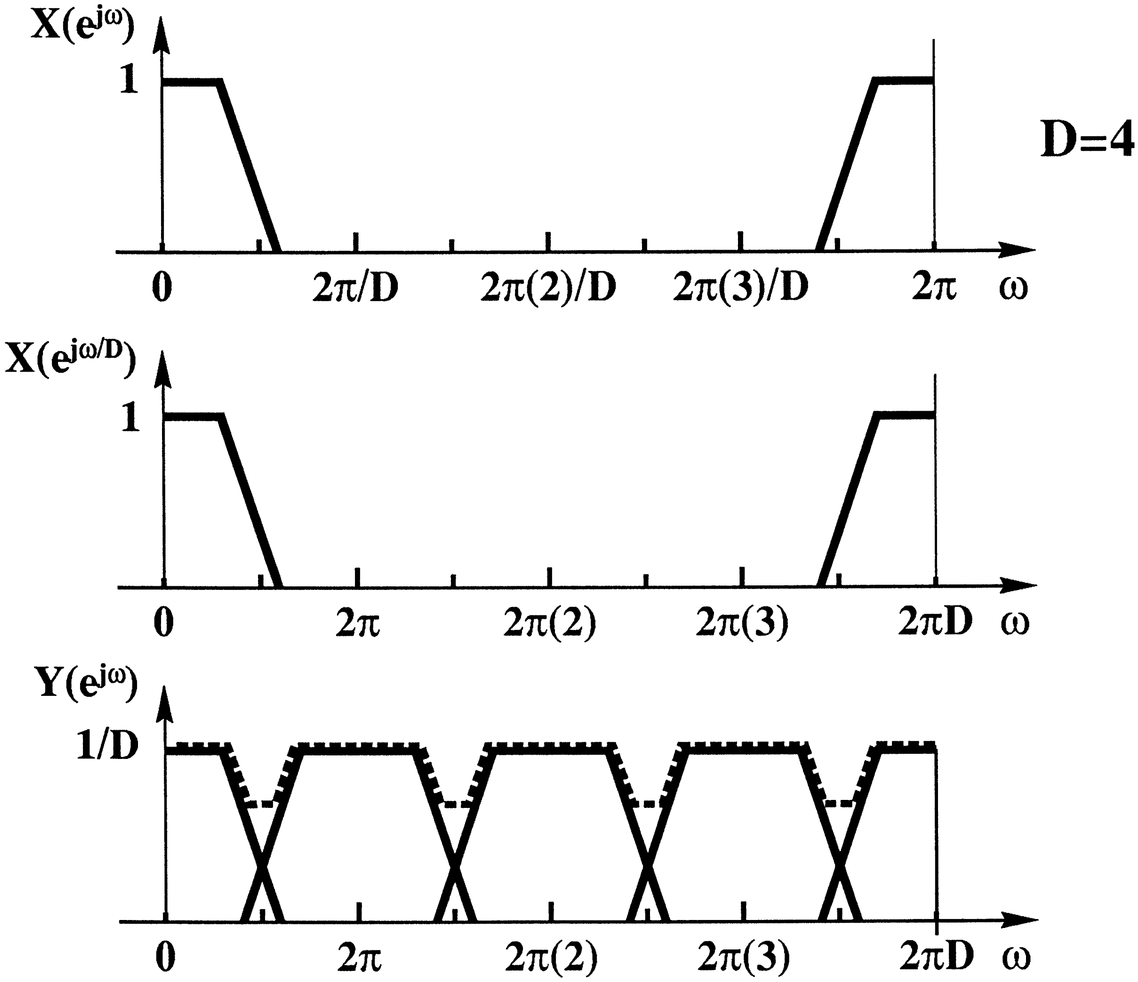
$$= \begin{cases} 1\,, & m/D \text{ is an integer} \\ 0\,, & \text{else} \end{cases}$$

