1.5.5 INVERSE Z TRANSFORM

- Formally, the inverse ZT may be written as

$$x(n) = \frac{1}{j2\pi} \oint_{c} X(z)\, z^{n-1} dz$$

where the integration is performed in a counter-clockwise direction around a closed contour in the region of convergence of X(z) and encircling the origin.

- If X(z) is a rational function of z, *i.e.* a ratio of polynomials, it is not necessary to evaluate this integral.

- Instead, we use a partial fraction expansion to express X(z) as a sum of simple terms for which the inverse transform may be recognized by inspection.

- The ROC plays a critical role in this process.

- We will illustrate the method via a series of examples.

Example 1

The signal $x(n) = (1/3)^n u(n)$ is input to a DT LTI system described by

$$y(n) = x(n) - x(n-1) + (1/2)\, y(n-1)\ .$$

Find the output y(n).

What are our options?

a. direct substitution

Assume $y(-1) = 0$

$$y(0) = x(0) - x(-1) + (1/2)\,y(-1) = 1 - 0 + (1/2)(0) = 1.000$$

$$y(1) = x(1) - x(0) + (1/2)\,y(0) = 1/3 - 1 + (1/2)(1.0) = -0.167$$

$$y(2) = x(2) - x(1) + (1/2)\,y(1) = 1/9 - 1/3 + (1/2)(-0.167) = -0.306$$

$$y(3) = x(3) - x(2) + (1/2)\,y(2) = 1/27 - 1/9 + (1/2)(-0.306) = -0.227$$

b. convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x(k)$$

find impulse response by direct substitution

$$h(n) = \delta(n) - \delta(n-1) + (1/2)\, h(n-1)$$

$$h(0) = 1 - 0 + (1/2)(0) = 1$$

$$h(1) = 0 - 1 + (1/2)(1) = -1/2$$

$$h(2) = 0 - 0 + (1/2)(-1/2) = -1/4$$

$$h(3) = 0 - 0 + (1/2)(-1/4) = -1/8$$

recognize $h(n) = \delta(n) - (1/2)^n u(n-1)$

$$y(n) = \sum_{k=-\infty}^{\infty} [\delta(n-k) - (1/2)^{(n-k)} u(n-k-1)]\, (1/3)^k u(k)$$

$$= (1/3)^n u(n) - (1/2)^n \sum_{k=0}^{n-1} (2/3)^k u(n-1)$$

$$= (1/3)^n u(n) - (1/2)^n \frac{1-(2/3)^n}{1-2/3}\, u(n)$$

$$= 4(1/3)^n u(n) - 3(1/2)^n u(n)$$

c. DTFT

$$x(n) = (1/3)^n u(n)$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}\, e^{-j\omega}}$$

$$h(n) = \delta(n) - (1/2)^n u(n-1)$$

$$H(e^{j\omega}) = 1 - \left[\frac{1}{1 - \frac{1}{2}\, e^{-j\omega}} - 1\right]$$

$$= \frac{1 - e^{-j\omega}}{1 - \frac{1}{2}\, e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$= \frac{1 - e^{-j\omega}}{(1 - \frac{1}{2}\, e^{-j\omega})(1 - \frac{1}{3}\, e^{-j\omega})}$$

$$= \frac{1 - e^{-j\omega}}{1 - \frac{5}{6}\, e^{-j\omega} + \frac{1}{6}\, e^{-j\omega 2}}$$

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - e^{-j\omega}}{1 - \frac{5}{6}\, e^{-j\omega} + \frac{1}{6}\, e^{-j\omega 2}}\, e^{j\omega n}\, d\omega$$

d. ZT

$$x(n) = (1/3)^n u(n)$$

$$X(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} \,, \qquad |z| > \frac{1}{3}$$

$$y(n) = x(n) - x(n-1) + (1/2)\, y(n-1)$$

$$Y(z) = X(z) - z^{-1}X(z) + (1/2)\, z^{-1}Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

What is region of convergence?

