1.5.1 DERIVATION OF THE Z TRANSFORM

- Recall sufficient conditions for existence of the DTFT

  1) $\sum_{n} |x(n)|^2 < \infty$

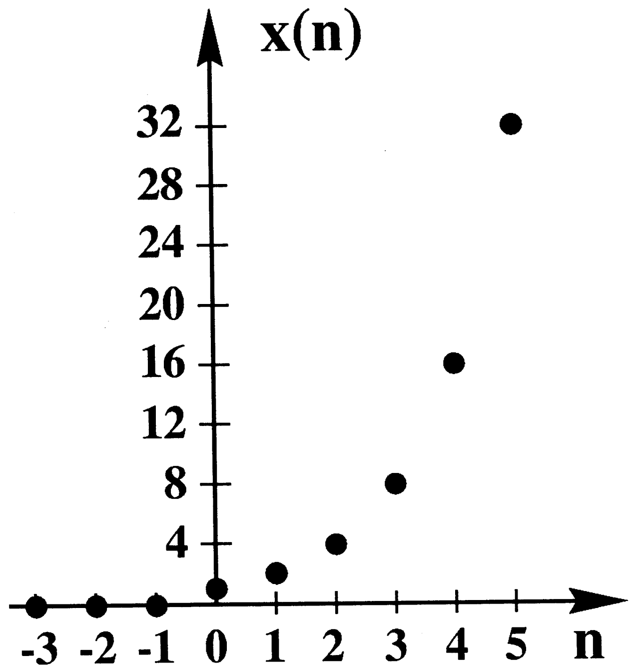
  or

  2) $\sum_{n} |x(n)| < \infty$

- By using impulses we were able to define the DTFT for periodic signals which satisfy neither of the above conditions.

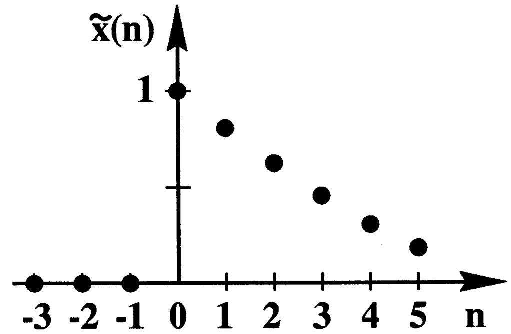
Example 1

Consider $\quad x(n) = 2^n \, u(n)$



- It also satisfies neither condition for existence of the DTFT.

Define $\tilde{x}(n) = r^{-n}\, x(n)\ ,\qquad r > 2$



Now $\sum_{n=0}^{\infty} |\tilde{x}(n)| = \sum_{n=0}^{\infty} |r^{-n}\, x(n)|$

$$= \sum_{n=0}^{\infty} (2/r)^n$$

$$\sum_{n=0}^{\infty} |\tilde{x}(n)| = \frac{1}{1 - 2/r} \quad , \quad (2/r) < 1$$

$$\text{or} \quad r > 2$$

$\therefore$ DTFT of $\tilde{x}(n)$ exists

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} \tilde{x}(n)\, e^{-j\omega n}$$

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} [2(re^{j\omega})^{-1}]^n$$

$$= \frac{1}{1 - 2(re^{j\omega})^{-1}} \quad , \quad | 2(re^{j\omega})^{-1} | < 1$$

$$\text{or} \quad r > 2$$

Now let's express $\tilde{X}(e^{j\omega})$ in terms of x(n)

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} r^{-n}\, x(n)\, e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} x(n)\, (re^{j\omega})^{-n}$$

Let $z = re^{j\omega}$ and define the Z Transform (ZT) of x(n) to be the DTFT of x(n) after multiplication by the convergence factor $r^{-n}u(n)$.