$$Y(e^{j\omega}) = \sum_{m} s_D(m)\, x(m)\, e^{-j\omega m/D}$$

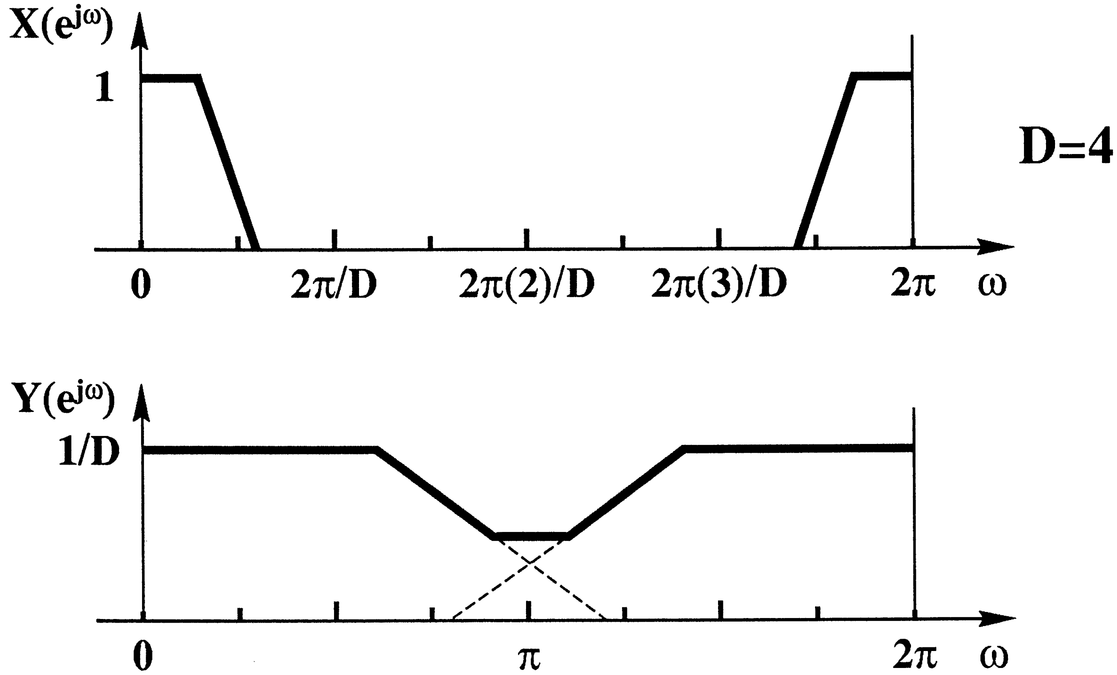
$$= \sum_{m} \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi km/D} x(m)\, e^{-j\omega m/D}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \sum_{m} x(m)\, e^{-j[(\omega+2\pi k)/D]m}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{j(\omega+2\pi k)/D})$$



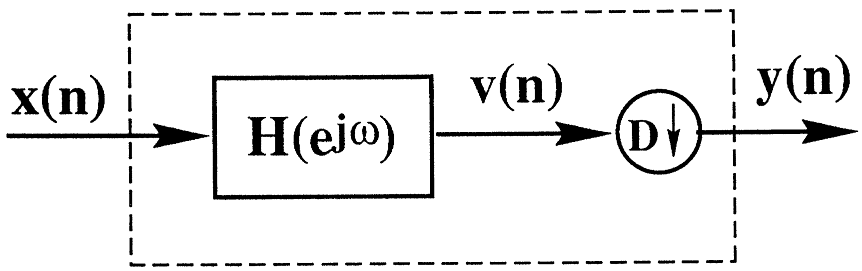
X(e^jω)
1
D=4
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
X(e^jω/D)
1
0
2π
2π(2)
2π(3)
2πD
ω
Y(e^jω)
1/D
0
2π
2π(2)
2π(3)
2πD
ω