Since system is causal, h(n) is a causal signal.

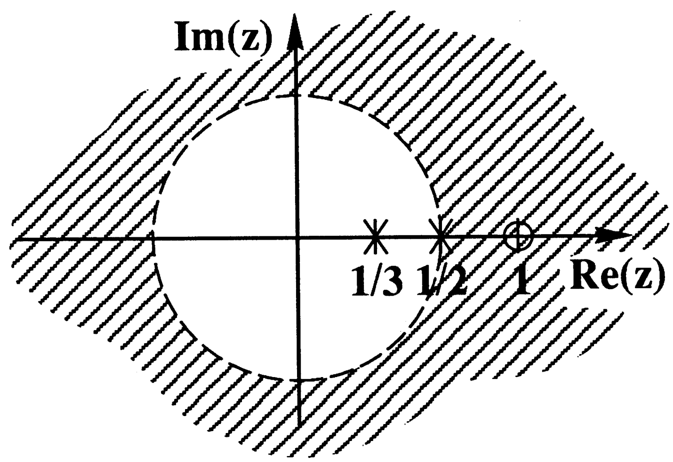
$\Rightarrow$ H(z) converges for $|z| > 1/2$.

$$Y(z) = H(z)\,X(z)$$

$$= \frac{1 - z^{-1}}{(1 - \frac{1}{2}\,z^{-1})(1 - \frac{1}{3}\,z^{-1})}$$

$$\mathrm{ROC}[Y(z)] = \mathrm{ROC}[H(z)] \;\cap\; \mathrm{ROC}[X(z)]$$

$$= \{z\colon\ |\,z\,| \ > 1/2\}$$



Partial Fraction Expansion (PFE)

(Two distinct poles)

$$\frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})} = \frac{A_1}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - \frac{1}{3} z^{-1}}$$

To solve for A_1 and A_2, we multiply both sides by $(1 - \frac{1}{2} z^{-1})\,(1 - \frac{1}{3} z^{-1})$ to obtain

$$1 - z^{-1} = A_1(1 - \frac{1}{3} z^{-1}) + A_2(1 - \frac{1}{2} z^{-1})$$

$$1 = A_1 + A_2 \qquad A_1 = -3$$

$$-1 = -\frac{1}{3} A_1 - \frac{1}{2} A_2 \qquad A_2 = 4$$

so

$$\frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})} = \frac{-3}{1 - \frac{1}{2} z^{-1}} + \frac{4}{1 - \frac{1}{3} z^{-1}}$$

check

$$1 - z^{-1} = -3(1 - \frac{1}{3} z^{-1}) + 4(1 - \frac{1}{2} z^{-1})$$

Now

$$y(n) = y_1(n) + y_2(n)$$

possible ROC's

where $$Y_1(z) = \frac{-3}{1 - \frac{1}{2} z^{-1}}, \qquad |z| < \frac{1}{2} \text{ or } |z| > \frac{1}{2}$$

$$Y_2(z) = \frac{4}{1 - \frac{1}{3} z^{-1}}, \qquad |z| < \frac{1}{3} \text{ or } |z| > \frac{1}{3}$$

and

$$ROC[Y(z)] = ROC[Y_1(z)] \cap ROC[Y_2(z)]$$

$$= \{z: |z| > \frac{1}{2}\}$$

$$\therefore \quad ROC[Y_1(z)] = \{z: |z| > \frac{1}{2}\}$$

$$ROC[Y_2(z)] = \{z: |z| > \frac{1}{3}\}$$

Recall transform pairs:

$$a^n u(n) \overset{ZT}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

$$-a^n u(-n-1) \overset{ZT}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \qquad |z| < |a|$$

$$\therefore \quad \frac{-3}{1 - \frac{1}{2} z^{-1}} \xrightarrow{ZT^{-1}} -3(\frac{1}{2})^n u(n)$$