$$X(z) = \tilde{X}(e^{j\omega}) = \sum_{n} x(n)\, z^{-n}$$

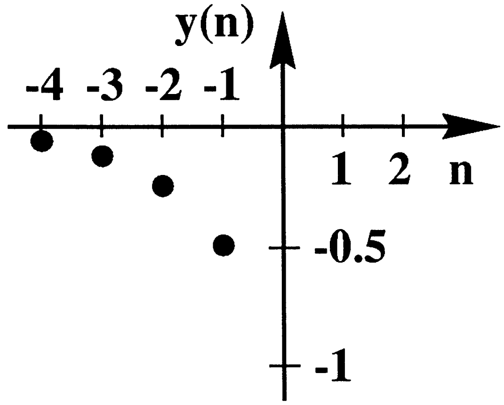
For the example $x(n) = 2^n\, u(n)$,

$$X(z) = \frac{1}{1 - 2z^{-1}} \quad , \quad |z| > 2$$

- It is important to specify the region of convergence since the transform is not uniquely defined without it.

Example 2

Let $y(n) = -2^n \, u(-n-1)$



$$Y(z) = \sum_{n} y(n) \, z^{-n}$$

$$= - \sum_{n=-\infty}^{-1} 2^n \, z^{-n}$$

$$Y(z) = -\sum_{n=-\infty}^{-1} (z/2)^{-n}$$

$$= -\sum_{n=1}^{\infty} (z/2)^{n}$$

$$= -\sum_{n=0}^{\infty} (z/2)^{n} + 1$$

$$Y(z) = 1 - \frac{1}{1 - z/2} \quad , \quad |z/2| < 1$$

$$\text{or} \quad |z| < 2$$

$$= \frac{-z/2}{1 - z/2}$$

$$= \frac{1}{1 - 2z^{-1}} \quad , \quad |z| < 2$$

so we have

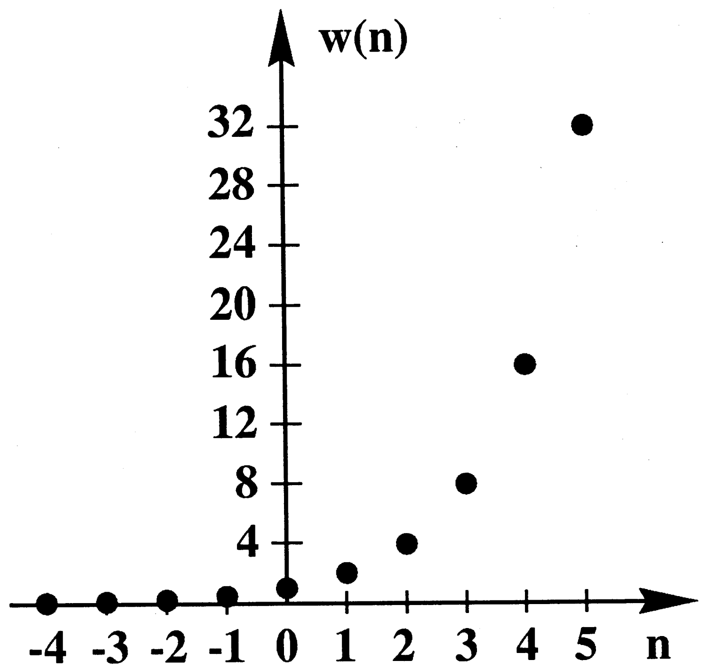
$$x(n) = 2^n u(n) \overset{ZT}{\longleftrightarrow} X(z) = \frac{1}{1 - 2z^{-1}} \quad , \quad |z| > 2$$

$$y(n) = -2^n \, u(-n-1) \overset{ZT}{\longleftrightarrow} Y(z) = \frac{1}{1 - 2z^{-1}} \quad , \quad |z| < 2$$

- The two transforms have the same functional form.
- They differ only in their regions of convergence.

Example 3

$w(n) = 2^n, \qquad -\infty < n < \infty$



$$w(n) = x(n) - y(n)$$

By linearity of the ZT,

$$W(z) = X(z) - Y(z)$$

$$= \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - 2z^{-1}} = 0 \; !$$

- But note that X(z) and Y(z) have no common region of convergence.

  $\therefore$ There is no ZT for $w(n) = 2^n$, $-\infty < n < \infty$.