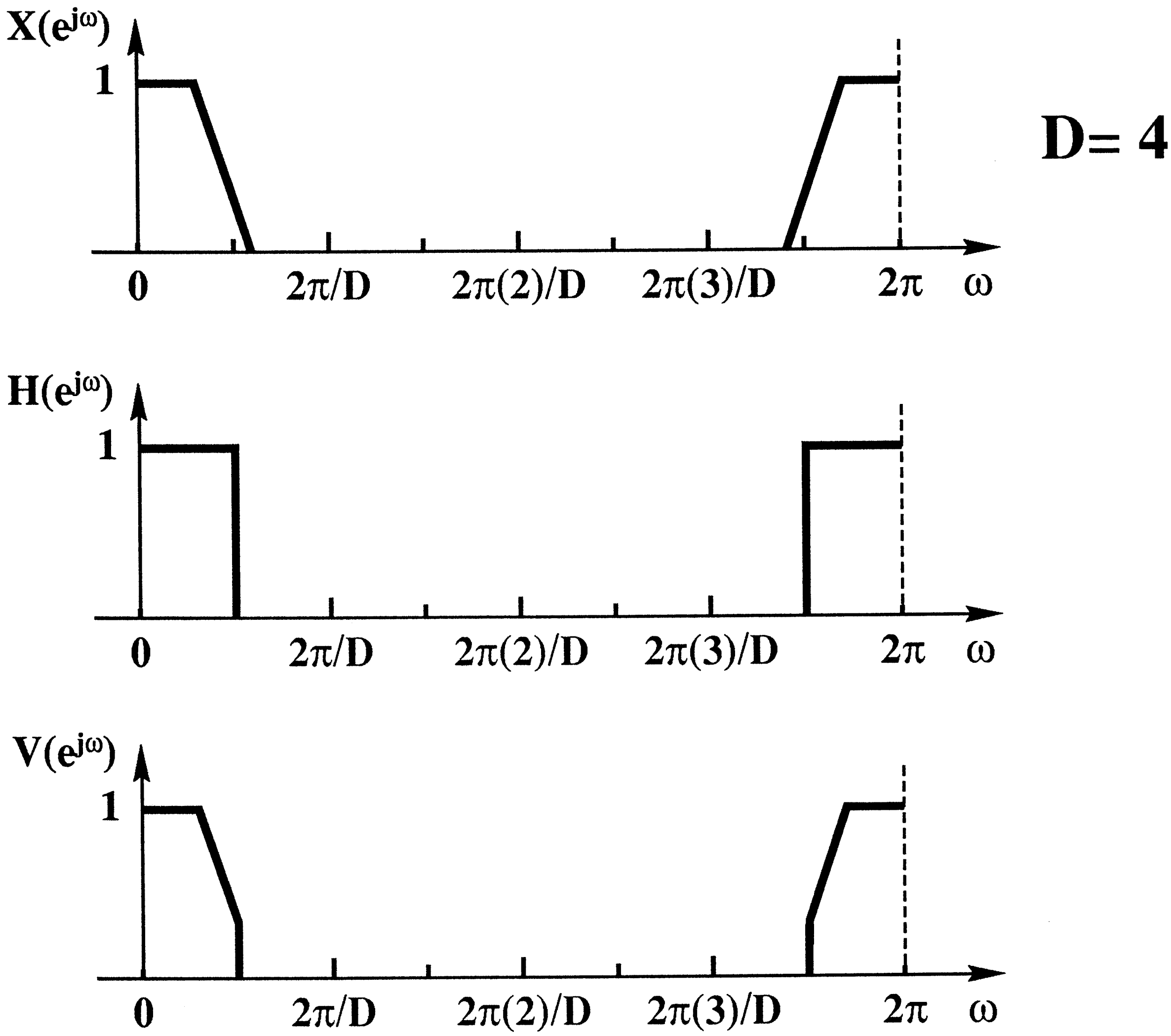

X(e^{jω})
1
D=4
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
Y(e^{jω})
1/D
0
π
2π
ω