$$\frac{4}{1 - \frac{1}{3} z^{-1}} \xrightarrow{ZT^{-1}} 4(\frac{1}{3})^n u(n)$$

and

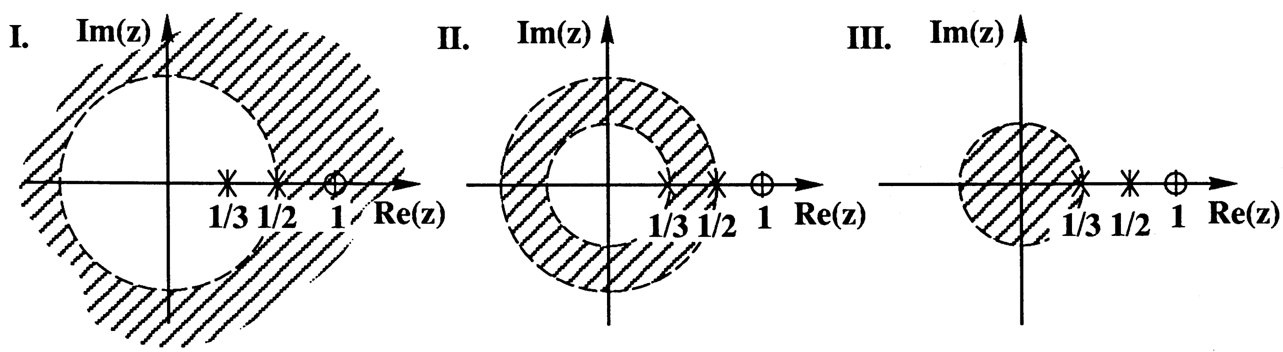
$$y(n) = -3(\frac{1}{2})^n u(n) + 4(\frac{1}{3})^n u(n)$$

Consider again

$$Y(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})}$$

$$= \frac{-3}{1 - \frac{1}{2} z^{-1}} + \frac{4}{1 - \frac{1}{3} z^{-1}}$$

There are 3 possible ROC's for a signal with this ZT.



Case II. $\frac{1}{3} < |z| < \frac{1}{2}$

possible ROC's

where $Y_1(z) = \dfrac{-3}{1 - \frac{1}{2} z^{-1}}$ $\quad |z| < \frac{1}{2}$ or $|z| > \frac{1}{2}$

$Y_2(z) = \dfrac{4}{1 - \frac{1}{3} z^{-1}}$ $\quad |z| < \frac{1}{3}$ or $|z| > \frac{1}{3}$

and

$$\mathrm{ROC}[Y(z)] = \mathrm{ROC}[Y_1(z)] \cap \mathrm{ROC}[Y_2(z)]$$

$$\therefore \quad \mathrm{ROC}[Y_1(z)] = \{z: |z| < \frac{1}{2}\}$$

$$\mathrm{ROC}[Y_2(z)] = \{z: |z| > \frac{1}{3}\} \qquad \text{as before}$$

$$\frac{-3}{1 - \frac{1}{2} z^{-1}} \xrightarrow{ZT^{-1}} -3(\frac{1}{2})^n u(-n-1)$$

$$\frac{4}{1 - \frac{1}{3} z^{-1}} \xrightarrow{ZT^{-1}} 4(\frac{1}{3})^n u(n) \qquad \text{as before}$$

$$y(n) = 3(\frac{1}{2})^n u(-n-1) + 4(\frac{1}{3})^n u(n)$$

Case III. $|z| < \frac{1}{3}$

$$\text{ROC}[Y_1(z)] = \{z: |z| < \frac{1}{2}\} \quad \text{as before}$$

$$\text{ROC}[Y_2(z)] = \{z: |z| < \frac{1}{3}\}$$

$$\frac{-3}{1 - \frac{1}{2} z^{-1}} \xrightarrow{ZT^{-1}} 3(\frac{1}{2})^n u(-n-1) \quad \text{as before}$$