Decimator

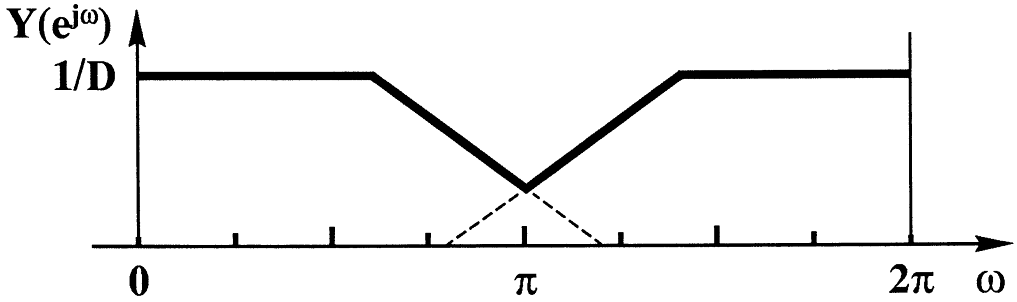
To prevent aliasing due to downsampling, we prefilter signal to bandlimit it to highest frequency $\omega_d = \pi/D$.

The combination of a lowpass filter followed by a downsampler is referred to as a *decimator.*

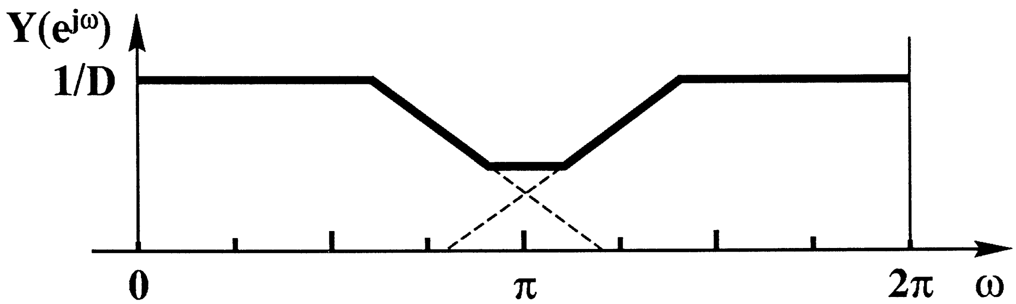



X(e^jω)
1
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
D= 4
H(e^jω)
1
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
V(e^jω)
1
0
2π/D
2π(2)/D
2π(3)/D
2π
ω

With prefilter



Without prefilter



UPSAMPLING

$$y(n) = \begin{cases} x(n/D) & , \text{ if } n/D \text{ is an integer} \\ 0 & , \text{ else} \end{cases}$$

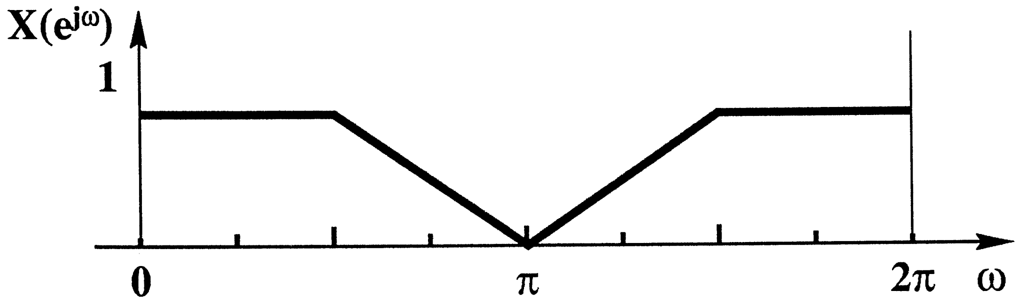
$$Y(e^{j\omega}) = \sum_{\substack{n \\ (n/D) \text{ is an integer}}} x(n/D)\, e^{-j\omega n}$$

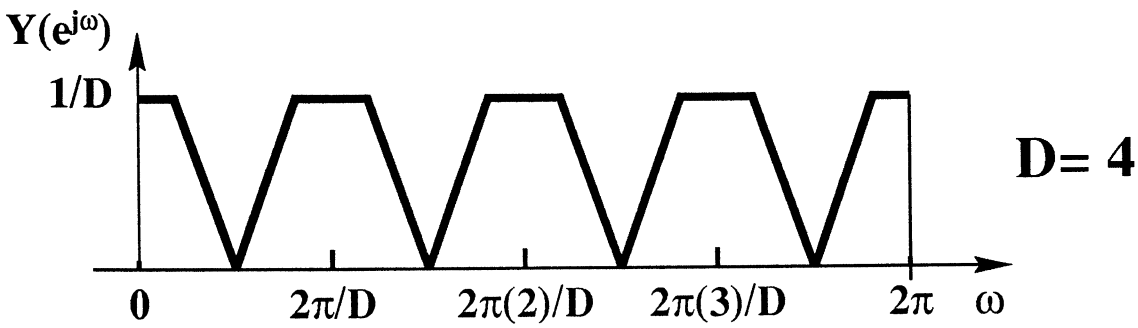
$$= \sum_{n} s_D(n)\, x(n/D)\, e^{-j\omega n}$$