$$\frac{4}{1 - \frac{1}{3} z^{-1}} \xrightarrow{ZT^{-1}} -4(\frac{1}{3})^n u(-n-1)$$

$$y(n) = 3(\frac{1}{2})^n u(-n-1) - 4(\frac{1}{3})^n u(-n-1)$$

Summary of Possible ROC's and Signals

$$Y(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})}$$

ROC	Signal
$\frac{1}{2} < \lvert z \rvert$	$-3(\frac{1}{2})^n u(n) + 4(\frac{1}{3})^n u(n)$
$\frac{1}{3} < \lvert z \rvert < \frac{1}{2}$	$3(\frac{1}{2})^n u(-n-1) + 4(\frac{1}{3})^n u(n)$
$\lvert z \rvert < \frac{1}{3}$	$3(\frac{1}{2})^n u(-n-1) - 4(\frac{1}{3})^n u(-n-1)$

Residue Method for Evaluating Coefficients of PFE

$$Y(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})} = \frac{A_1}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - \frac{1}{3} z^{-1}}$$

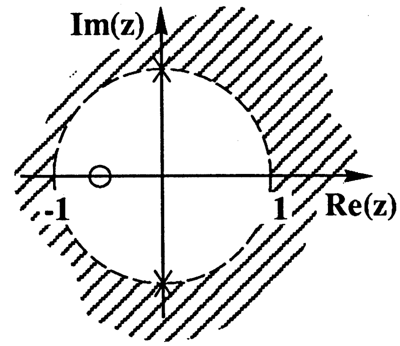
$$A_1 = (1 - \frac{1}{2} z^{-1})\, Y(z)\Big|_{z=\frac{1}{2}} = \frac{1 - z^{-1}}{1 - \frac{1}{3} z^{-1}}\Bigg|_{z=\frac{1}{2}} = \frac{1 - 2}{1 - 2/3} = -3$$

$$A_2 = (1 - \frac{1}{3} z^{-1})\, Y(z)\Big|_{z=\frac{1}{3}} = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}}\Bigg|_{z=\frac{1}{3}} = \frac{1 - 3}{1 - 3/2} = 4$$

Example 2 (complex conjugate poles)

$$Y(z) = \frac{1 + \frac{1}{2} z^{-1}}{(1 - jz^{-1})(1 + jz^{-1})}$$

$|z| > 1$



$$X(z) = \frac{A_1}{1 - jz^{-1}} + \frac{A_2}{1 + jz^{-1}}$$

$$A_1 = \left. \frac{1 + \frac{1}{2} z^{-1}}{1 + jz^{-1}} \right|_{z=j} \qquad A_2 = \left. \frac{1 + \frac{1}{2} z^{-1}}{1 - jz^{-1}} \right|_{z=-j}$$

$$= \frac{1}{2}(1 - \frac{1}{2} j) \qquad = \frac{1}{2}(1 + \frac{1}{2} j)$$

check

$$\frac{\frac{1}{2}(1 - \frac{1}{2}\,j)}{1 - jz^{-1}} + \frac{\frac{1}{2}(1 + \frac{1}{2}\,j)}{1 + jz^{-1}}$$

$$= \frac{\frac{1}{2}(1 - \frac{1}{2}\,j)(1 + jz^{-1}) + \frac{1}{2}(1 + \frac{1}{2}\,j)(1 - jz^{-1})}{(1 - jz^{-1})(1 + jz^{-1})}$$

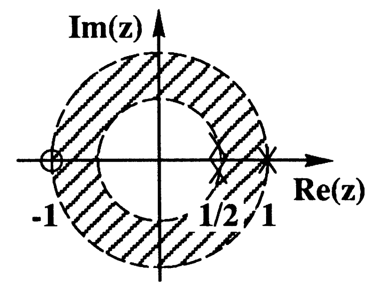
$$= \frac{1 + \frac{1}{2}\,z^{-1}}{(1 - jz^{-1})(1 + jz^{-1})}$$

- Note that $A_1 = A_2^*$.
- This is necessary for y(n) to be real-valued.
- Use it to save computation.

Example 3 (poles with multiplicity > 1)

$$Y(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})^2 (1 - z^{-1})}$$

$$\frac{1}{2} < |z| < 1$$



recall

$$n\, a^n u(n) \overset{ZT}{\longleftrightarrow} \frac{a\, z^{-1}}{(1 - az^{-1})^2}\,, \qquad |z| > |a|$$

Try

$$\frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})^2 (1 - z^{-1})} = \frac{A_1}{(1 - \frac{1}{2} z^{-1})^2} + \frac{A_2}{1 - z^{-1}}$$

$$A_1 = \left. \frac{1 + z^{-1}}{1 - z^{-1}} \right|_{z=\frac{1}{2}} \qquad A_2 = \left. \frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})^2} \right|_{z=1}$$

$$= -3 \qquad\qquad = 8$$

check

$$\frac{-3}{(1-\frac{1}{2}\,z^{-1})^2}+\frac{8}{1-z^{-1}}=\frac{-3(1-z^{-1})+8(1-z^{-1}+\frac{1}{4}\,z^{-2})}{(1-\frac{1}{2}\,z^{-1})^2(1-z^{-1})}$$

$$=\frac{5-5z^{-1}+2z^{-2}}{(1-\frac{1}{2}\,z^{-1})^2(1-z^{-1})}$$

$$\neq\frac{1+z^{-1}}{(1-\frac{1}{2}\,z^{-1})(1-z^{-1})}$$

General form of terms in PFE for pole p_ℓ with multiplicity m_ℓ

$$\frac{A_{\ell,m_\ell}}{(1-p_\ell z^{-1})^{m_\ell}} + \frac{A_{\ell,m_\ell-1}}{(1-p_\ell z^{-1})^{m_\ell-1}} + \ldots + \frac{A_{\ell,1}}{1-p_\ell z^{-1}}$$

Now have

$$\frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})^2 (1 - z^{-1})} = \frac{A_{1,2}}{(1 - \frac{1}{2} z^{-1})^2} + \frac{A_{1,1}}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$A_{1,2} = -3 \qquad A_2 = 8 \qquad$ as before

To solve for $A_{1,1}$, multiply both sides by $(1 - \frac{1}{2} z^{-1})^2$

$$\frac{1 + z^{-1}}{1 - z^{-1}} = A_{1,2} + A_{1,1}(1 - \frac{1}{2}\, z^{-1}) + A_2 \frac{(1 - \frac{1}{2}\, z^{-1})^2}{1 - z^{-1}}$$

differentiate with respect to z^{-1}

$$\frac{2}{(1 - z^{-1})^2} = 0 + A_{1,1}(-\frac{1}{2}) + A_2(1 - \frac{1}{2}\, z^{-1})\, P(z)$$

let $\quad z = \frac{1}{2}$

$$A_{1,1} = -2\, \frac{2}{(1 - 2)^2} = -4$$

Example 4 (numerator degree $\geq$ denominator degree)

$$Y(z) = \frac{1 + z^{-2}}{1 - \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}}$$

reduce degree of numerator by long division

$$\begin{array}{r|l} & \quad -2 \\ \hline -\frac{1}{2} z^{-2} - \frac{1}{2} z^{-1} + 1 & z^{-2} + 1 \\ & z^{-2} + z^{-1} - 2 \\ \hline & -z^{-1} + 3 \end{array}$$

so

$$\frac{1 + z^{-2}}{1 - \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}} = -2 + \frac{3 - z^{-1}}{1 - \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}}$$

In general, will have a polynomial of degree M − N

$$\sum_{k=0}^{M-N} B_k z^{-k} \xrightarrow{ZT^{-1}} \sum_{k=0}^{M-N} B_k \delta(n-k)$$