Let $m = n/D \quad \Rightarrow n = mD$

$$Y(e^{j\omega}) = \sum_{m} s_D(mD)\, x(m)\, e^{-j\omega mD} \quad \text{but} \quad s_D(mD) \equiv 1$$

$$\therefore \quad Y(e^{j\omega}) = X(e^{j\omega D})$$

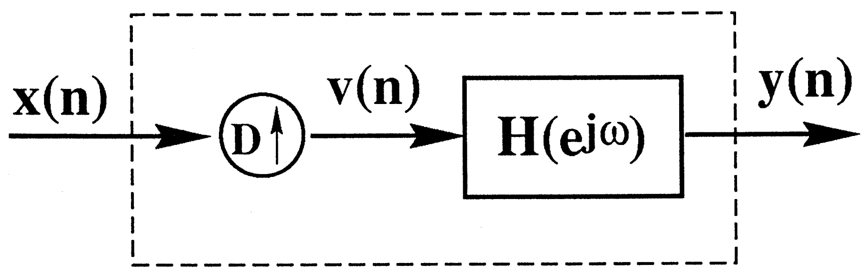


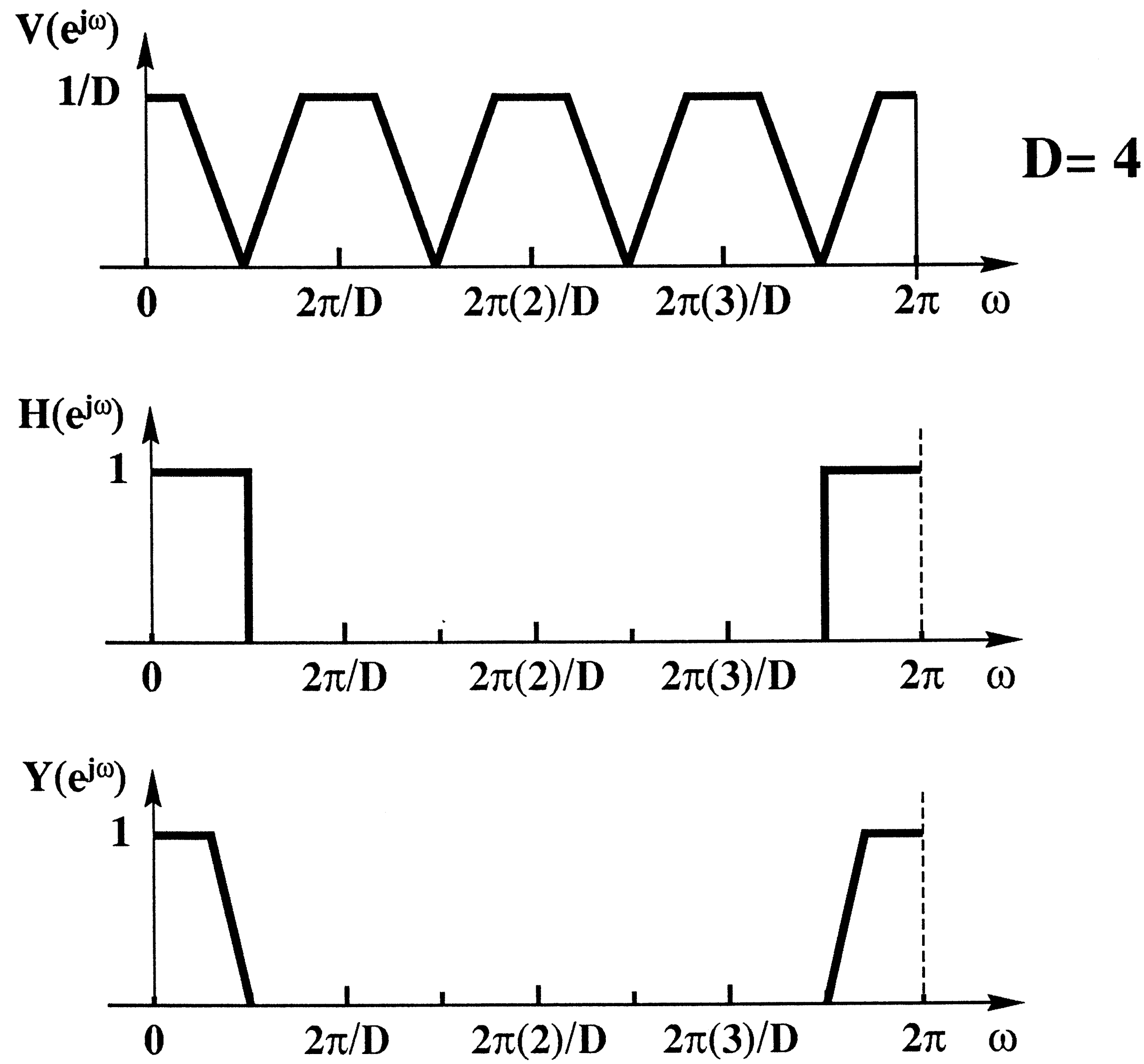


Interpolator

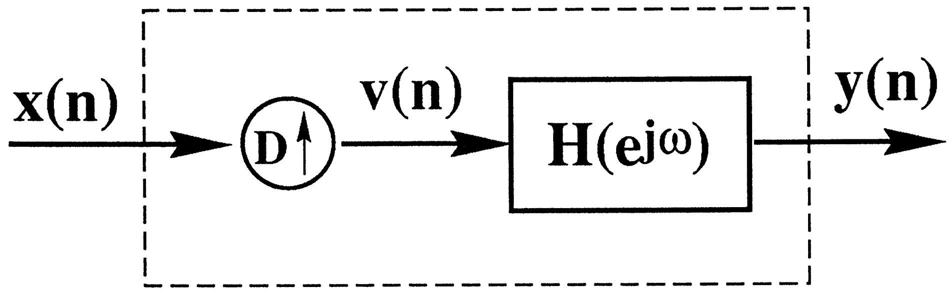
To interpolate between the nonzero samples generated by upsampling, we use a lowpass filter with cutoff at $\omega_d = \pi/D$.

The combination of an upsampler followed by a lowpass filter is referred to as an *interpolator*




V(e^jω)
1/D
D= 4
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
H(e^jω)
1
0
2π/D
2π(2)/D
2π(3)/D
2π
ω
Y(e^jω)
1
0
2π/D
2π(2)/D
2π(3)/D
2π
ω

Time Domain Analysis of Interpolator



$$y(n) = \sum_{k} v(k)\, h(n - k)$$

$$v(k) = s_D(k)\, x(k/D)$$

$$\text{let} \quad \ell = k/D \quad \Rightarrow \quad k = \ell D$$

$$y(n) = \sum_{\ell} x(\ell)\, h(n - \ell D)$$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/D}^{\pi/D} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\, CTFT^{-1}\{rect\,(\frac{\omega}{2\pi/D})\}\,\Big|_{\,t=n/2\pi}$$

$$= \frac{1}{2\pi}\, \frac{2\pi}{D}\, sinc\,[\frac{2\pi}{D}\,(\frac{n}{2\pi})]$$

$$= \frac{1}{D}\, sinc\,(\frac{n}{D})$$

$$y(n) = \sum_{\ell} x(\ell)\ h(n - \ell D)$$

$$= \sum_{\ell} x(\ell)\ \text{sinc}\left(\frac{n - \ell D}{D}\right)$$